# Blocking Designs

Design of Experiments - Montgomery
Section 4-1



# Randomized Complete Block Design

- Nuisance factor
  - Has effect on response but effect not of interest
  - If unknown → Protection through randomization
  - If known but uncontrollable → Analysis of Covariance
  - If known and controllable → Blocking Design
- Randomized Complete Block Design (RCBD)
  - $b$ blocks each consisting of $a$ experimental units
  - $a$ treatments are randomly assigned to EUs within block
  - results in restriction on randomization
  - extension of paired t-test where pairs=blocks



# Statistical Model

- $b$ blocks and $a$ treatments
- Statistical model is

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

$\mu$ - grand mean

$\tau_i$ - $i$th treatment effect

$\beta_j$ - $j$th block effect

$\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

- Assume additional additive effect due to block

  Block effect the same for all EUs in block
- Treatment and block effects deviations from mean

  Common parameter restrictions are then

$$\sum \tau_i = 0 \qquad \sum \beta_j = 0$$



# Partitioning the SS

- Rewrite observation as:

$$\begin{aligned} y_{ij} &= \overline{y}_{..} + (\overline{y}_{i.} - \overline{y}_{..}) + (\overline{y}_{.j} - \overline{y}_{..}) + (y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..}) \\ &= \widehat{\mu} + \widehat{\tau}_i + \widehat{\beta}_j + \widehat{\epsilon}_{ij} \end{aligned}$$

- Can partition $\mathrm{SS_T} = \sum\sum(y_{ij} - \overline{y}_{..})^2$ into

$$b\sum(\overline{y}_{i.} - \overline{y}_{..})^2 + a\sum(\overline{y}_{.j} - \overline{y}_{..})^2 + \sum\sum(y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..})^2$$

$$\mathrm{SS_{Treatment}} + \mathrm{SS_{Block}} + \mathrm{SS_E}$$

- Under $H_0$, all SS/$\sigma^2$ independent $\chi^2$
- Ratio of SS will be F distributed

## Hypothesis Testing

- Can show (in the fixed case):

  E(MS$_E$)=$\sigma^2$

  E(MS$_{\text{Treatment}}$) $= \sigma^2 + b\sum \tau_i^2/(a-1)$

  E(MS$_{\text{Block}}$) $= \sigma^2 + a\sum \beta_j^2/(b-1)$

- Use F-test to test equality of treatment effects

$$F_0 = \frac{SS_{\text{Treatment}}/(a-1)}{SS_E/((a-1)(b-1))}$$

- Caution testing block effects
  - Usually not of interest (blocked for a reason)
  - Blocks not randomized to experimental units (restriction)
  - Differing opinions on using F-test
  - Can view ratio to see if blocking successful



## Analysis of Variance Table

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F_0$ |
|---|---|---|---|---|
| Blocks | SS$_{\text{Block}}$ | $b-1$ | MS$_{\text{Block}}$ | |
| Treatment | SS$_{\text{Treatment}}$ | $a-1$ | MS$_{\text{Treatment}}$ | $F_0$ |
| Error | SS$_E$ | $(b-1)(a-1)$ | MS$_E$ | |
| Total | SS$_T$ | $ba-1$ | | |

SS$_T$ $= \sum\sum y_{ij}^2 - y_{..}^2/N$

SS$_{\text{Treatment}}$ $= \frac{1}{b}\sum y_{i.}^2 - y_{..}^2/N$

SS$_{\text{Block}}$ $= \frac{1}{a}\sum y_{.j}^2 - y_{..}^2/N$

SS$_E$=SS$_T$ - SS$_{\text{Treatment}}$- SS$_{\text{Block}}$

If $F_0 > F_{\alpha,a-1,(b-1)(a-1)}$ then reject $H_0$



## Example

An experiment was designed to study the performance of four different detergents for cleaning clothes. The following "cleanness" readings (higher=cleaner) were obtained with specially designed equipment for three different types of common stains. Is there a difference among the detergents?

| | Stain 1 | Stain 2 | Stain 3 |
|---|---|---|---|
| Detergent 1 | 45 | 43 | 51 |
| Detergent 2 | 47 | 46 | 52 |
| Detergent 3 | 48 | 50 | 55 |
| Detergent 4 | 42 | 37 | 49 |

$\sum\sum y_{ij} = 565$ and $\sum\sum y_{ij}^2 = 26867$

$y_{1.} = 139$, $y_{2.} = 145$, $y_{3.} = 153$ and $y_{4.} = 128$

$y_{.1} = 182$, $y_{.2} = 176$, and $y_{.3} = 207$

SS$_T$ $= 26867 - 565^2/12 = 265$

SS$_{\text{Trt}}$ $= (139^2 + 145^2 + 153^2 + 128^2)/3 - 565^2/12 = 111$

SS$_{\text{Block}}$ $= (182^2 + 176^2 + 207^2)/4 - 565^2/12 = 135$

SS$_E$ $= 265 - 111 - 135 = 19$

$F_0 = (111/3)/(19/6) = 11.6$

P-value $< 0.01$ (Reject $H_0$ - There is a detergent difference)



## Diagnostics

Assumptions

Model is correct (additivity assumption)

Errors independent, Normally distributed

Constant variance

Normality

Histogram, normal probability plot of residuals

Variance and Unusual Observations

Residuals vs blocks

Residuals vs treatments

Residuals vs $\hat{y}_{ij}$

Additivity

Residuals vs $\hat{y}_{ij}$

If curvilinear, interaction likely exists

Interaction: Block effect different for different treatments

Tukey's Test of Non-additivity (Formal Test)

Usually transform to eliminate interaction

# Comparisons of Treatments

- Multiple Comparisons/Contrasts
  - Similar procedures as CRD
  - $n$ is replaced by $b$ in all formulas
  - Degrees of freedom error is $(b-1)(a-1)$

- Example : Comparison of Detergents
  - Duncan's Multiple Range ($\alpha = .05$)

    6 degrees of freedom error $\rightarrow r_\alpha = (3.46, 3.58, 3.64)$

    $s_{\overline{y}} = \sqrt{\text{MSE}/3} = \sqrt{(19/6)/3} = 1.03$

    Least Significant Ranges are (3.56,3.68,3.74)

Comparison of Means

Treatments

| 4 | 1 | 2 | 3 |
|---|---|---|---|
| 42.67 | 46.33 | 48.33 | 51.00 |
| A | | C | C |
| | B | B | |



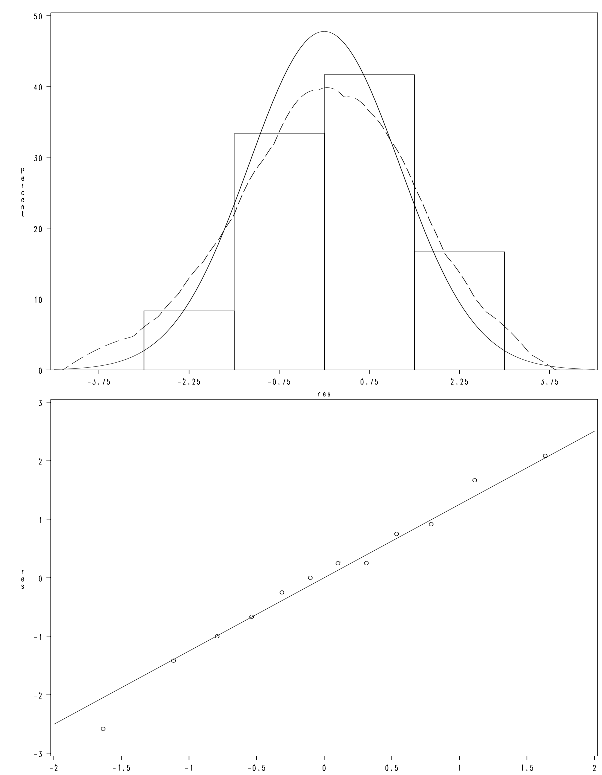
# Using SAS

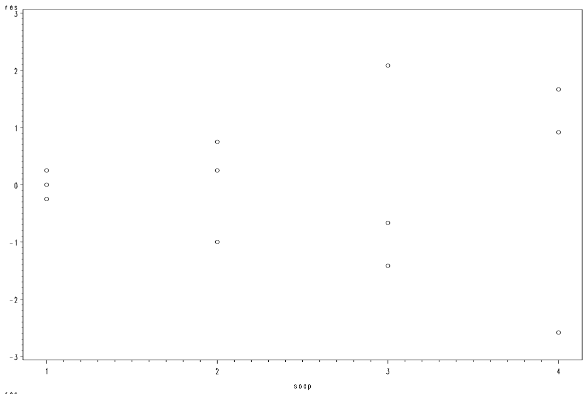
```
options nocenter ls=78;
goptions colors=(none);
symbol1 v=circle; axis1 offset=(5);

data wash;
 input stain soap y @@;
 cards;
1 1 45 1 2 47 1 3 48 1 4 42
2 1 43 2 2 46 2 3 50 2 4 37
3 1 51 3 2 52 3 3 55 3 4 49
;

proc glm;
 class stain soap;
 model y = soap stain;
 means soap / duncan lines;
 output out=diag r=res p=pred;

proc univariate noprint;
 qqplot res / normal (L=1 mu=0 sigma=est);
 hist res /normal (L=1 mu=0 sigma=est) kernel(L=2 K=quadratic);
run;

proc gplot;
 plot res*soap / haxis=axis1;
 plot res*stain / haxis=axis1;
 plot res*pred;
run;
```

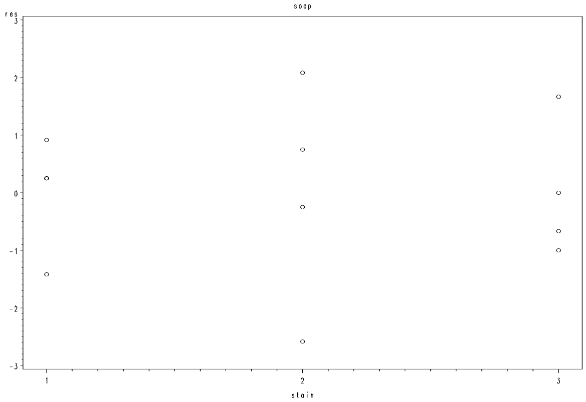


```
Dependent Variable: y
                                          Sum of
Source                      DF          Squares     Mean Square    F Value    Pr > F
Model                        5      246.0833333      49.2166667      15.68    0.0022
Error                        6       18.8333333       3.1388889
Corrected Total             11      264.9166667

R-Square     Coeff Var      Root MSE        y Mean
0.928908      3.762883      1.771691      47.08333

Source                      DF      Type I SS     Mean Square    F Value    Pr > F
soap                         3    110.9166667      36.9722222      11.78    0.0063
stain                        2    135.1666667      67.5833333      21.53    0.0018

Source                      DF    Type III SS     Mean Square    F Value    Pr > F
soap                         3    110.9166667      36.9722222      11.78    0.0063
stain                        2    135.1666667      67.5833333      21.53    0.0018

Duncan's Multiple Range Test for y

Alpha                          0.05
Error Degrees of Freedom          6
Error Mean Square          3.138889

Number of Means          2          3          4
Critical Range       3.540      3.669      3.732

                Mean      N    soap

     A        51.000      3    3
     A
B    A        48.333      3    2
B
B             46.333      3    1

     C        42.667      3    4
```

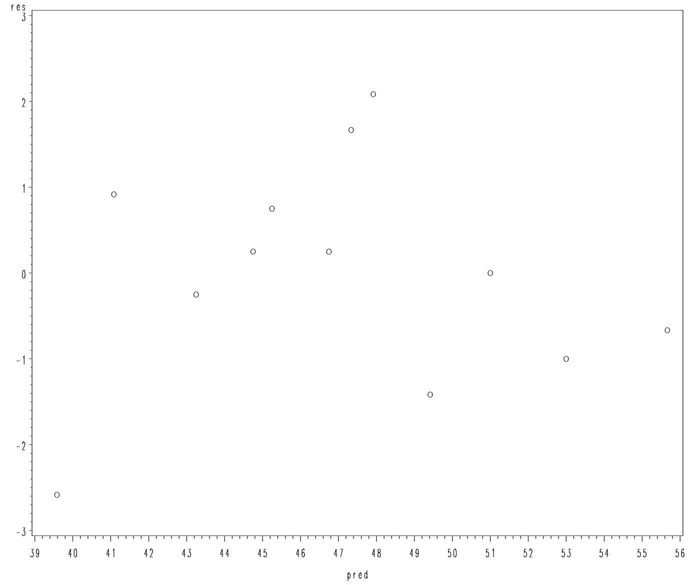











# Regression Model

- Regression model simple extension of CRD
- Add additional $b-1$ columns to represent block
- Block columns orthogonal to treatment columns
- Thus, order of fit does not matter

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$



# Missing Values

- When missing

  Orthogonality lost - missing row in $X$

  Design unbalanced → Order of fit important

- Procedures

  1 Exact (Regression) approach

  Use Type III SS's (general regression signif test)

  2 Approximate approach : Estimate missing value

  Choose value to minimize $SS_E$ (minimum contribution)

  Take derivative and set equal to zero

$$\begin{aligned} SS_{\text{E}} &= \sum\sum y_{ij}^2 - y_{..}^2/ab - \frac{1}{b}\sum y_{i.}^2 + y_{..}^2/ab - \frac{1}{a}\sum y_{.j}^2 + y_{..}^2/ab \\ &= x^2 - \frac{1}{b}(y'_{i.} + x)^2 - \frac{1}{a}(y'_{.j} + x)^2 + \frac{1}{ab}(y'_{..} + x)^2 + R \end{aligned}$$

$$x = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)}$$

# Example

- Consider detergent comparison example
- Suppose $y_{4,2} = 37$ is missing
- Regression approach
  - Same procedure in SAS - look at Type III result
- Estimate Approach

$$y'_{4.} = 91 \quad y'_{..} = 528 \quad y'_{.2} = 139$$

  - Estimate is

$$x = \frac{4(91) + 3(139) - 528}{6} = 42.17$$

  - Do analysis but adjust error degrees of freedom
- Regression: $\hat{\sigma}^2 = 1.097$
- Estimate: $\hat{\sigma}^2 = 1.097$ (must divide by 5 not 6)



```
options nocenter ps=60 ls=78;

data wash;
 input stain soap y @@;
 if y=37 then y=.;
 cards;
1 1 45 1 2 47 1 3 48 1 4 42
2 1 43 2 2 46 2 3 50 2 4 37
3 1 51 3 2 52 3 3 55 3 4 49
;

proc glm;
 classes stain soap;
 model y = soap stain;
 output out=diag r=res p=pred;
 means soap / lsd lines;
 lsmeans soap / stderr;

data new1;
 set wash;
 if y=. then y=42.1666666666;

proc glm;
 classes stain soap;
 model y = soap stain;
 output out=diag r=res p=pred;
 means soap / lsd lines;
run;
```



# Regression - Type III

```
Dependent Variable: y
                                  Sum of
Source                     DF    Squares     Mean Square  F Value   Pr > F
Model                       5    148.5138889  29.7027778    27.07   0.0013
Error                       5      5.4861111   1.0972222
Corrected Total            10    154.0000000

Source                     DF    Type I SS   Mean Square  F Value   Pr > F
soap                        3    48.1666667   16.0555556    14.63   0.0066
stain                       2   100.3472222   50.1736111    45.73   0.0006

Source                     DF  Type III SS   Mean Square  F Value   Pr > F
soap                        3    58.9305556   19.6435185    17.90   0.0042
stain                       2   100.3472222   50.1736111    45.73   0.0006
---------------------------------------------------------------

t Tests (LSD) for y
                Mean      N    soap
     A       51.0000      3    3

     B       48.3333      3    2
     B
C    B       46.3333      3    1
C
C            45.5000      2    4

Least Squares Means

                            Standard
soap         y LSMEAN          Error    Pr > |t|
1          46.3333333      0.6047650      <.0001
2          48.3333333      0.6047650      <.0001
3          51.0000000      0.6047650      <.0001
4          44.3888889      0.7807483      <.0001
```



# Estimate - Must adjust F by hand

```
Dependent Variable: y
                                  Sum of
Source                     DF    Squares     Mean Square  F Value   Pr > F
Model                       5    179.7060185  35.9412037    39.31   0.0002
Error                       6      5.4861111   0.9143519
Corrected Total            11    185.1921296

R-Square     Coeff Var      Root MSE        y Mean
0.970376      2.012501      0.956217      47.51389

Source                     DF    Type I SS   Mean Square  F Value   Pr > F
soap                        3    71.9513889   23.9837963    26.23   0.0008
stain                       2   107.7546296   53.8773148    58.92   0.0001

Source                     DF  Type III SS   Mean Square  F Value   Pr > F
soap                        3    71.9513889   23.9837963    26.23   0.0008
stain                       2   107.7546296   53.8773148    58.92   0.0001

LSD Test
          Mean      N    soap
A       51.0000      3    3   **Must use correct error in comparisons
B       48.3333      3    2
C       46.3333      3    1
D       44.3889      3    4   **Really only ``two'' observations here
```

$$F_0 = \frac{71.95/3}{5.49/5} = 21.84$$

$$\text{Pvalue} = 0.0027$$

## Tukey's Test for Non-additivity

- Consider special type of interaction
- Assume following model (page 190-193)

$$y_{ij} = \mu + \tau_i + \beta_j + \gamma\tau_i\beta_j + \epsilon_{ij}$$

- $H_0 : \gamma = 0$
- Use regression approach to test significance
- Procedure
  1. Fit additive model $y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$
  2. Obtain $\hat{y}_{ij}$ and $y_{ij} - \hat{y}_{ij}$
  3. Fit additive model $\hat{y}_{ij}^2 = q_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$
  4. Regress $y_{ij} - \hat{y}_{ij} = q_{ij} - \hat{q}_{ij}$

     Partitioning $SS_E$ into $SS_N$ and remainder

     The parameter $\hat{\gamma}$ is the slope estimate

     $SS_{\text{non-additivity}} = \hat{\gamma}^2 \sum\sum (q_{ij} - \hat{q}_{ij})^2$

$$F_0 = \frac{SS_N/1}{(SS_E - SS_N)/((a-1)(b-1)-1)}$$



## Example 5-2 from Montgomery

- Impurity in chemical product is affected by temperature and pressure. We will assume temperature is the blocking factor. The data is shown below. We will test for non-additivity.

| | Pressure | | | | |
|---|---|---|---|---|---|
| Temp | 25 | 30 | 35 | 40 | 45 |
| 100 | 5 | 4 | 6 | 3 | 5 |
| 125 | 3 | 1 | 4 | 2 | 3 |
| 150 | 1 | 1 | 3 | 1 | 2 |

- Can use SAS to compute SS
- Must divide by proper degrees of freedom

$$F_0 = \frac{.0985/1}{1.9015/7} = .36$$

$F_0 < F_{1,7}$ - Do Not Reject.



## SAS Procedures

```
options nocenter ls=75;

data impurity;
input trt blk y @@;
cards;
1 1 5 1 2 3 1 3 1 2 1 4 2 2 1 2 3 1 3 1 6 3 2 4 3 3 3
4 1 3 4 2 2 4 3 1 5 1 5 5 2 3 5 3 2
;

proc glm;
class blk trt;
model y=blk trt;
output out=resid1 r=res1 p=pred1;

data predsq;
set resid1;
predsq1 = pred1*pred1;

proc glm;
 class blk trt;
 model predsq1=blk trt;
 output out=resid2 r=res2 p=pred2;

proc glm;
 model res1=res2;
run;
```



```
Dependent Variable: y
                                  Sum of
Source                  DF       Squares   Mean Square  F Value  Pr > F
Model                    6   34.93333333    5.82222222    23.29  0.0001
Error                    8    2.00000000    0.25000000
Corrected Total         14   36.93333333

Source                  DF     Type I SS   Mean Square  F Value  Pr > F
blk                      2   23.33333333   11.66666667    46.67  <.0001
trt                      4   11.60000000    2.90000000    11.60  0.0021


---------------------------------------------------------------
---------------------------------------------------------------


Dependent Variable: res1
                                  Sum of
Source                  DF       Squares   Mean Square  F Value  Pr > F
Model                    1    0.09852217    0.09852217     0.67  0.4266
Error                   13    1.90147783    0.14626753
Corrected Total         14    2.00000000

Source                  DF     Type I SS   Mean Square  F Value  Pr > F
res2                     1    0.09852217    0.09852217     0.67  0.4266

                                Standard
Parameter        Estimate          Error    t Value    Pr > |t|
Intercept     -.0000000000     0.09874800     -0.00      1.0000
res2          0.0369458128     0.04501655      0.82      0.4266
```

# Random Block/Treatment Effects

- Could randomly select trts and/or blocks
- Do not need to worry about interaction
- Interaction considered random effect
- Interaction variance appears in all EMS
- Perform usual F-test (ratio of MS)
- Use Proc Mixed instead of Proc Glm
- Otherwise underestimate variability in trt means

# Choice of Sample Size

- Same as determining the number of blocks ($b$)
- Use same tables/procedures with $b$ replacing $n$



```
options nocenter ls=75;

data wash;
input stain soap y @@;
cards;
1 1 45 1 2 47 1 3 48 1 4 42
2 1 43 2 2 46 2 3 50 2 4 37
3 1 51 3 2 52 3 3 55 3 4 49
;
proc glm;
 class stain soap;
 model y = soap stain soap*stain;
 random stain soap*stain;
run;
-------------------------------------------

General Linear Models Procedure

Dependent Variable: Y
                                    Sum of           Mean
Source             DF          Squares         Square   F Value     Pr > F
Model              11      264.9166667     24.0833333       .          .
Error               0           .               .
Corrected Total    11      264.9166667

Source             DF      Type III SS    Mean Square   F Value     Pr > F
SOAP                3      110.9166667     36.9722222       .          .
STAIN               2      135.1666667     67.5833333       .          .
STAIN*SOAP          6       18.8333333      3.1388889       .          .


Source       Type III Expected Mean Square

SOAP         Var(Error) + Var(STAIN*SOAP) + Q(SOAP)
STAIN        Var(Error) + Var(STAIN*SOAP) + 4 Var(STAIN)
STAIN*SOAP   Var(Error) + Var(STAIN*SOAP)
```



```
options nocenter ls=75;

data wash;
input stain soap y @@;
cards;
1 1 45 1 2 47 1 3 48 1 4 42
2 1 43 2 2 46 2 3 50 2 4 37
3 1 51 3 2 52 3 3 55 3 4 49
;


proc glm;
 class stain soap;
 model y = soap stain;
 random stain;
 lsmeans soap / stderr tdiff;


proc mixed;
 class stain soap;
 model y = soap;
 random stain;
 lsmeans soap / stderr tdiff;
run;
```



```
The GLM Procedure
                                 Sum of
Source              DF          Squares    Mean Square   F Value   Pr > F
Model                5      246.0833333     49.2166667     15.68   0.0022
Error                6       18.8333333      3.1388889
Corrected Total     11      264.9166667

Source              DF      Type III SS    Mean Square   F Value   Pr > F
soap                 3      110.9166667     36.9722222     11.78   0.0063
stain                2      135.1666667     67.5833333     21.53   0.0018

                              Standard                   LSMEAN
soap         y LSMEAN            Error    Pr > |t|      Number
1          46.3333333        1.0228863      <.0001           1
2          48.3333333        1.0228863      <.0001           2
3          51.0000000        1.0228863      <.0001           3
4          42.6666667        1.0228863      <.0001           4

            Least Squares Means for Effect soap
          t for H0: LSMean(i)=LSMean(j) / Pr > |t|

                  Dependent Variable: y
i/j              1             2             3             4
   1                    -1.38257        -3.226      2.534715
                          0.2161        0.0180        0.0444
   2      1.382572                    -1.84343      3.917286
            0.2161                      0.1148        0.0078
   3      3.226001      1.843429                    5.760715
            0.0180        0.1148                      0.0012
   4      -2.53471      -3.91729      -5.76072
            0.0444        0.0078        0.0012
```

```
The Mixed Procedure

Covariance Parameter
      Estimates

Cov Parm     Estimate
stain         16.1111
Residual       3.1389


        Type 3 Tests of Fixed Effects
              Num     Den
Effect         DF      DF    F Value    Pr > F
soap            3       6      11.78    0.0063

                            Least Squares Means
                                 Standard
Effect     soap    Estimate        Error       DF    t Value    Pr > |t|
soap       1        46.3333       2.5331        6      18.29      <.0001
soap       2        48.3333       2.5331        6      19.08      <.0001
soap       3        51.0000       2.5331        6      20.13      <.0001
soap       4        42.6667       2.5331        6      16.84      <.0001

                         Differences of Least Squares Means

                                          Standard
Effect     soap    _soap     Estimate        Error       DF    t Value    Pr > |t|
soap       1       2          -2.0000       1.4466        6      -1.38      0.2161
soap       1       3          -4.6667       1.4466        6      -3.23      0.0180
soap       1       4           3.6667       1.4466        6       2.53      0.0444
soap       2       3          -2.6667       1.4466        6      -1.84      0.1148
soap       2       4           5.6667       1.4466        6       3.92      0.0078
soap       3       4           8.3333       1.4466        6       5.76      0.0012
```



# RCBD with Replication

- What if multiple trt observation per block?
  - $b$ blocks
  - $a$ treatments
  - $n$ observations/block

$$y_{ijl} = \mu + \tau_i + \beta_j + \epsilon_{ijl} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ l = 1, 2, \ldots, n \end{cases}$$

- When would this occur?
  - Have large field with very gradual slope
  - Blocks expensive but observations cheap
- Increases $df_E$ (or allows interaction)

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F$ |
|---|---|---|---|---|
| Blocks | $SS_{Block}$ | $b-1$ | $MS_{Block}$ | |
| Treatment | $SS_{Treatment}$ | $a-1$ | $MS_{Treatment}$ | $F_0$ |
| Error | $SS_E$ | $abn-b-a+1$ | $MS_E$ | |
| Total | $SS_T$ | $abn-1$ | | |



# RCBD with Replication

- Usual diagnostics checks
- Replace $b$ by $bn$ in multiple comparisons or power
- Allows for easier assessment of additivity
  - More error degrees of freedom
  - Interaction and error not confounded
  - Can separate error and interaction SS

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F$ |
|---|---|---|---|---|
| Blocks | $SS_{Block}$ | $b-1$ | $MS_{Block}$ | |
| Treatment | $SS_{Treatment}$ | $a-1$ | $MS_{Treatment}$ | $F_0$ |
| Blk*Trt | $SS_{Blk*Trt}$ | $(b-1)(a-1)$ | $MS_{Blk*Trt}$ | |
| Error | $SS_E$ | $ab(n-1)$ | $MS_E$ | |
| Total | $SS_T$ | $abn-1$ | | |



# Example

You have been asked to design an experiment to compare four varieties of seed corn. You have a field consisting of sixteen subplots (in a 4x4 grid) at your disposal. If you were told that one side of the field is next to a highway and the side directly across from this one is next to a river, how would you design the experiment?

**If we feel pretty certain that plots near the road or river will "behave" differently than plots in the middle, we might want to create $b = 3$ blocks. Block 1 consists of the four plots along the road. Block 2 consists of the 4 plots along the river and Block 3 consists of the eight plots in the middle. Thus, we have two blocks which only have $n = 1$ observation per treatment and one block that has $n = 2$ observations per treatment.**

- Statistical model is (with interaction)

$$y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, 3, 4 \\ j = 1, 2, 3 \\ k = 1, ..., n_j \end{cases}$$

where $n_j = \begin{cases} 1 & \text{if } j = 1, 2 \\ 2 & \text{if } j = 3 \end{cases}$

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F_0$ |
|---|---|---|---|---|
| Blocks | $SS_{Block}$ | 2 | $MS_{Block}$ | |
| Interaction | $SS_{Trt*Blk}$ | 6 | $MS_{Trt*Blk}$ | |
| Treatment | $SS_{Treatment}$ | 3 | $MS_{Treatment}$ | $F_0$ |
| Error | $SS_E$ | 4 | $MS_E$ | |
| Total | $SS_T$ | 15 | | |

- If four blocks, cannot separate error and interaction
- $SS_E$ based on obs within block 3 (has replicates)