Dependent percolation: some examples and multi-scale tools

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I. Motivation

Classical Ising model (spins ± 1) in two dimensions

$$H(\sigma) = - \sum_{|x-y|=1} J_{\{x,y\}} \sigma_x \sigma_y$$

$\sigma \in \{-1, +1\}^{\Lambda}$; $\Lambda \subset \mathbb{Z}^2$ (finite)

$J_{\{x,y\}} \equiv J > 0$ ferromagnetic interaction; $x, y \in \mathbb{Z}^2$

$$\mu_\Lambda(\sigma) = \frac{1}{Z_\Lambda} \exp\{-\beta H_\Lambda(\sigma)\} \quad \text{(probability measure)}$$

Phase transition at sufficiently low temperature

$$\beta_c = \frac{1}{2J} \log(1 + \sqrt{2})$$

multiple limits of μ_Λ as $\Lambda \to \mathbb{Z}^2$ if $\beta > \beta_c$.

Peierls (1936); Onsager (1944); Yang (1952)

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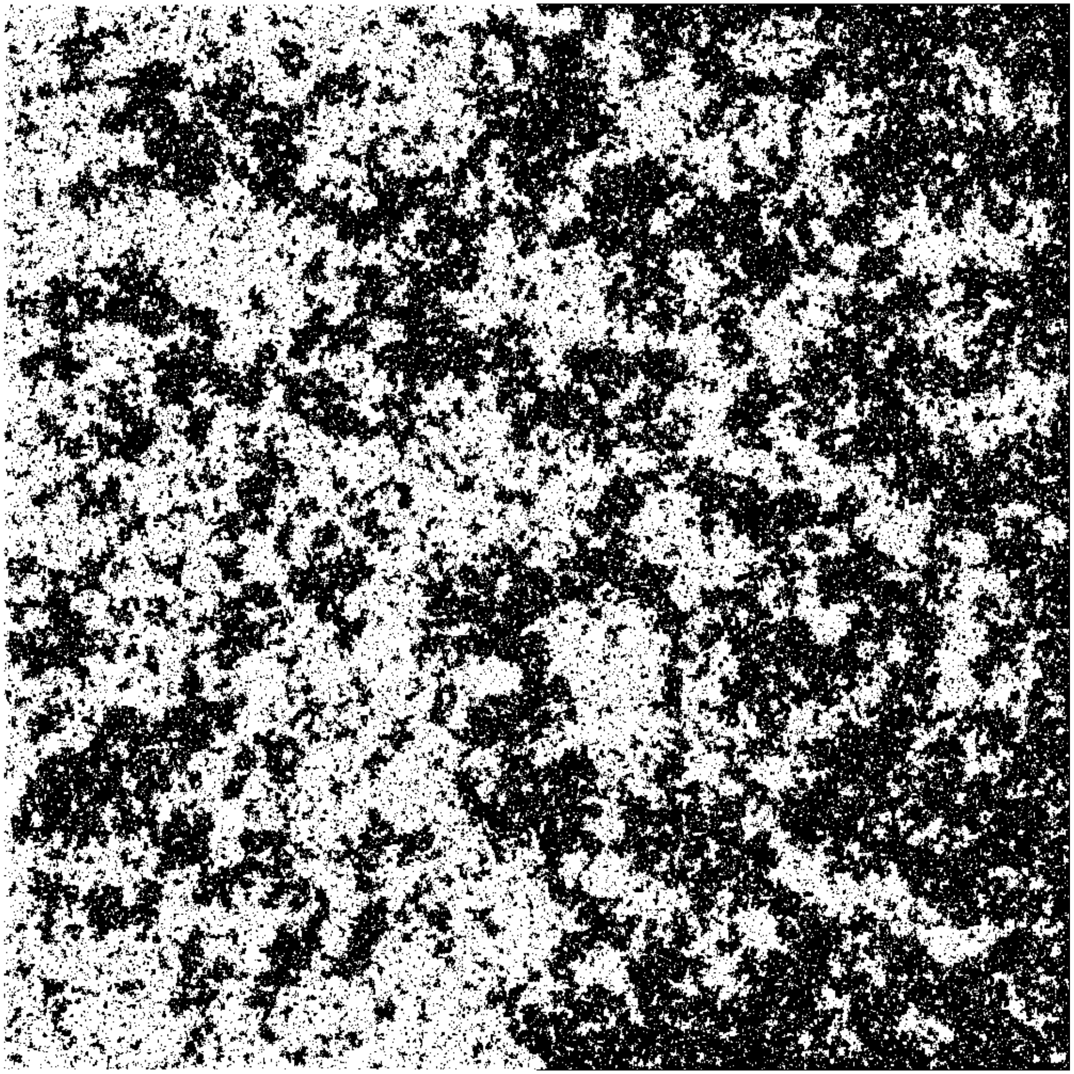

Simulations by Vincent Beffara

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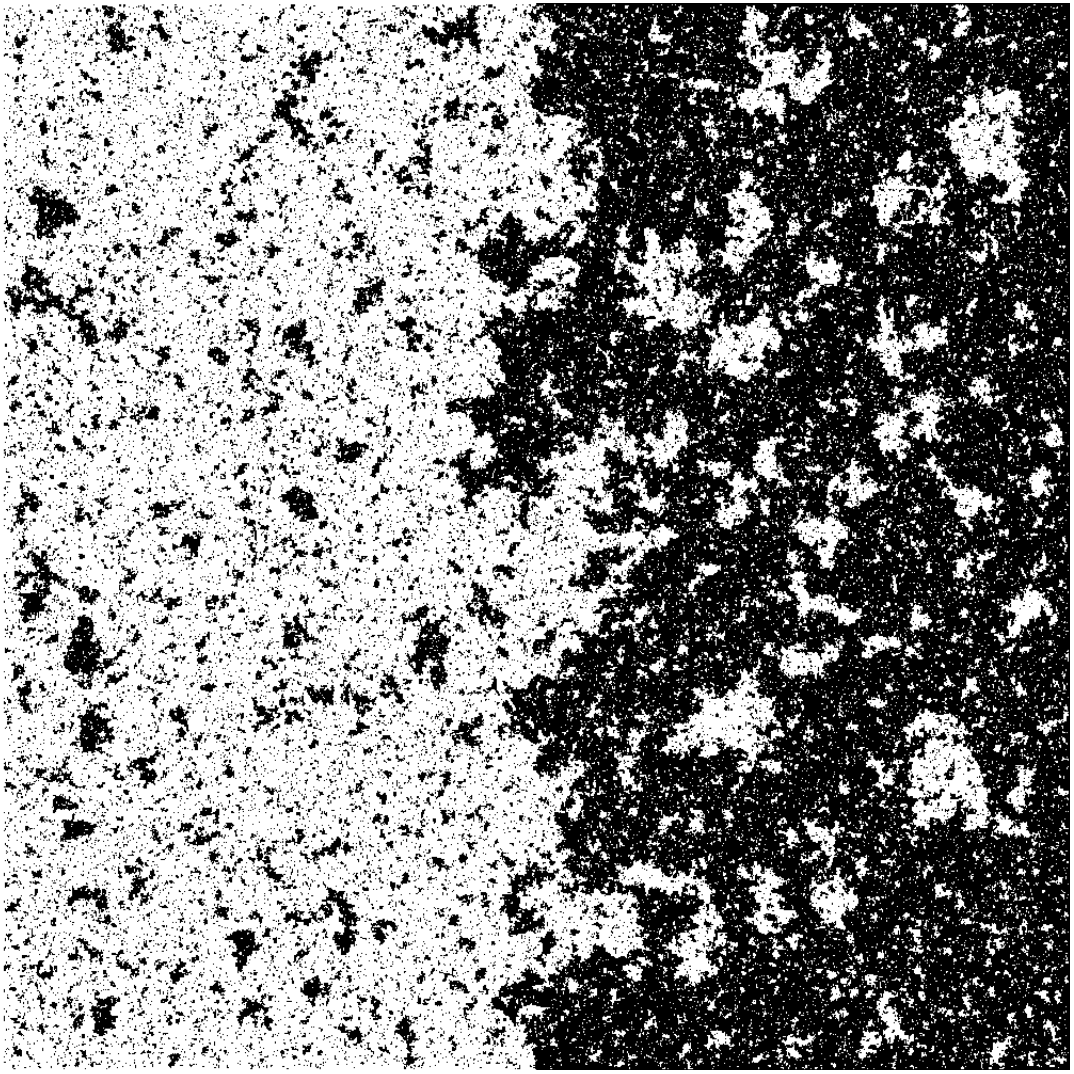

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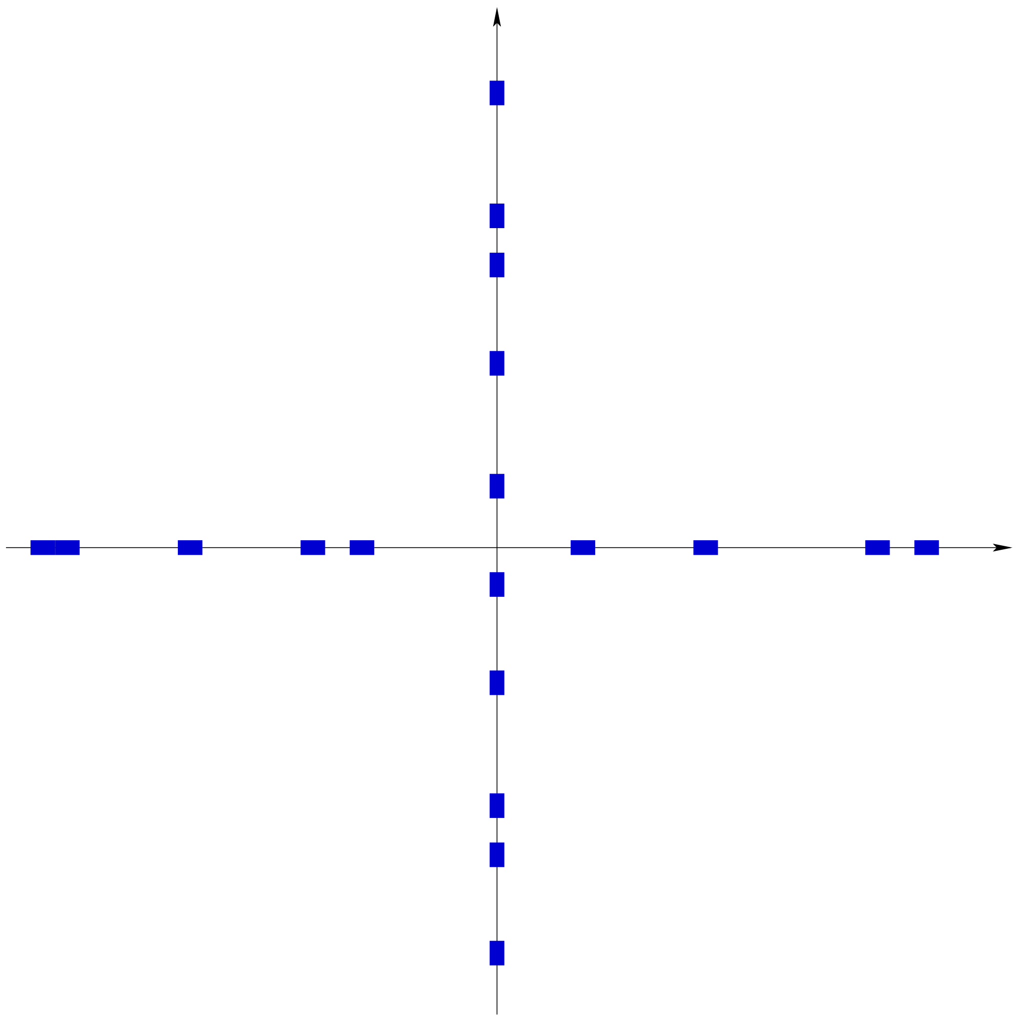
McCoy and Wu (1968) investigated the effect of random impurities



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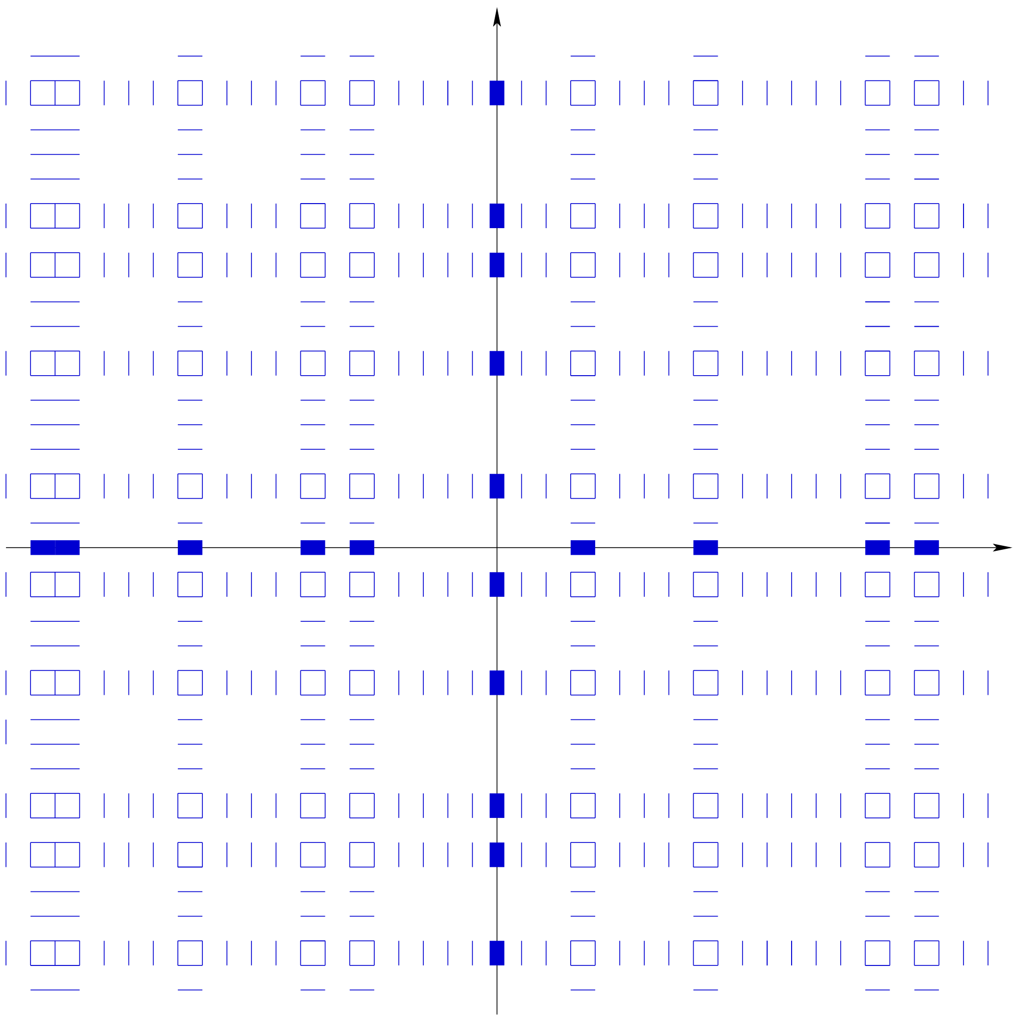

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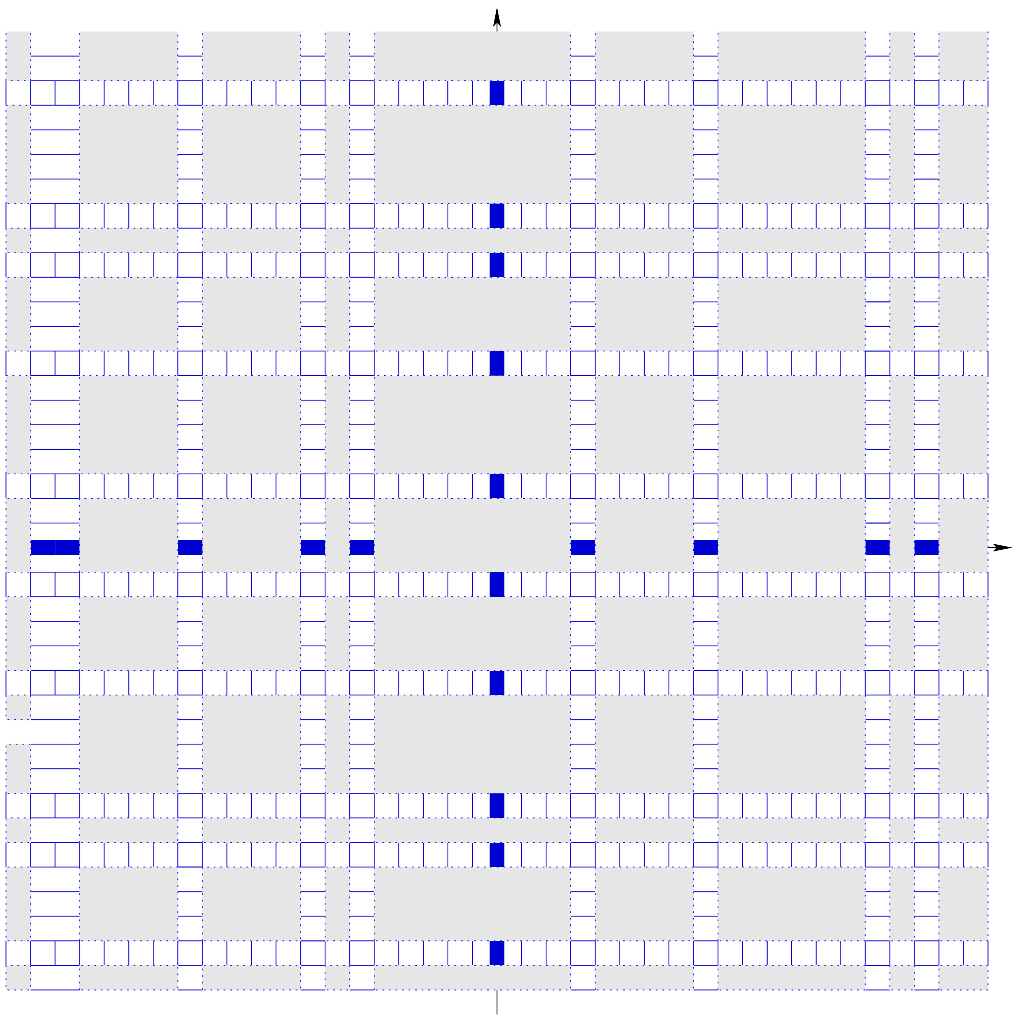

Disordered ferromagnets – randomly layered environment

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Phase transitions for systems in a (randomly) layered environment

- Growth processes in random environment
- Some forms of coordinate percolation. Winkler's compatibility problem.

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Based on joint results with H. Kesten, B. Lima, V. Sidoravicius

II. Oriented percolation in a randomly layered environment

On $\widetilde{\mathbb{Z}}^2_+ := \{(x, y) \in \mathbb{Z} \times \mathbb{Z}_+ \colon x + y \text{ is even}\}$

consider the following oriented (NW, NE) site percolation model:

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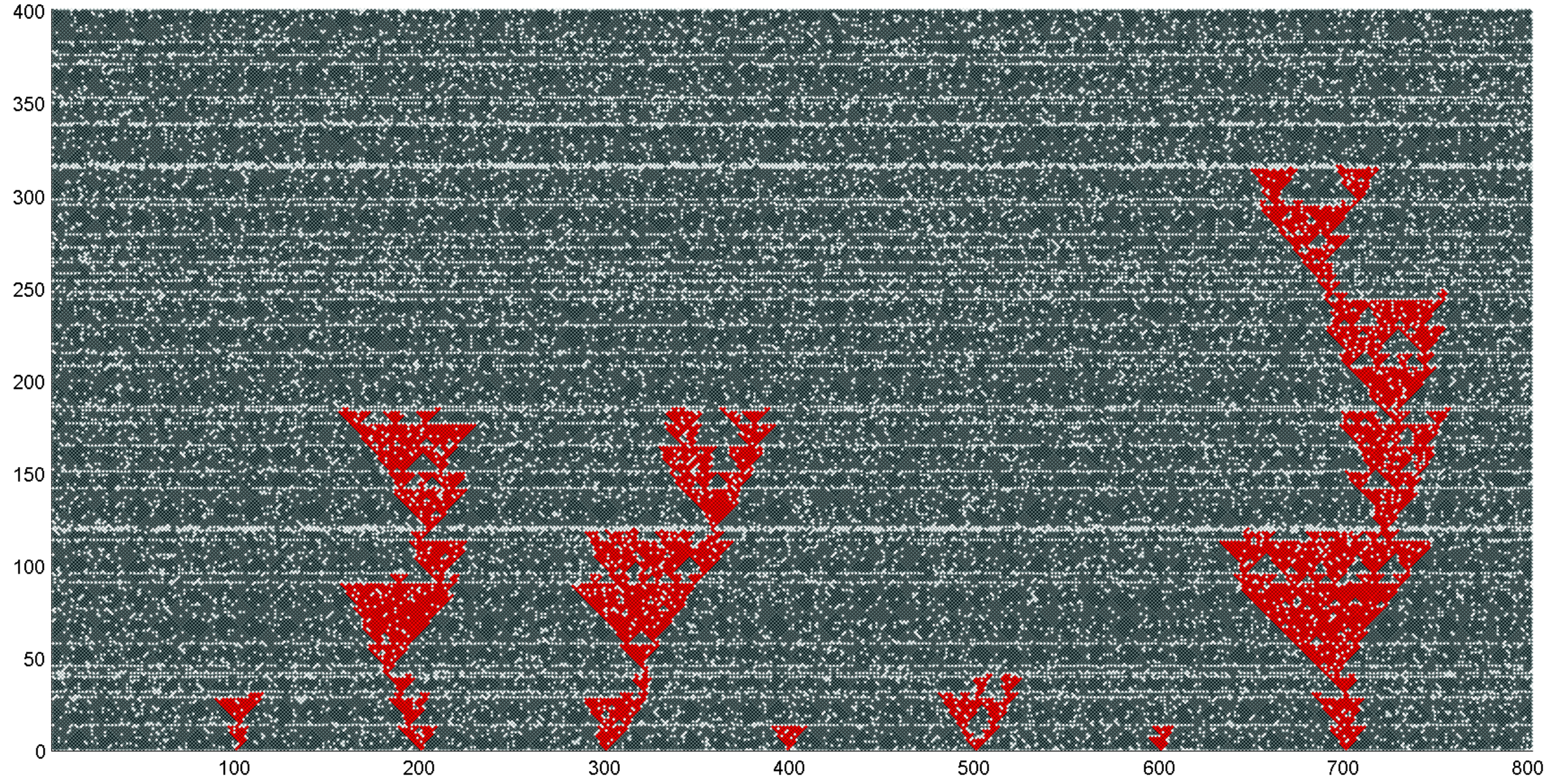

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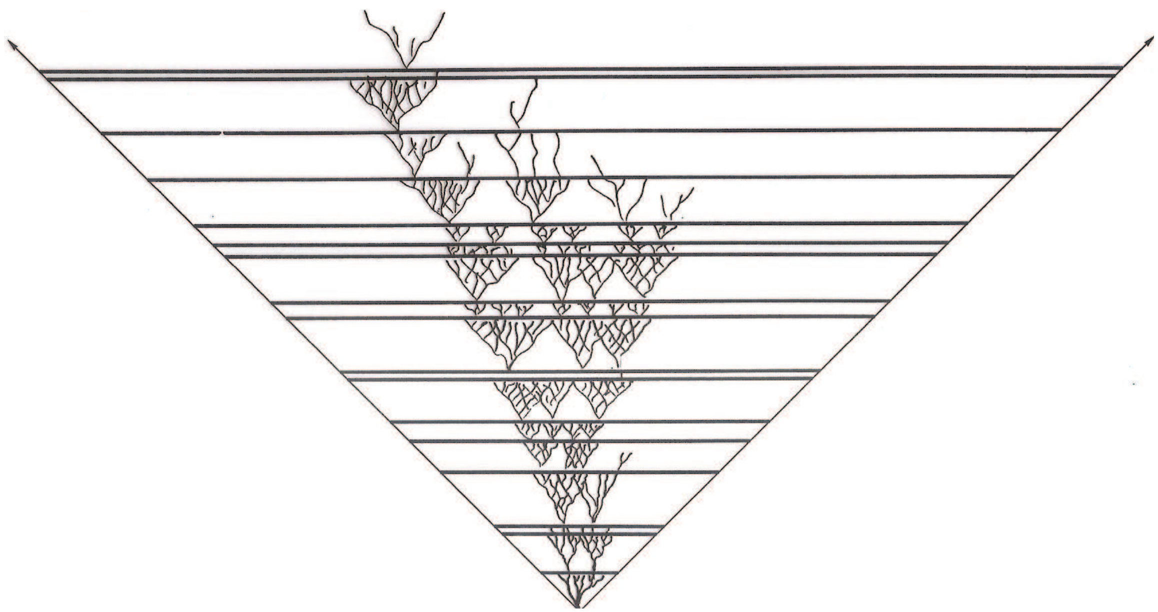

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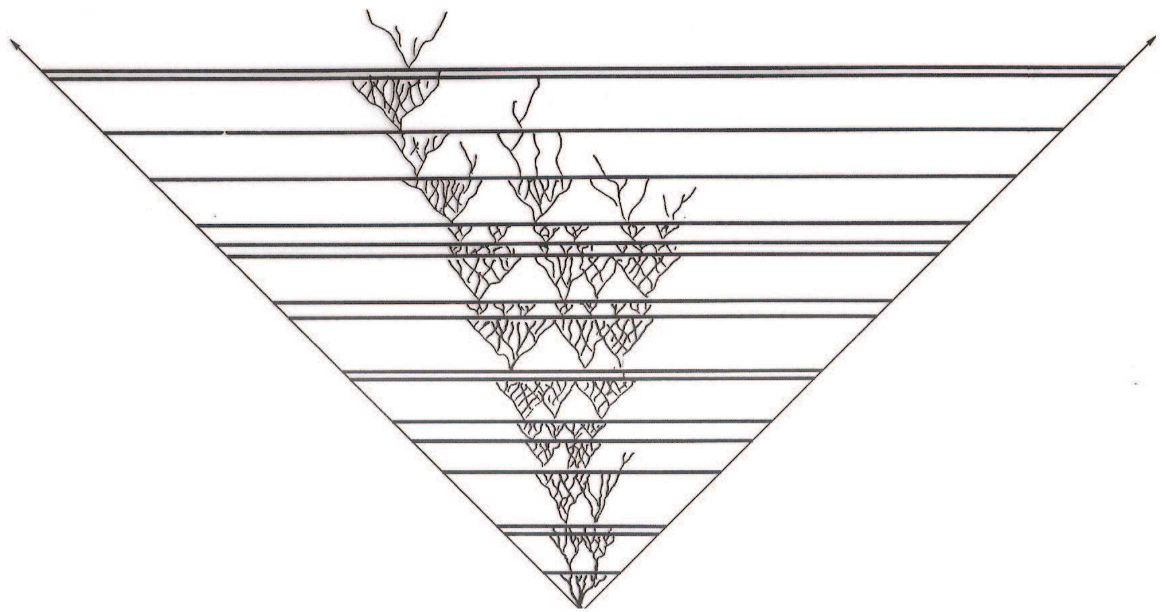

$$C^{+}(\mathbf{0}) = \{v\colon \exists \text{ open oriented path from } 0 \text{ to } v\}$$

Let $\Theta(p_G, p_B, \delta) = \mathbb{P}(C^{+}(0) \text{ is infinite })$.

Theorem (Kesten, Sidoravicius, V.)

$\forall p_G > p_c$, $\forall p_B > 0$, $\exists \delta_0 > 0$ so that $\Theta(p_G, p_B, \delta) > 0$ if $\delta \leq \delta_0$. In fact

$$\mathbb{P}(C^{+}(0) \text{ is infinite } |\xi) > 0 \text{ a.s. in } \xi \qquad (\xi \text{ configuration of lines})$$

.

Basic tool: multi-scale analysis

Get started with a **very simple situation:**

hierarchical model, L large (depending on p_G, p_B),

$$\zeta_j := \begin{cases} k, & \text{if } L^k | j \text{ but } L^{k+1} \nmid j \, , \\ 0, & \text{if } L \nmid j. \end{cases}$$

Replace each entry $\zeta_j = k$ by k consecutive bad lines (shift the rest to the right)

- bad walls of thickness k: k consecutive bad lines;
- such bad walls at distance of order L^k from each other.

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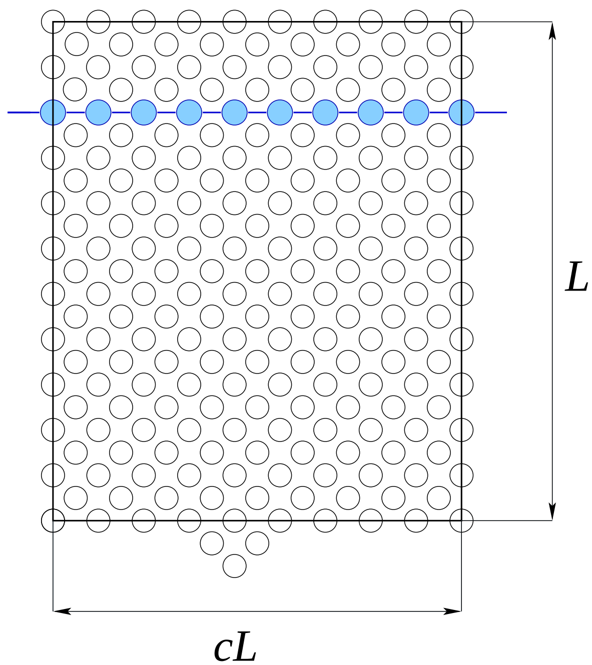
This immediately calls to the consideration of rescaled lattices, all similar to the original one, and adapted to deal with bad lines of thickness k.



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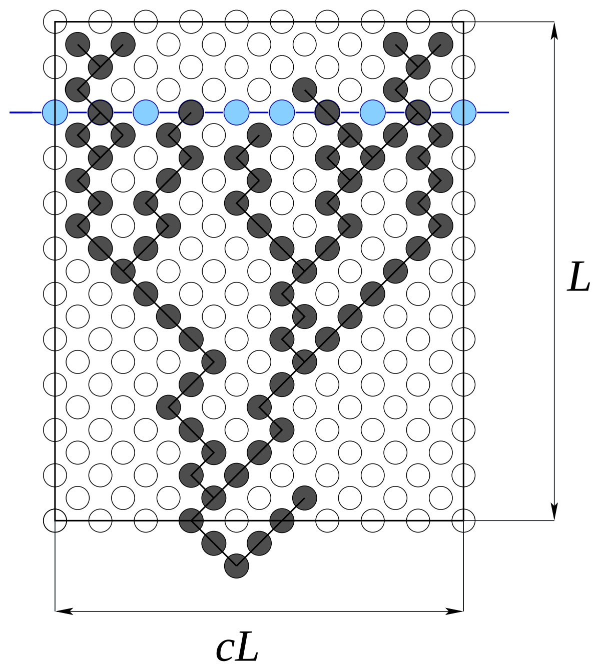

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$p_G > p^*$ so that $\mathbb{P}($**event in the picture**$|$**seed**$) \geq 1 - (1 - p_G)^2$

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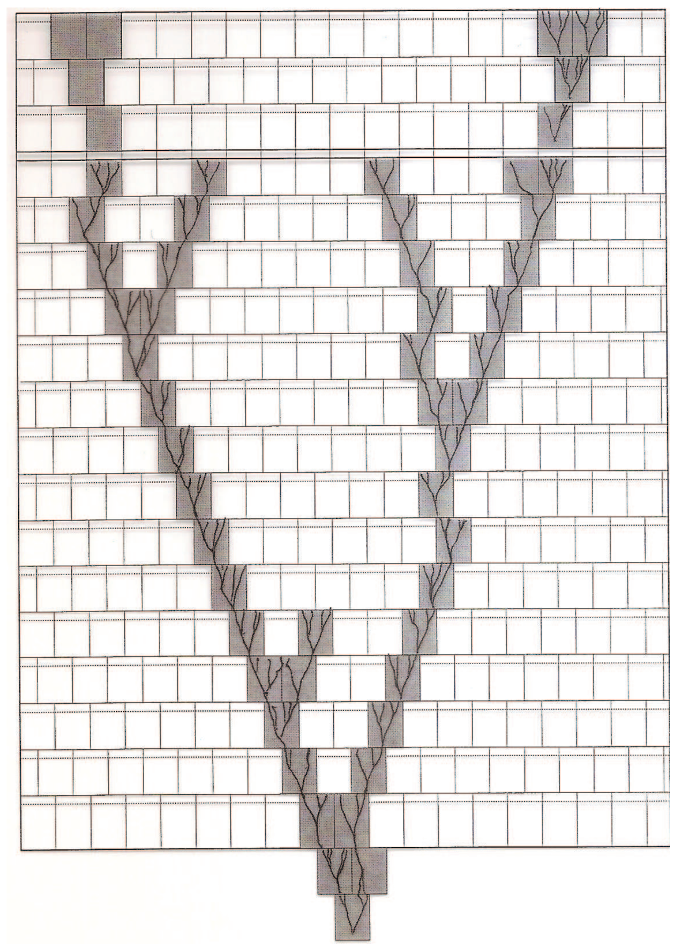

Renormalized k-sites $S^k_{(i,j)}$, $(i,j) \in \widetilde{\mathbb{Z}}^2_+$.

Recursively define the notion of a good k-site S^k being passable from a $(k-1)$-seed.

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It should:

(a) guarantee the existence of open oriented paths that cross it from bottom to top

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Proposition. Given $p_B > 0$ and $p_G > p^*$, there exists L large enough such that (*) holds with $p_k = 1 - q_k$ and

$$q_k \leq q_{k-1}^2 \qquad \text{for all } k \geq 1,$$

and where $q_0 = 1 - p_G$.

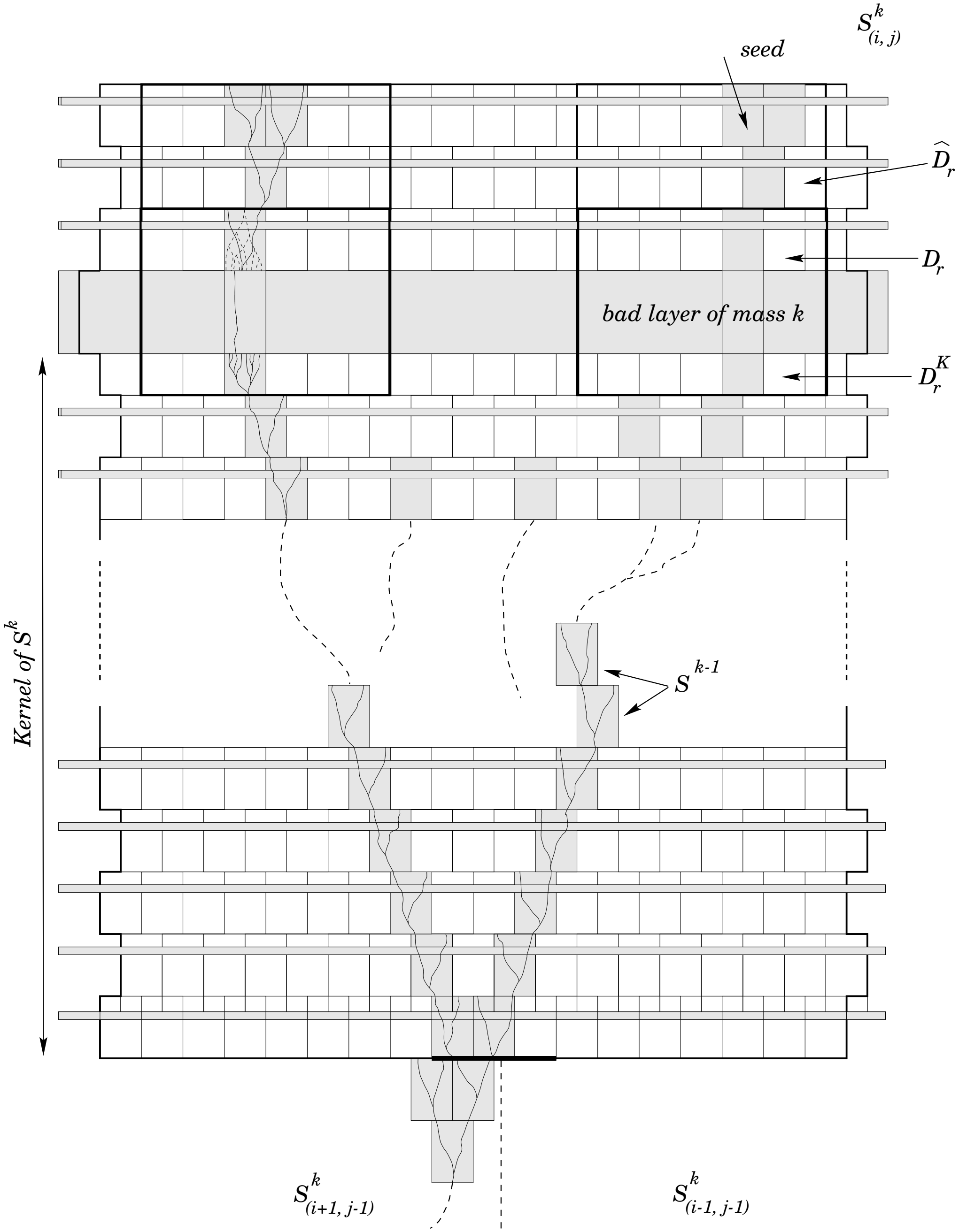
The above estimate clearly implies that for L large enough

$$P(\text{there exists an infinite cluster starting from the origin}) \geq \prod_{k=0}^{+\infty} p_k^3 > 0.$$

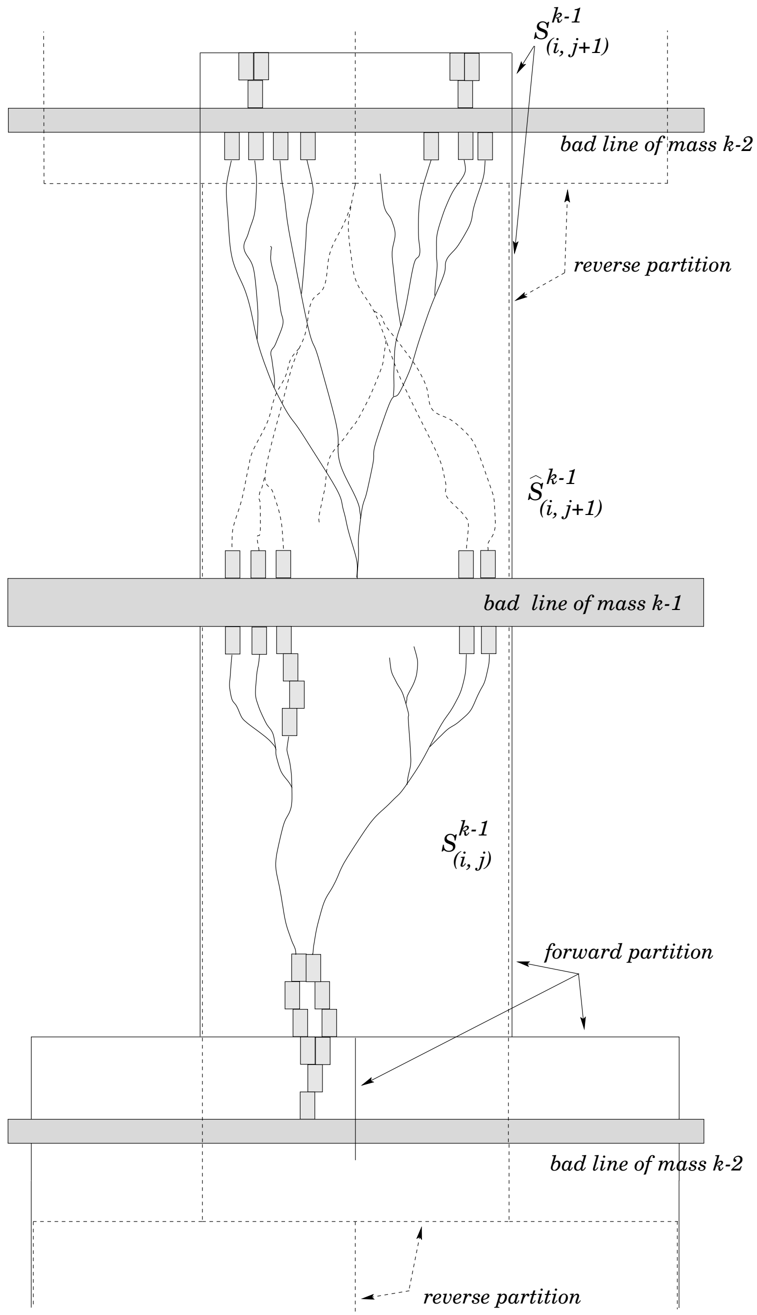
- The main difficulty in pushing the estimate at each step is when one faces the bad wall of larger mass.

- Planarity plays important role in these arguments.

- Enlarging the seeds and taking some extra care replace p^* by p_c.



$S^k_{(i,\,j)}$
seed
$\widehat{D}_r$
D_r
bad layer of mass k
D^K_r
Kernel of S^k
$S^{k\text{-}1}$
$S^k_{(i+1,\,j\text{-}1)}$
$S^k_{(i\text{-}1,\,j\text{-}1)}$

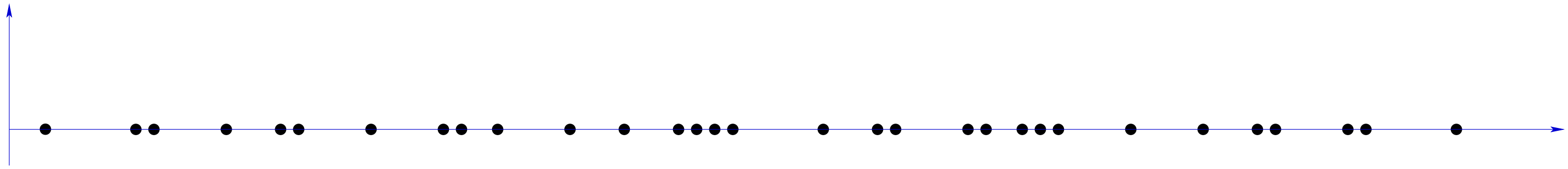


$S^{k-1}_{(i,\,j+1)}$
bad line of mass k-2
reverse partition
$\widehat{S}^{k-1}_{(i,\,j+1)}$
bad line of mass k-1
$S^{k-1}_{(i,\,j)}$
forward partition
bad line of mass k-2
reverse partition

Dealing with random layers

- Step 1: Devise a suitable grouping procedure
- Step 2: Perform the recursive (**much more involved!**) estimates

Step 1: Grouping procedure



$\xi \in \{0,1\}^{\mathbb{N}}$ sampled from the Bernoulli distribution $\mathbb{P}_\delta$ with low density δ.

$\Gamma = \{i\colon \xi_i = 1\}$ - correspond to the "bad lines" (of level 0):

$L \geq 3$ integer.

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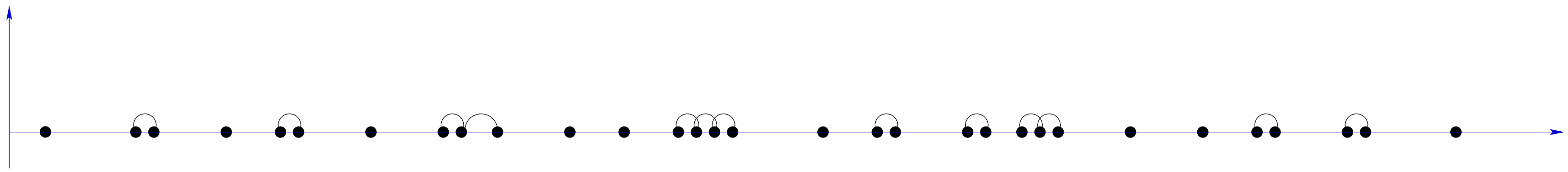

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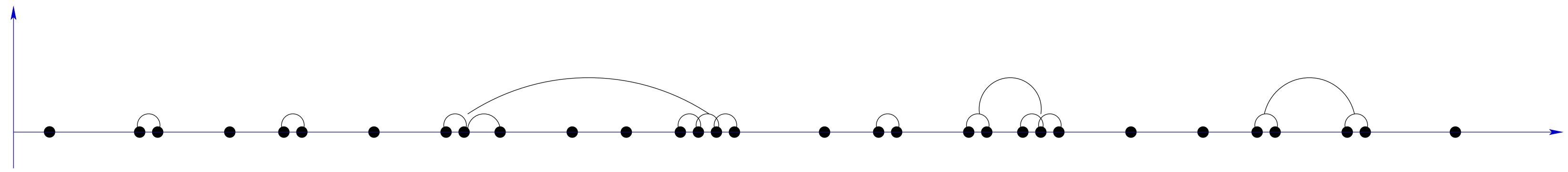

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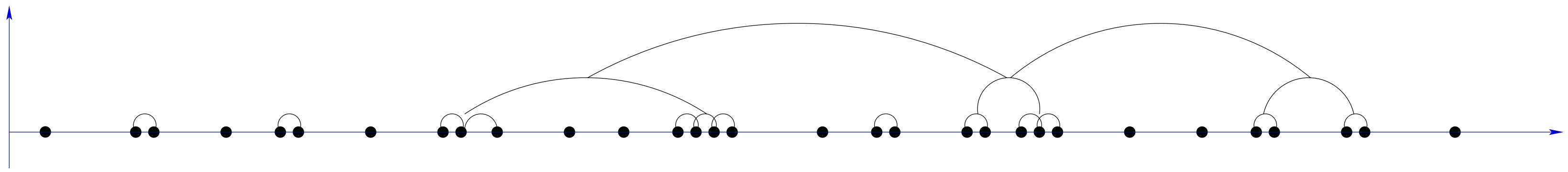

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Assuming $64\delta L^2 < 1$ the procedure converges:

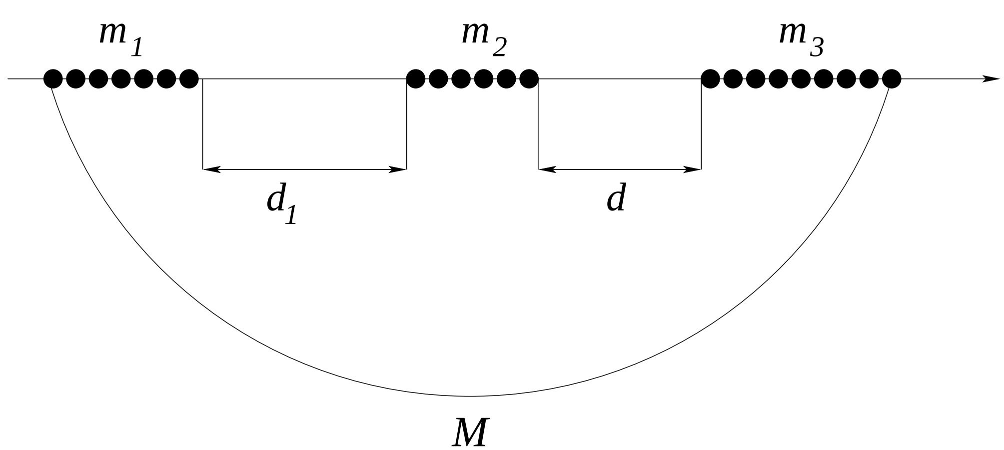
Each $x \in \Gamma$ will be "re-incorporated" finitely many times a.s.; the final partition is well defined.

Step 1: Grouping procedure

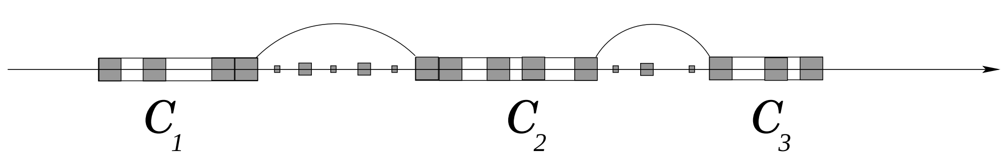
On a set $\Xi(\delta)$ of full measure we can decompose Γ into sets $\mathcal{C}_i$, called *clusters*, to which an $\mathbb{N}$-valued mass $m(\mathcal{C}_i)$ is attributed $(m(\mathcal{C}_i) \leq |\mathcal{C}_i|)$ in a way that

$$d(\mathcal{C}_i, \mathcal{C}_j) \geq L^{\min\{m(\mathcal{C}_i), m(\mathcal{C}_j)\}}, \quad \text{for all } i, j$$

The $\mathcal{C}_i := \mathcal{C}_{\infty,i}$ are obtained by the (limiting) recursive procedure outlined above. Each constructed cluster has a level (the step when it was born!) and a mass.



$$M = m_1 + m_2 + m_3 - \lfloor \log_L d_1 \rfloor - \lfloor \log_L d_2 \rfloor$$



Step 2: Multi-scale analysis for fixed realization of good/bad lines

Assume $3 \leq L,\ 64\delta L^2 < 1$.

$$\chi(\xi) := \inf\{k \geq 0 \colon d(\mathcal{C}, 0) \geq M^{m(\mathcal{C})} \text{ for all } \mathcal{C} \in \mathbf{C}_\infty \text{ with } m(\mathcal{C}) > k\}$$

with $\chi(\xi) = \infty$ if the above set is empty or $\xi \notin \Xi(\delta)$

Then:

$$\mathbb{P}_\delta\{\xi \colon \chi(\xi) < \infty\} = 1 \text{ and } \mathbb{P}_\delta\{\xi \colon \chi(\xi) = 0\} > 0.$$

We prove

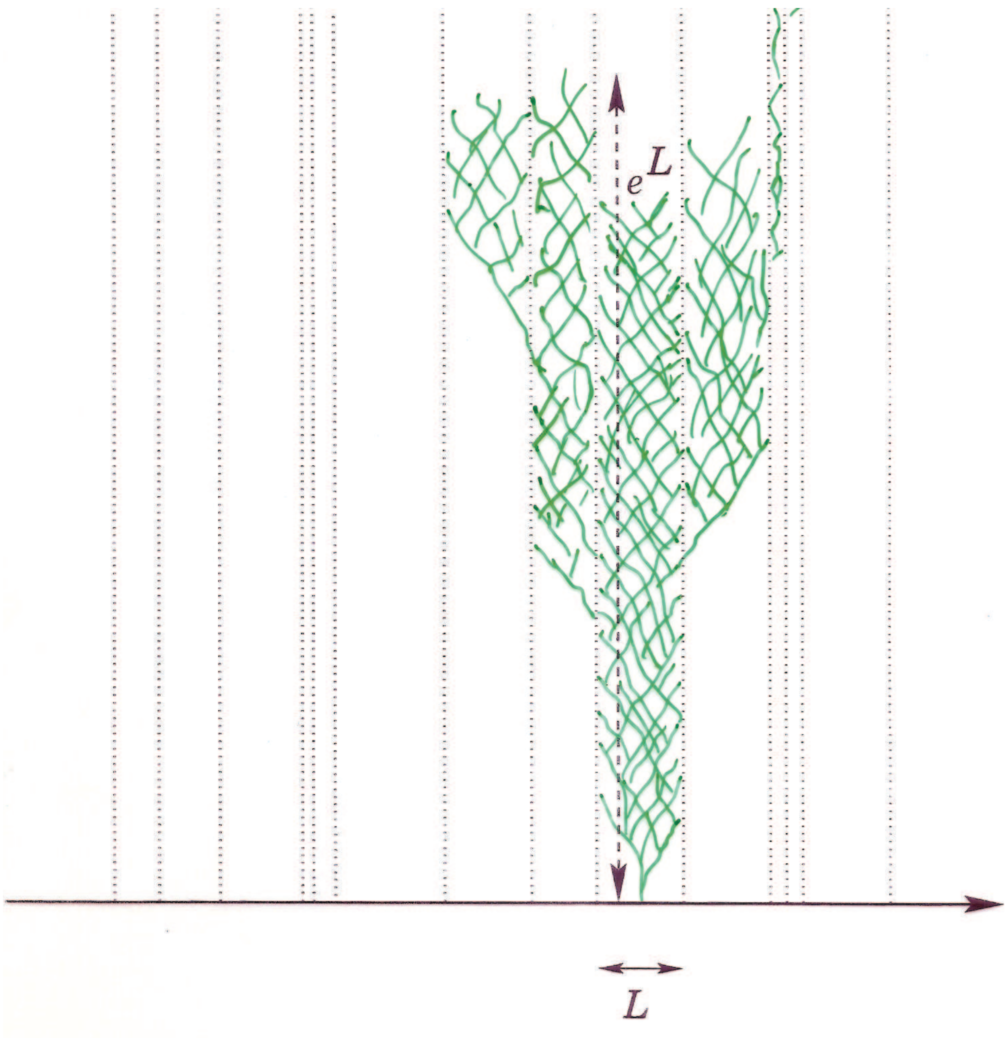
$$\mathbb{P}\left(C^+(0) \text{ is infinite}) | \chi(\xi) < \infty\right) > 0.$$

Conceptually, the structure is similar to that in the simple hierarchical situation, but:

- rescaled lattices depend also on ξ;
- main estimate (drilling through the higher mass) within a good k-site S^k is much more involved; our estimates require L somehow larger ($L \geq 192$ suffices);
- $p_k \nearrow 1$ exponentially in k.

III. Some related results

Bramson, Durrett, Schonmann (1991)



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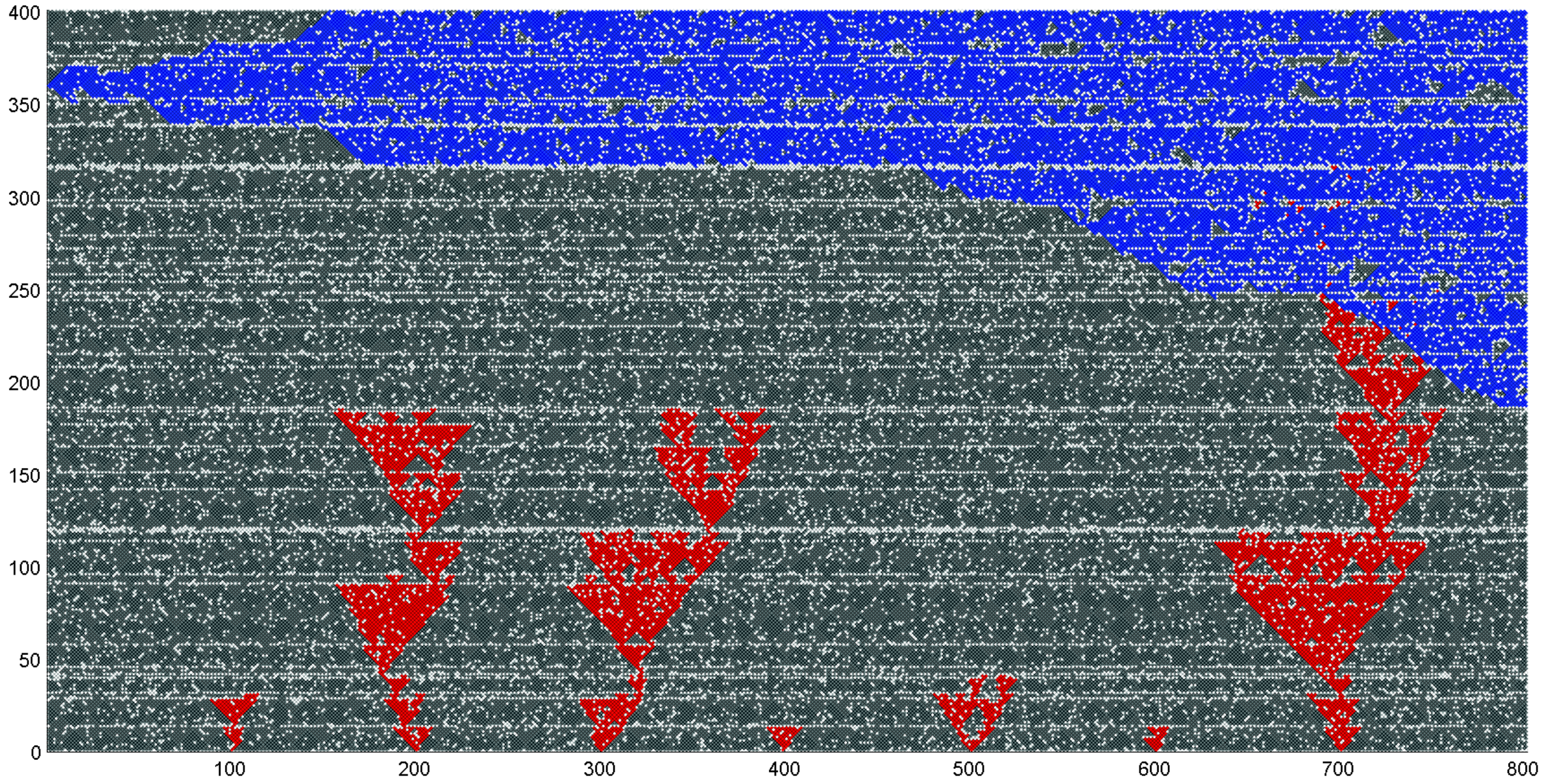

IV. Coordinate percolation. Winkler's compatibility

(η, ξ) a pair of sequences in $\Xi = \{0, 1\}^{\mathbb{N}}$

Allowed to: remove ones from η; remove zeros from ξ

Can one map both sequences to the same semi-infinite sequence?

If YES, say that (η, ξ) is compatible.

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Winkler's compatibility question:

Does it exist $(p', p) \in (0,1)^2$ such that

$$\mathbb{P}_{p'} \otimes \mathbb{P}_p \{(\eta, \xi) \in \Xi \times \Xi \colon (\eta, \xi) \text{ are compatible}\} > 0\,?$$

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Let $p \in (0,1)$. Say that $\eta \in \Xi$ is p-compatible if

$$\mathbb{P}_p\{\xi \in \Xi \colon (\eta, \xi) \text{ is compatible}\} > 0.$$

IV. Coordinate percolation. Winkler's compatibility

Theorem (Kesten, Lima, Sidoravicius, V.)

For every $\epsilon > 0$ there exist $0 < p_\epsilon < 1$ and a binary sequence $\eta \equiv \eta_\epsilon \in \Xi$, such that $\mathcal{Z}_{\eta_\epsilon}$ is a discrete fractal with Hausdorff dimension $d_H(\mathcal{Z}_\eta) \geq 1 - \epsilon$, and such that

$$\mathbb{P}_p\{\xi \in \Xi \colon (\eta, \xi) \text{ is compatible}\} > 0$$

for any $p < p_\epsilon$.

Notation: $\mathcal{Z}_\eta = \{i \geq 1 \colon \eta_i = 0\}$

For the proof

- exploit a representation as coordinate oriented percolation;
- essential ingredient: the grouping procedure mentioned before.

Move from ξ to $\psi \in \mathbb{Z}_+^{\mathbb{N}}$: $\psi_i \geq 1$ representing the length of the corresponding run of 1s.

$$\Psi = \left\{ \psi \in \mathbb{Z}_+^{\mathbb{N}} \colon \psi_i \geq 1 \text{ implies } \psi_{i+1} = 0 \right\}.$$

IV. Coordinate percolation. Winkler's compatibility

Coordinate percolation process.

Oriented graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, where

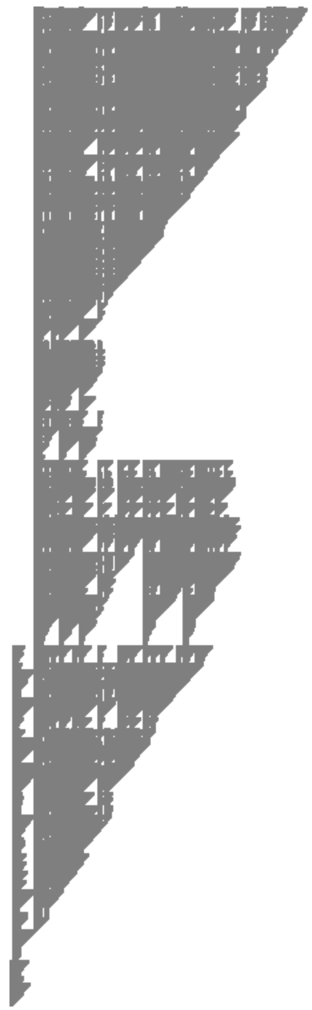
$$\mathbb{V} = \mathbb{Z}_+^2 \quad \mathbb{E} = \{ \text{ vertical n.n., northeast diagonals } \} \qquad \text{oriented upwards}$$

Given $\zeta,\ \psi \in \Psi$, define the site configuration $\omega_{\zeta,\psi}$ on $\mathcal{G}$: for $v = (v_1, v_2)$ with $v_1, v_2 \geq 1$

$$\omega_{\zeta,\psi}(v) = \begin{cases} 1 & \text{if } \zeta_{v_1} \geq \psi_{v_2}, \\ 0 & \text{otherwise.} \end{cases}$$

$v \in \mathbb{V}$ open iff $\omega_{\zeta,\psi}(v) = 1$

$(\omega_{\zeta,\psi}(0,0) = 1, \quad \omega_{\zeta,\psi}(v) = 0 \text{ if } v_1 \wedge v_2 = 0)$



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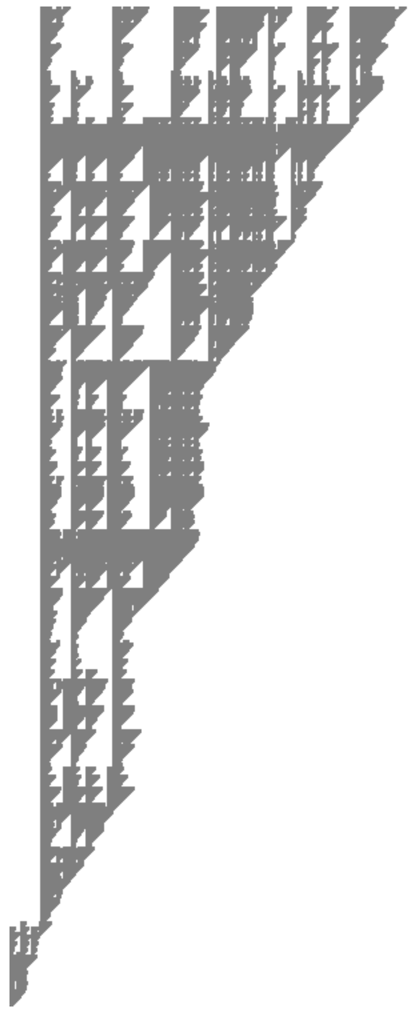

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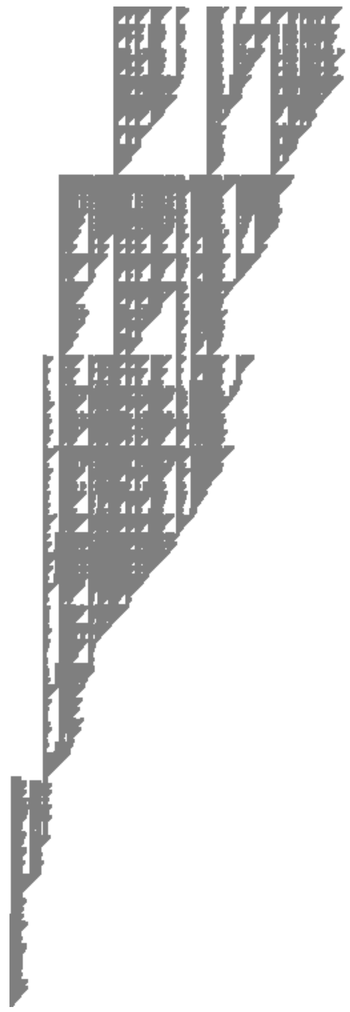

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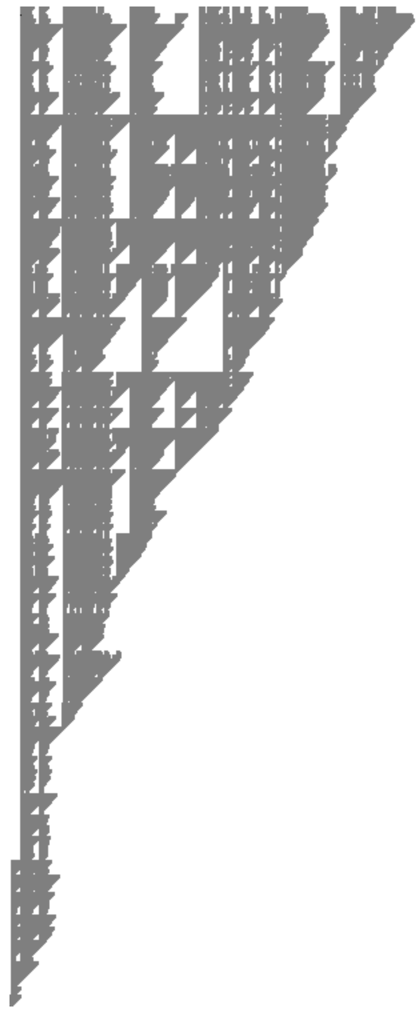

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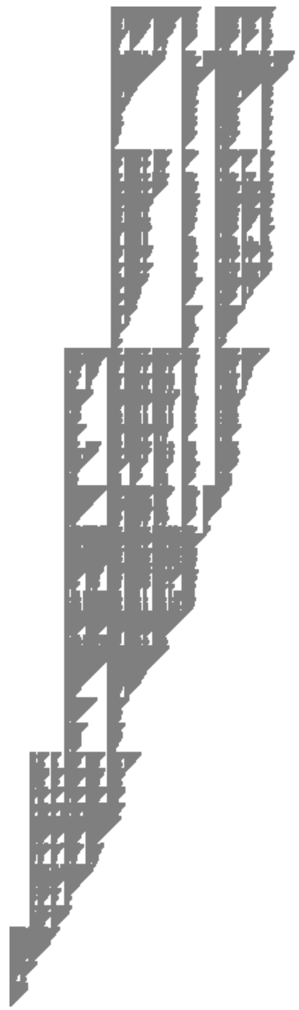

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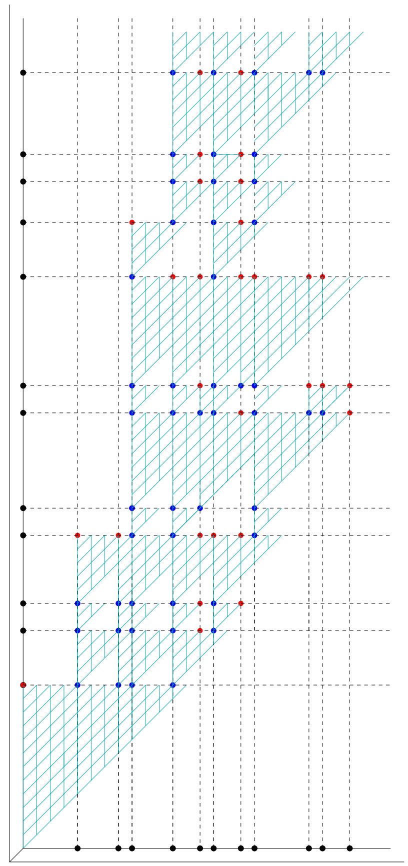

Figure 1: oriented cluster of the origin

For the compatibility question, we need more than an open oriented path.

A vertex $v = (v_1, v_2)$ with $v_1, v_2 \geq 1$ is heavy if $\psi_{v_2} \geq 1$.

IV. Coordinate percolation. Winkler's compatibility

Permitted path: does not cross two heavy vertices with the same first coordinate.

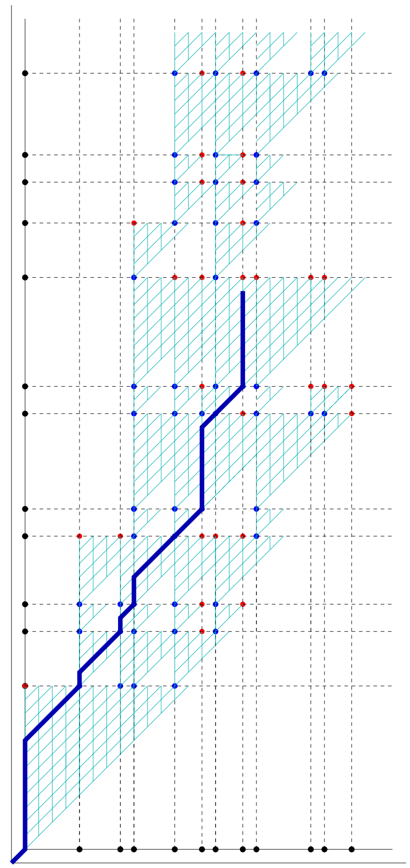

Figure 2: open permitted path from the origin

IV. Coordinate percolation. Winkler's compatibility

Lemma

Let $\zeta,\ \psi \in \Psi$. If there exists an infinite open permitted path π starting from the origin for the percolation configuration $\omega_{\zeta,\psi}$, then the pair (ζ, ψ) is compatible.

IV. Coordinate percolation. Winkler's compatibility

M-spaced sequences

$M \geq 2$ integer. A sequence $\psi \in \Psi$ is M-spaced if:

a) $i_j(\psi) \geq M^j$ for all $j \geq 1$, where

$$i_j(\psi) = \inf\{n \in \mathbb{N}: \ \psi_n \geq j\} \quad (+\infty \text{ if } \{\ \} = \emptyset)$$

b) $j - i \geq M^{\min\{\psi_i, \psi_j\}}$, for all $1 \leq i < j$.

$$\Psi_M := \{\xi \in \Psi : \xi \text{ is } M\text{-spaced}\}$$

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Theorem

Let $L \geq 2$ and $M \geq 3(L+1)$ be integers, $\zeta(L)$ the hierarchical sequence as before, and $\psi \in \Psi_M$. Then the configuration $\omega_{\zeta(L),\psi}$ has an infinite open permitted path π starting from the origin.

Corollary

If $\psi \in \Psi_M$ with $M = 3(L+1)$ the pair $(\zeta(L), \psi)$ is compatible.

IV. Coordinate percolation. Winkler's compatibility

Using this and the grouping lemma discussed before, one gets:

Theorem

Let $L \geq 2$ and $M = 3(L+1)$. If $p < \frac{1}{64M^2}$, $\tilde{\zeta}(L) \in \Psi$ is given by

$$(\tilde{\zeta}(L))_j = \begin{cases} 3M^{k-1}, & \text{if } L^k | j \text{ and } L^{k+1} \nmid j, \\ 0\ , & \text{if } L \nmid j, \end{cases}$$

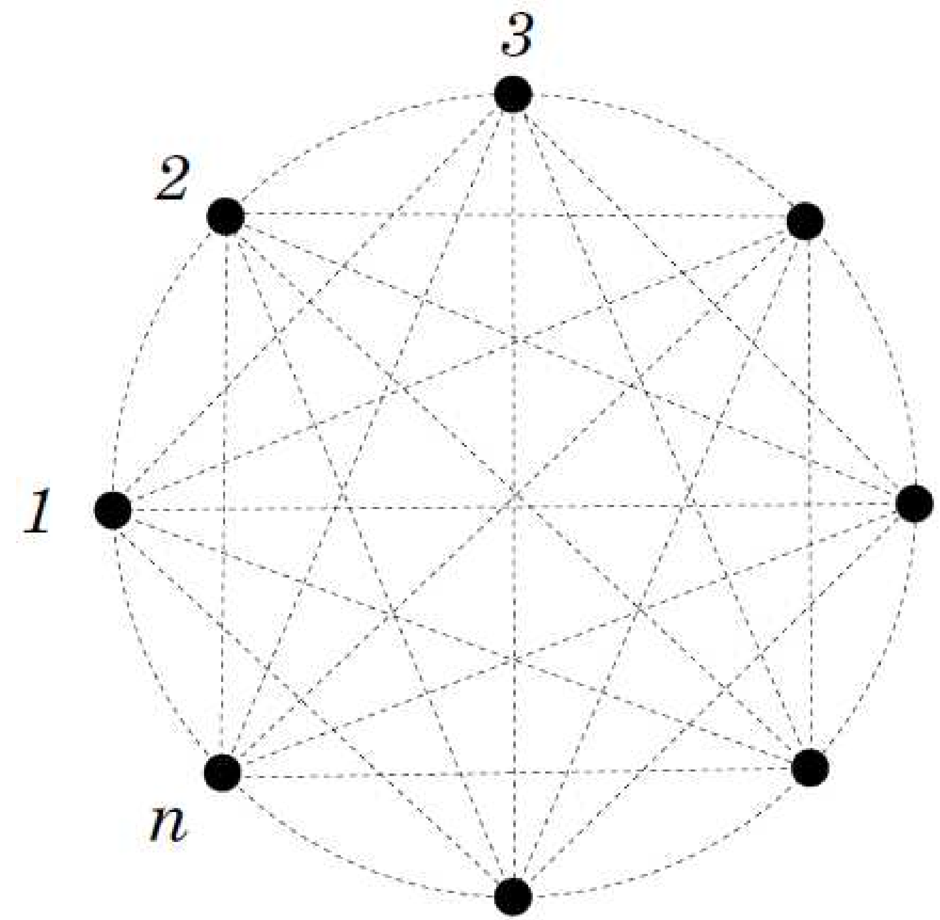
and $\eta(L)$ is the corresponding binary sequence, then

$$\mathbb{P}_p\{\xi \in \Xi\colon (\eta(L), \xi) \text{ is compatible}\} > 0.$$

Remark. The statement about the zero set of $\eta(L)$ is simple to verify, by classical results (Barlow and Taylor).

V. Related problems

Winkler's Clairvoyant Demon problem



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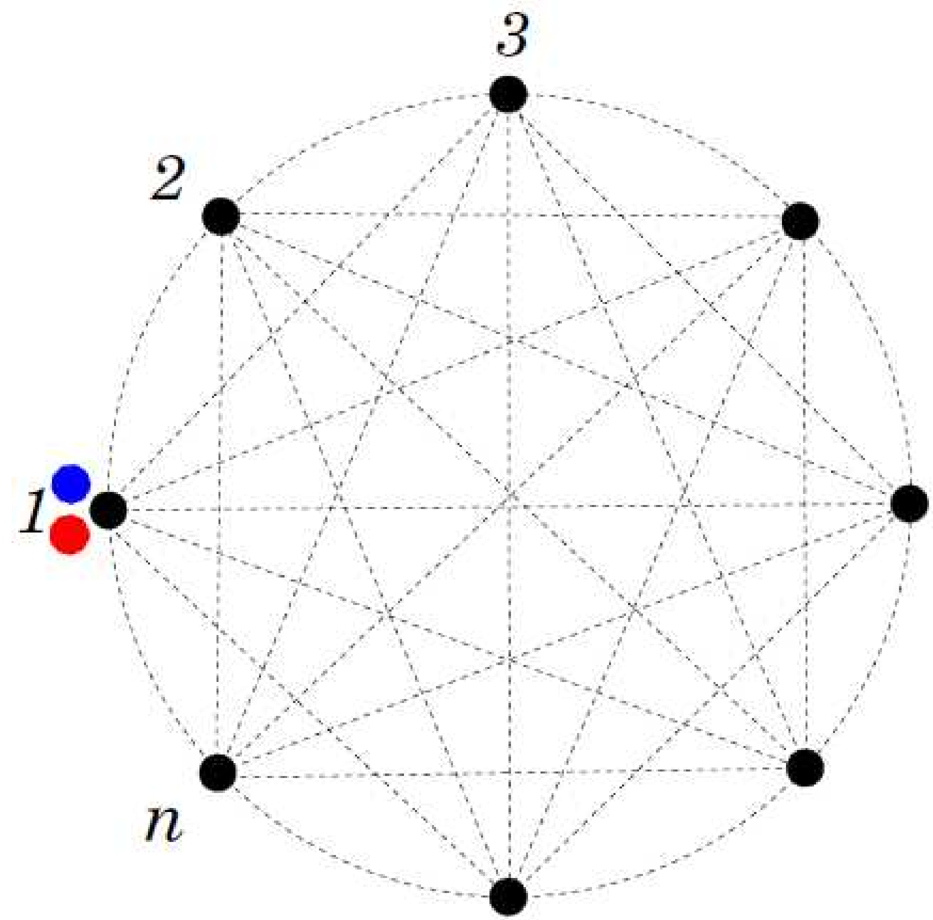

V. Related problems

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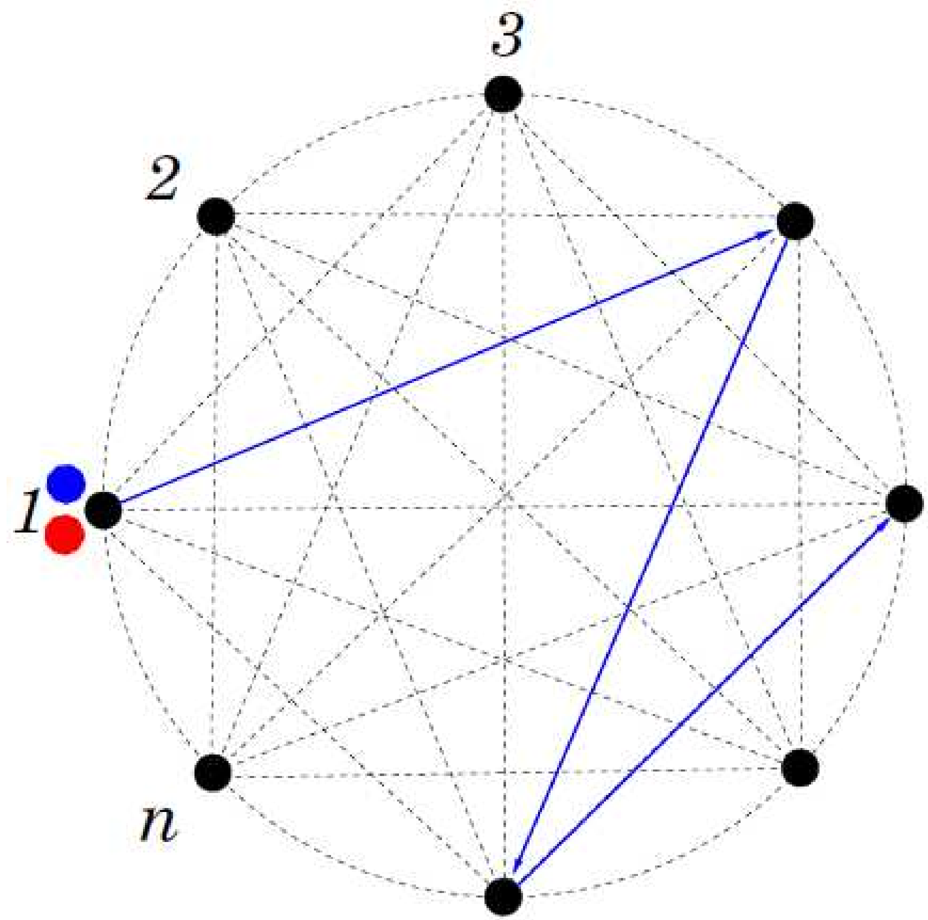

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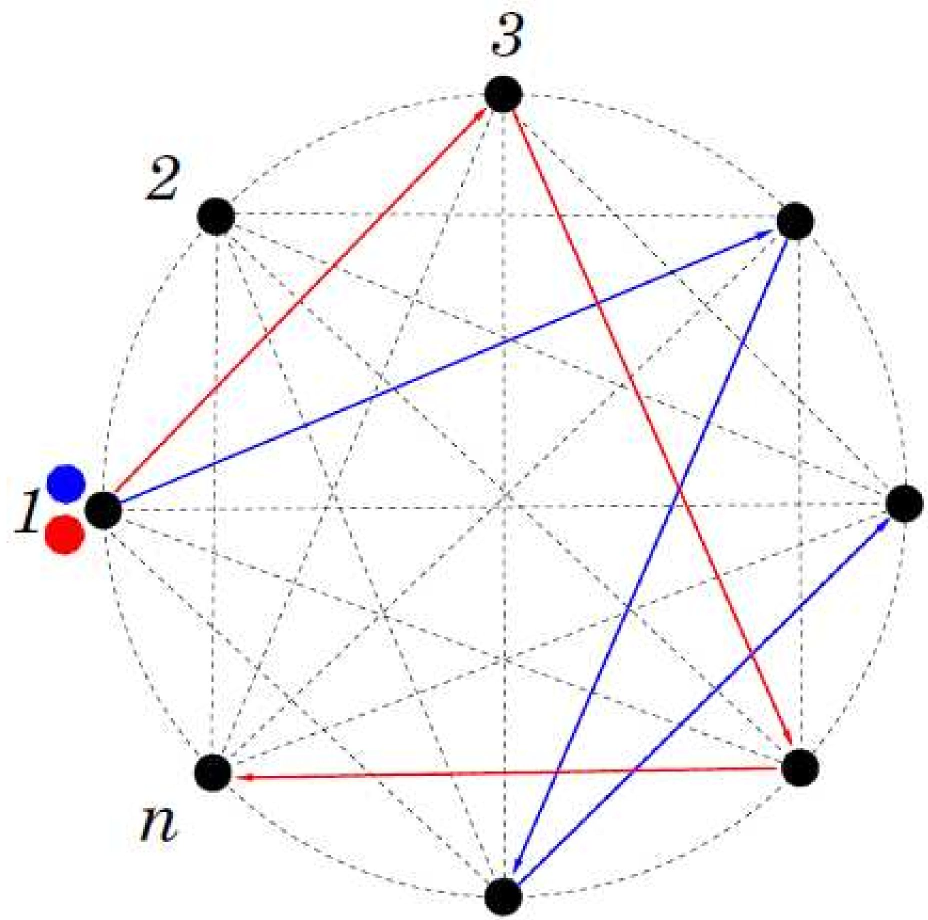

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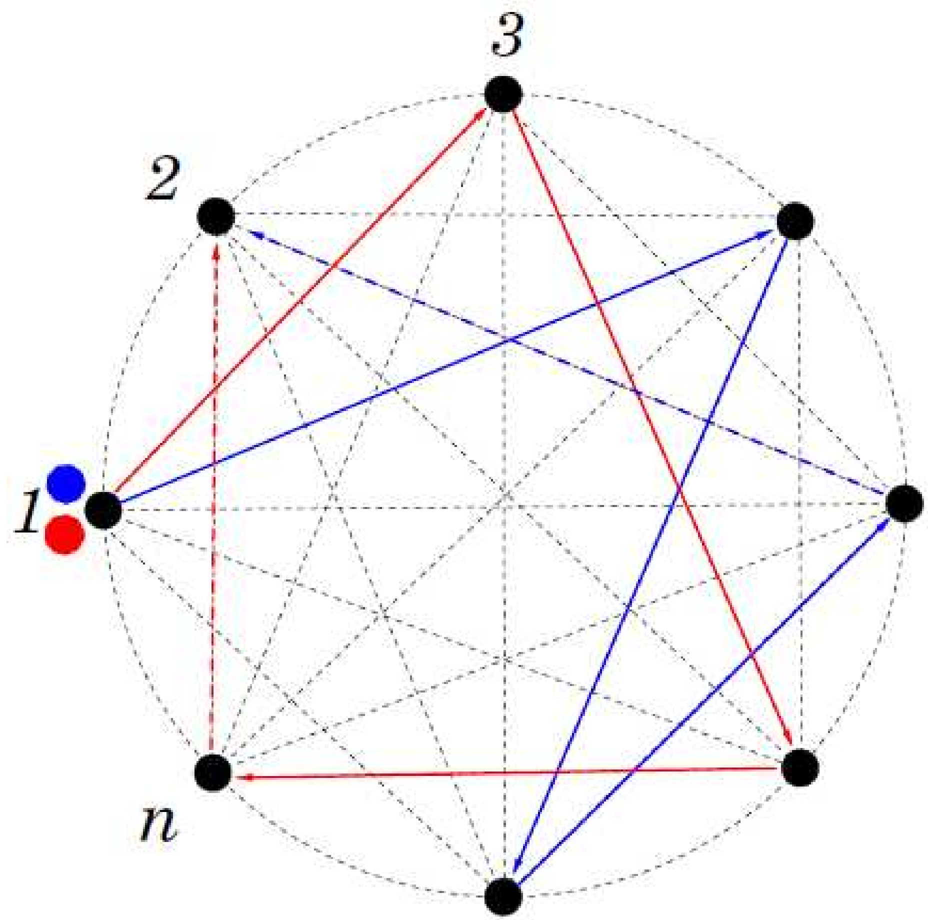

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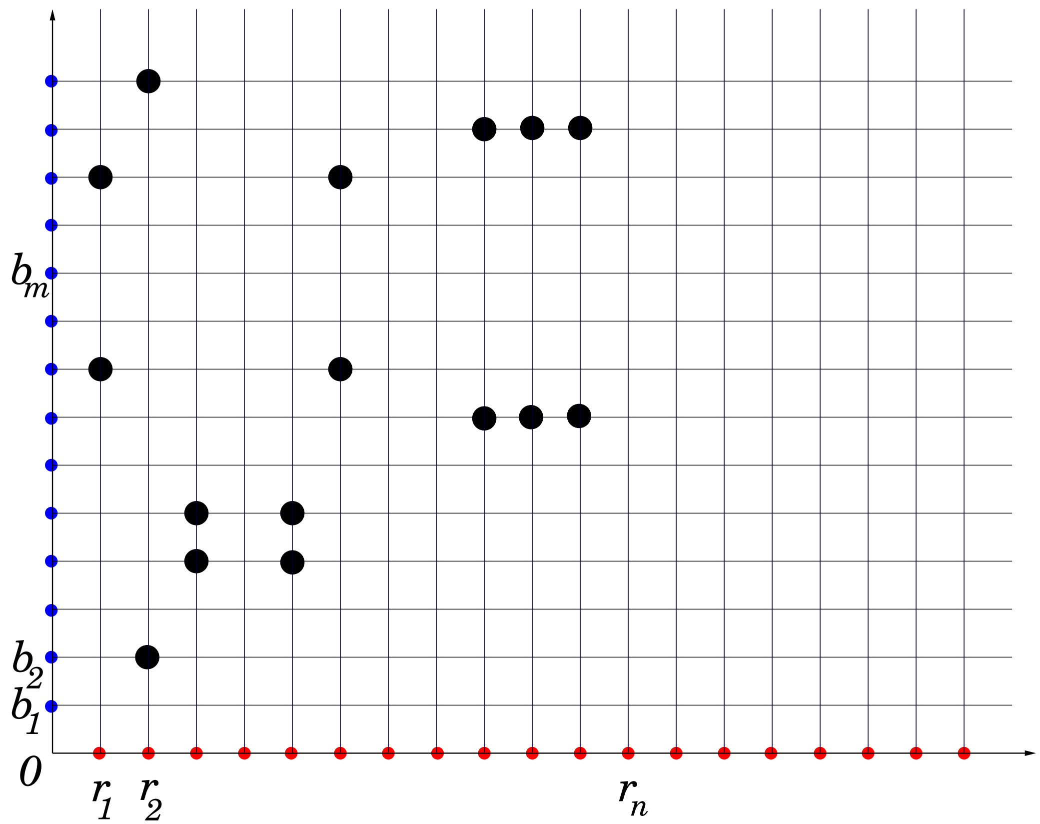
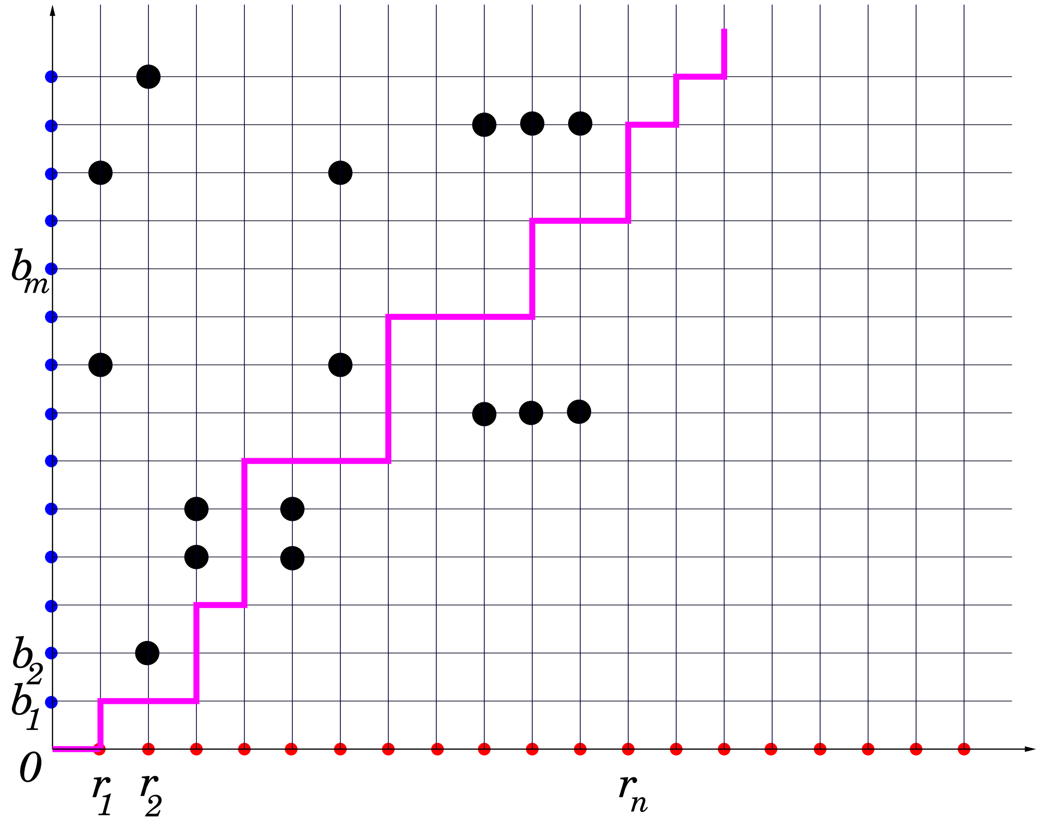


Formulation as oriented percolation problem - Noga Alon

V. Related problems

Winkler's Clairvoyant Demon problem



Formulation as oriented percolation problem - Noga Alon

V. Related problems

Winkler's Clairvoyant Demon problem

Negative answer if $n = 2$ or $n = 3$;

$n \geq 4$ - open question

Positive answer in the unoriented case for $n \geq 4$: P Winkler (2000); Balister, Bollobas, Stacey (2000)

For the oriented model: Gács (2000): The Clairvoyant Demon has a hard task ...

V. Related problems

- Stretched lattices: Jonasson, Mossel, Peres ; Hoffman

- Unoriented percolation; Potts model: Kesten, Lima, Sidoravicius, V.

- Percolation of words: Grimmett, Liggett, Richthammer (2008), Lima (2008, 2009)

- Rough isometries: Peled (2010), recent work: Basu, Sly, Sidoravicius.

References

On the compatibility of binary sequences. H. Kesten, B. Lima, V. Sidoravicius, M.E. Vares. (preprint on arXiv)

Oriented percolation in a random environment. H. Kesten, V. Sidoravicius, M.E. Vares. (preprint on arXiv)

Dependent percolation on $\mathbb{Z}^2$. H. Kesten, B. Lima, V. Sidoravicius, M.E. Vares. (preprint)

Lipschitz embeddings of random sequences. R. Basu, A. Sly. (preprint on arXiv)