Alternatives to Linear Models

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Outline

- Linear Model Usefulness
- Alternative Approaches
- Examples
    - Generalized least squares
    - Piecewise linear model

Material mentioned in Chapter 1 of Faraway textbook

Linear Models' Usefulness

- Focus is on relating the mean to a set of predictors
- Models generally easy to fit $\rightarrow$ no convergence issues
- Sampling dists rely more so on means being Normally distributed rather than the data themselves (CLT)
- For skewed data, variance often increases/decreases with mean so a transformation will stabilize variance and improve Normality.
- Nonlinear relationships often well-described by polynomials
- All models are wrong. Are the violations such that useful information cannot be extracted?

Alternative Approaches

- What to do if model conds are not approximately met?
  - Transformation of Y or X's
  - Weighted least squares
  - Bootstrapping
  - Robust regression
  - Regression trees
  - Piecewise linear model
  - Additive models
  - Nonparameteric Regression
  - Penalized regression
  - Linear mixed model
  - Generalized linear model
  - Generalized linear mixed model

Case #1 - Generalized Least Squares

- Normal error regression model is reasonable except there are correlated errors and unequal variances
- Consider $\mathbf{Y} = \mathbf{X}\beta + \varepsilon$ where $\boldsymbol{\sigma}^2(\varepsilon) = \mathbf{W}^{-1}$ is known
- Can transform Y based on $\mathbf{W}$

$$\begin{aligned} \mathbf{W}^{1/2}\mathbf{Y} &= \mathbf{W}^{1/2}\mathbf{X}\beta + \mathbf{W}^{1/2}\varepsilon \\ \mathbf{Y}_w &= \mathbf{X}_w\beta + \varepsilon_w \end{aligned}$$

- Results in

$$\mathrm{E}(\varepsilon_w) = 0 \text{ and } \boldsymbol{\sigma}^2(\varepsilon_w) = \mathbf{I}$$

- Use OLS to fit $\mathbf{Y}_w$ to $\mathbf{X}_w$

$$\hat{\beta} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{Y}$$

Case #2 - Weighted Least Squares

- Normal error regression model is reasonable except for unequal error variances
- This is just a special case of Case #1 where **W** is a diagonal matrix.
- In terms of maximum likelihood

$$Y_i \sim \mathrm{N}(\mathbf{X}_i\boldsymbol{\beta}, \sigma_i^2) \qquad (\sigma_i\text{'s known})$$

$$\downarrow$$

$$f_i = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left\{-\frac{1}{2\sigma_i^2}(Y_i - \mathbf{X}_i\boldsymbol{\beta})^2\right\}$$

- Want β that minimizes

$$\sum_{i=1}^{n} \frac{1}{\sigma_i^2}(Y_i - \mathbf{X}_i\boldsymbol{\beta})^2$$

Weighted Least Squares

- Can be implemented in R using lm() with weight variable
- Optimal weights $\propto 1/\text{var}(Y_i)$
- In practice, rarely know the variances *apriori*
- If they were known, define

$$\mathbf{W} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \cdots & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1/\sigma_{n-1}^2 & 0 \\ 0 & \cdots & \cdots & 0 & 1/\sigma_n^2 \end{bmatrix}$$

Determining the Weights

- Methods depend on data available
  - Use grouped data or approximately grouped data to estimate the variance
  - Compute the absolute (or squared) residual and determine a function for the standard deviation (variance).
  - Function could depend on predictors, fitted value/mean. Could be parametric (regression) or nonparametric (LOESS)
- Because weights often based on residuals, suggests iterative approach but convergence is quick

Linear Mixed Model Approach

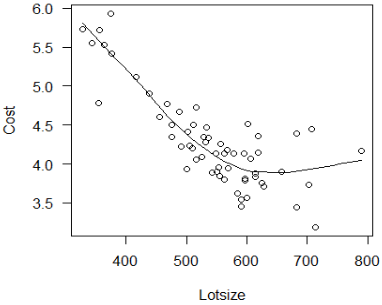
- Often assuming variance/covariance matrix is a diagonal matrix whose values along the main diagonal (the variances) are either a
  - Linear function of $\mathbf{X}$
  - Quadratic function of $\mathbf{X}$
- This relationship along with the estimation of parameters can be done simultaneously using the appropriate covariance structure
- Will discuss in more detail when addressing linear mixed models

Case #3 - Piecewise Linear Model

- Lead-in to nonparametric regression
- At some point or points, the slope of the relationship changes
- When points known, can build into standard regression framework
- If points unknown, becomes a more difficult problem
    - How many change points?
    - Where is the location of each change point?

Lotsize versus Cost to Produce

- Expect cost to go down with size due to operating efficiencies
- Relationship may plateau or alter at high lot sizes



Possible Models

- Will assume there is a single changepoint
- Two possible models

$$\text{Model 1: } E(\text{Cost}) = \beta_0 + \beta_1 \text{lotsize} + \beta_2 \max(0, \text{lotsize} - C_x)$$

$$\text{Model 2: } E(\text{Cost}) = \begin{cases} \beta_0 + \beta_1 \text{lotsize} & \text{lotsize} \leq C_x \\ \beta_0 + \beta_1 C_x & \text{lotsize} > C_x \end{cases}$$

- Cycle through possible C_x to find the best changepoint
- Best refers to the model with the smallest s
- This is assuming σ remains constant. Reasonable?

R Code

```
regsigma = numeric(length=150)
regsigma1 = numeric(length=150)
for(i in 1:150){
 chngpt = 499+i

 cslope=0*(Lotsize<=chngpt)+(Lotsize-chngpt)*(Lotsize>chngpt)
 Lotsize1 = Lotsize*(Lotsize<=chngpt)+chngpt*(Lotsize>chngpt)

 jnk = lm(Cost~Lotsize+cslope, manufacturing)
 regsigma[i] <- summary(jnk)$sigma

 jnk1 = lm(Cost~Lotsize1, manufacturing)
 regsigma1[i] <- summary(jnk1)$sigma
}
```

R Code

```
chngpt = 499+which.min(regsigma1)

Lotsize1 = Lotsize*(Lotsize<=chngpt)+chngpt*(Lotsize>chngpt)
model1 = lm(Cost~Lotsize1, manufacturing)

plot(Lotsize,Cost,las=1)

critpts <- c(min(Lotsize)-5,chngpt,max(Lotsize)+5)

ypred = model1$coefficients[1]+critpts*model1$coefficients[2]
ypred[3]=ypred[2]

lines(critpts,ypred,col="red")
```

Lotsize versus Cost to Produce