POSITIVE-INTEGER-VALUED INFINITELY DIVISIBLE DISTRIBUTIONS

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A probability distribution is called infinitely divisible if for each $n = 2,3,\ldots$ it can be represented as the n-fold convolution of some probability distribution. Equivalently, a random variable is infinitely divisible if for each n it can be represented as the sum of n independent identically distributed random variables.

The class of all infinitely divisible distributions has a prominent place in Probability Theory, being, in fact, the class of all possible limit laws in the celebrated (generalized) Central Limit Problem. See, for example, Lévy (1937), Gnedenko and Kolmogorov (1954), or Breiman (1968).

The Normal and Poisson laws are the most notable examples of infinitely divisible distributions. Another is any compound Poisson distribution--i.e. any distribution of a sum of "Poissonly-many" independent, identically distributed random variables. In fact every infinitely divisible distribution is a limit (in the usual sense) of compound Poisson distributions. See, for example, Feller (1971) pp. 557.

In the special case of distributions concentrated on the non-negative integers, the situation is much simpler because then every infinitely divisible distribution is itself compound Poisson. See Feller (1968) pp. 290. We shall give a new proof of this fact as part of the following--

Theorem: Let X be a non-negative integer-valued random variable. Then the following are equivalent:

(a) X is infinitely divisible;

(b) The λ_j's, j = 1,2,... which satisfy the system of equations

(1) $$kP(X=k) = \sum_{j=1}^{k} j\lambda_j \, P(X=k-j) \qquad k = 1,2,\ldots$$

are all non-negative;

(c) X can be represented as $\sum_{i=1}^{\infty} i\, Z_i$ where the Z_i's are independent Poisson random variables with parameters $\lambda_i \geq 0$ and $\sum_{i=1}^{\infty} \lambda_i \equiv \lambda < \infty$. (Hence, necessarily, $\lambda = -\ln P(X=0)$.)

(d) X has the compound Poisson distribution represented by $\sum_{j=1}^{N} U_j$ where N is independent of the U_j's and Poisson with parameter $\lambda = -\ln P(X=0)$ and the U_j's are independent and identically distributed with $P(U_j=i) = \lambda_i/\lambda \quad i = 1,2,\ldots$.

In the proof which follows, properties (b) and (c) of the Theorem--which have inherent interest--provide the link between (a) and (d). Equivalence of (c) and (d) is well-known. (See, for example, Feller (1968) pp. 217.) It is essentially the statement that, in "Poissonly-many" multinomial trials, the numbers of occupancies of each of the cells are independent Poisson random variables. Equivalence of (a) and (b) was shown by Katti (1967). His proof uses generating functions as does the usual proof that (a) implies (d). Ours does not, which may provide some satisfaction to the many probabilists who try to avoid them whenever they can. The deepest facts we use are that the mean of a sum of random variables is the sum of the means and the (unconditional) mean is the mean of the conditional mean.

Proof of the Theorem.

(a) implies (b): By hypothesis, for each n, X can be represented as

$X = X_{n1}+\dots+X_{nn}$, the X_{ni}'s being independent and identically distributed. Let X_{n0} be a random variable independent of the X_{ni}'s, $i = 1,2,\dots,n$, but with the same distribution. Let I_A denote the indicator function of the set A. Then for $k = 1,2,\dots$,

$$\sum_{i=0}^{n} X_{ni} I_{\{X+X_{n0}=k\}} = k I_{\{X+X_{n0}=k\}}.$$

Taking expectations of both sides of this trivial identity, we have:

$$(2) \qquad (n+1)\sum_{j=1}^{k} jP(X_{n1}=j)P(X=k-j) = k\sum_{j=0}^{k} P(X_{n1}=j)P(X=k-j).$$

Of course

$$P(X_{n1}=0) = [P(X=0)]^{1/n} > 0,$$

so $P(X_{n1}=0) \to 1$ as $n \to \infty$. Hence, by induction on k in (2), $\lim_{n\to\infty} nP(X_{n1}=k)$ exists for each $k = 1,2,\dots$. Call the limit λ_k. Now take limits of both sides of (2), which gives (1).

<u>(b) implies (c)</u>: Since in (1), non-negativity of the λ_j's implies $\lambda_k < P(X=k)/P(X=0)$ so $\sum\lambda_j < \infty$, and since, for any non-negative, summable sequence of λ_j's, there is at most one probability distribution which satisfies (1), it suffices to prove that (c) implies (1).

Taking expectations of both sides of the trivial identity--

$$kI_{\{X\equiv\Sigma iZ_i=k\}} = \sum jZ_j I_{\{\Sigma iZ_i=k\}}$$

--we have (using "mean = mean of conditional mean")

$$(3) \qquad k\,P(X=k) = E[\sum_{j\geq 1} E(jZ_j\, I_{\{\Sigma iZ_i=k\}}|Z_j)]$$

$$= \sum_{j=1}^{k}\sum_{z\geq 1} jzP(X-jZ_j=k-jz)P(Z_j=z)$$

(ignoring some of the zero terms).

Now we exploit the Poisson distribution of Z_j:

$$zP(Z_j=z) = z\lambda_j^z e^{-\lambda_j}/z! \qquad z \geq 1$$

$$= \lambda_j P(Z_j=z-1).$$

Also,

$$\sum_{z\geq 1} P(X-jZ_j = k-jz)\, P(Z_j = z-1) = P(X=k-j).$$

Substituting back into (3) yields (1).

The usual proofs of (c) implies (d) and (d) implies (a) are quite routine and cannot be improved upon.

REFERENCES

[1] Breiman, L. (1968). Probability. Addison-Wesley. Reading, Mass.

[2] Feller, W. (1968), (1971). An Introduction to Probability Theory and its Applications 1, 3rd ed., 2, 2nd ed., Wiley, New York.

[3] Gnedenko, B. V. and Kolmogorov, A. N. (1954). Limit Distributions for Sums of Independent Random Variables. Addison-Wesley. Reading, Mass.

[4] Katti, S. K. (1967). Infinite divisibility of integer-valued random variables. Ann. Math. Statist. 38. 1306-1308.

[5] Lévy, P. (1937). Théorie de l'Addition des Variables Aléatoires. Bautier-Villars. Paris.