Sample Size/Power Calculations

Design of Experiments - Montgomery
Sections 3.7, 13.4

Choice of Sample Size

Fixed Effects Model

- Can determine sample size for
  - Overall F test
  - Constrast of interest
- For simplicity, typically assume n_i's constant
- Power = 1 − P(Type II error) = $1-\beta$

– Determining power for F test

$\beta = \Pr(F_0 < F_{\alpha,a-1,N-a} | H_0$ false)

Need to know distribution of F_0 when H_0 false

Can show $F_0 \sim F_{a-1,N-a}(\delta)$

$\delta = n\sum \tau_i^2/\sigma^2$ (non-centrality parameter)

Recall E(MS$_{\text{Trt}}$)=$\sigma^2 + n\sum \tau_i^2/(a-1)$

Need to specify the τ_i's

Power will vary for different choices

Power calculation for F test

- Given α, a, and n, can determine $F_{\alpha,a-1,N-a}$
- Given some value of δ, can use noncentral F to compute power

  In SAS, use function probf

  Power=1-probf($F_{\alpha,a-1,N-a}$,$a-1$,$N-a$,δ)

- Montgomery: OCC given in Chart V

  Plots β vs Φ

  $\Phi^2 = n\sum \tau_i^2/a\sigma^2 = \delta/a$

  Can use charts to determine power or sample size

Methods to determine the τ_i's

1 Choose treatment means ($\mu + \tau_i$)

- Solve for $\tau_i's$ and compute Φ^2 or δ
- Difficult to know what means to select

2 Take a mimimum difference approach

- Rejection if any $|\tau_i - \tau_j| > D$
- Use $\Phi^2 = nD^2/2a\sigma^2$ (i.e., $\{\tau_i\} = \{-D/2, 0, \ldots, 0, D/2\}$)
- Power of test at least $1-\beta$

3 Specify a std dev percentage increase (P)

- Under H_1, variance of random y_i is $\sigma_y^2 = \sigma^2 + \sum \tau_i^2/a$
- Randomly chosen τ_i has mean 0 and variance $\sum \tau_i^2/a$
- $(\sigma^2 + \sum \tau_i^2/a)/\sigma^2 = (1+.01P)^2$
- $\delta = an((1+.01P)^2 - 1)$
- $\Phi^2 = n((1+.01P)^2 - 1)$

Power calculation for specific contrast

- Often with an experiment, a researcher is primarily interested in just a few comparisons or contrasts. In these cases, it can be preferable to determine sample size for these rather than the overall F test.
- This reduces problem back to the t test situation
- Need to determine
  - Difference of importance
  - Standard error of comparison
- May want/need to adjust for multiple comparisons
- Montgomery describes confidence interval approach
  - Consider pairwise difference in treatment means
  - Specify length of (1-α)% confidence interval
  - Length/2 = $t_{\alpha/2,N-a}\sqrt{\frac{2\text{MS}_\text{E}}{n}}$
  - Based on choice of MS_E, find n

Example 3.1 - Etch Rate (page 70)

- Consider new experiment to investigate 5 RF power settings equally spaced between 180 and 200 W
- Wants to determine sample size to detect a mean difference of D=30 (Å/min) with 80% power
- Will use Example 3.1 estimates to determine new sample size

  $\hat{\sigma}^2 = 333.7$, $D = 30$, and $\alpha = .05$

- Using Table V : $\Phi^2 = 900n/2(5)(333.7) \approx .27n$

n	9	10	11
Φ	$\sqrt{2.43}$	$\sqrt{2.70}$	$\sqrt{3.0}$
df_E	40	45	50
β	26%	20%	15%

- Using SAS : $\delta = a\Phi^2$

n	9	10	11
δ	12.14	13.49	14.83
df_E	40	45	50
β	24.7%	19.2%	14.8%

- Appears $n = 10$ is proper choice

Example 3.1 - Etch Rate (page 70)

- Suppose the researcher was interested primarily in comparing each setting to 200 W.
- He'd like to detect a difference if the true difference is more than 30 (Å/min) with 80% power
- A pairwise comparison has standard error

$$\text{SE} = \sqrt{\frac{2\sigma^2}{n}}$$

- Consider using Dunnett's adjustment
- Thus reject when $|\text{Diff}| > d_{.05}(4, df_e)$SE
- Can construct power curve in SAS


```
data new1;
set params;
delta=30;
do n=5 to 15;
    df = a*(n-1); nc = delta/sqrt(var*2/n);
    crit = probmc("dunnett2",.,1-alpha,df,a-1);
    power=1-probt(crit,df,nc)+probt(-crit,df,nc);
    output;
 end;

proc print;
var n df power;
run;

-----------------------------------------------------------

bs     n    df     power

  1     5    20    0.49271
  2     6    25    0.59807
  3     7    30    0.68794
  4     8    35    0.76201
  5     9    40    0.82136  ***Only 9 replicates needed here
  6    10    45    0.86780
  7    11    50    0.90341
  8    12    55    0.93024
  9    13    60    0.95015
 10    14    65    0.96472
 11    15    70    0.97525
```

Choice of Sample Size

Random Effects

- Can use central F distribution

  $(N-a)\text{MS}_\text{E}/\sigma^2 \sim \chi^2_{N-a}$

  $(a-1)\text{MS}_\text{Trt}/(\sigma^2+n\sigma^2_\tau) \sim \chi^2_{a-1}$

  Thus $F_0/(1+n\sigma^2_\tau/\sigma^2) \sim F_{a-1,N-a}$

  Can specify ratio of σ^2_τ/σ^2

  Can specify percentage increase P

- OCC given in Chart VI

  Plots β vs λ

  $\lambda^2 = 1 + n\sigma^2_\tau/\sigma^2$

- Use SAS function probf

  power = 1-probf($F_{\alpha,a-1,N-a}/\lambda^2$,$a-1$,$N-a$)

Example - Batch Example on Slide 7-7

- Consider new experiment a random effects problem
  - The variance estimate is $\sigma^2 = 1.8$
  - Desire to detect situation when $\sigma^2_\tau \geq 3.6 = 2\sigma^2$
  - Set power at 80% and $\alpha = .05$
  - Using Table VI : $\lambda = \sqrt{1+2n}$

n	3	4	5
λ	$\sqrt{7}$	$\sqrt{9}$	$\sqrt{11}$
df_E	10	15	20
β	28%	18%	15%

  - Using SAS : use λ^2

n	3	4	5
λ^2	7	9	11
df_E	10	15	20
β	26.1%	15.3%	10.0%

  - Appears $n = 4$ gives appropriate power


```
options nocenter ps=35 ls=72;
/* This is how you can calculate the power in one data set */

data params;
  input a alpha ratiovar;
cards;
  5 .05 2.0
;

data new;
 set params;
 do n=2 to 10;
    df = a*(n-1);
    lambdasq = 1+ratiovar*n;
    fcut  = finv(1-alpha,a-1,df);
    beta=probf(fcut/lambdasq,a-1,df);
    output;
 end;

proc print;
var n beta;
run;
------------------------------------------------------------

OBS     N      BETA
 1      2    0.52933
 2      3    0.26112
 3      4    0.15292
 4      5    0.10027
 5      6    0.07081
 6      7    0.05267
 7      8    0.04072
 8      9    0.03242
 9     10    0.02643
```