

Introductory Probability for Diverse Applications

Anirban DasGupta

Contents

I	Part I: Introductory Probability	6
1	Introducing Probability	7
1.1	Experiments and Sample Spaces	8
1.2	Set Theory Notation and Axioms of Probability	9
1.3	How to Interpret a Probability	11
1.4	Calculating Probabilities	13
1.4.1	Manual Counting	14
1.4.2	General Counting Methods	16
1.5	Inclusion Exclusion Formula	19
1.6	* Bounds on the Probability of a Union	22
1.7	Exercises	23
1.8	References	29
2	Three Classics	30
2.1	The Birthday Problem	30
2.1.1	Stirling's Approximation	31
2.1.2	* Some Extensions	32
2.2	The Matching Problem	33
2.3	* Probability of Occurrence of k of n Events	34
2.3.1	The Coupon Collection Problem	36
2.4	Exercises	37
2.5	References	38
3	Conditional Probability and Independence	40
3.1	Basic Formulas and First Examples	40
3.2	More Advanced Examples	42
3.3	Independent Events	45
3.4	Bayes Theorem	48
3.5	Exercises	50
4	Integer Valued and Discrete Random Variables	55
4.1	Mass Function	55
4.2	CDF and Median of a Random Variable	57

4.2.1	Functions of a Random Variable	64
4.2.2	Independence of Random Variables	66
4.3	Expected Value of a Discrete Random Variable	67
4.4	Basic Properties of Expectations	69
4.5	Illustrative Examples	71
4.6	Using Indicator Variables to Calculate Expectations	72
4.7	The Tail Sum Method to Calculate Expectations	74
4.8	Variance and Basic Inequalities	75
4.9	Illustrative Examples	78
4.9.1	Variance of the Sum of Independent Random Variables	79
4.10	Utility of μ, σ as Summaries	80
4.10.1	Chebyshev's Inequality and Weak Law of Large Numbers	80
4.10.2	Better Inequalities	82
4.11	* Other Fundamental Moment Inequalities	83
4.12	Truncated Distributions	86
4.13	Exercises	87
4.14	References	92
5	Generating Functions, Patterns, and Coincidences	93
5.1	The Generating Function	93
5.2	The Moment Generating Function and Cumulants	96
5.2.1	* Cumulants	99
5.3	* Applications of Generating Functions	100
5.3.1	Success Runs	100
5.3.2	* More General Patterns	102
5.3.3	Multinomial Trials and Expected Waiting Times	103
5.4	Exercises	106
5.5	References	108
6	Standard Discrete Distributions	109
6.1	Introducing the Special Distributions	109
6.2	Binomial Distribution	112
6.3	Geometric and Negative Binomial Distribution	116
6.4	Hypergeometric Distribution	119
6.5	Poisson Distribution	122
6.5.1	Mean Absolute Deviation and the Mode	126

6.6	Poisson Approximation to Binomial	127
6.7	* Miscellaneous Poisson Approximations	130
6.8	Distribution of Sums and Differences	132
6.9	* Discrete Does Not Mean Integer Valued	135
6.10	Exercises	136
6.11	References	141
7	Moments, the Moment Problem and Stirling Numbers	143
7.1	Stirling Numbers and Connections to Moments	143
7.2	* Generalized Stirling Numbers and Connections to Moments	147
7.3	* Determining Distributions from Moments	149
7.4	Exercises	152
7.5	References	154
8	Urn Models in Physics and Genetics	155
8.1	Urn Models in Quantum Mechanics	155
8.2	* Poisson Approximations	160
8.3	Pólya's Urn	162
8.4	Pólya-Eggenberger Distribution	164
8.5	* de Finetti's Theorem and Pólya Urns	165
8.6	* Urn Models in Genetics	167
8.6.1	Wright-Fisher Model	168
8.6.2	Time till Allele Uniformity	170
8.7	Mutation and Hoppe's Urn	171
8.8	Ewens Sampling Formula	173
8.9	Exercises	176
8.10	References	178
9	Continuous Random Variables	180
9.1	The Density Function and the CDF	180
9.1.1	Quantiles	186
9.2	Generating New Distributions from the Old.	188
9.3	Normal and Other Symmetric Unimodal Densities	191
9.4	Functions of a Continuous Random Variable	194
9.4.1	Quantile Transformation	198
9.4.2	Cauchy density	199

9.5	Expectation of Functions and Moments	200
9.6	The Tail Probability Method to Calculate Expectations	208
9.6.1	* Survival and Hazard Rate	209
9.6.2	Moments and the Tail	209
9.7	* Moment Generating Function and Fundamental Tail Inequalities .	211
9.7.1	Chernoff-Bernstein Inequality	212
9.7.2	Lugosi's Improved Inequality	214
9.8	* Jensen and Other Moment Inequalities and a Paradox	215
9.9	Exercises	217
9.10	References	221
10	Some Special Continuous Distributions	223
10.1	The Uniform Distribution	223
10.2	The Exponential and Weibull Distribution	225
10.3	The Gamma and the Inverse Gamma Distribution	229
10.4	The Beta Distribution	233
10.5	Exercises	236
10.6	References	240
11	Normal Distribution	241
11.1	Definition and Basic Properties	241
11.2	Working with a Normal Table	245
11.3	Additional Examples and the Lognormal Density	246
11.4	Sums of Independent Normal Variables	249
11.5	Mill's Ratio and Approximations for the Standard Normal CDF . .	252
11.6	* Stein's Lemma	254
11.7	* Chernoff's Variance Inequality	256
11.8	* Characterizations	257
11.9	Exercises	258
11.10	References	263
12	Normal Approximations and Central Limit Theorem	265
12.1	Some Motivating Examples	265
12.2	Central Limit Theorem	267
12.3	Normal Approximation to Binomial	269
12.4	Examples of the General CLT	275

12.5	Normal Approximation to Poisson and Gamma	280
12.6	* Convergence of Densities and Higher Order Approximations . . .	284
12.6.1	Refined Approximations	285
12.7	Practical Recommendations for Normal Approximations	287
12.8	Exercises	288
12.9	References	293

II Part II: Intermediate Probability 294

13 Multivariate Discrete Distributions 295

13.1	Bivariate Joint Distributions and Expectations of Functions	295
13.2	Conditional Distributions and Conditional Expectations	302
13.3	Using Conditioning to Evaluate Mean and Variance	306
13.4	Covariance and Correlation	309
13.5	Multivariate Case	314
13.5.1	* Joint MGF	315
13.5.2	Multinomial Distribution	317
13.6	* The Poissonization Technique	320
13.7	Exercises	321

14 Markov Chains and Applications 325

14.1	Notation and Basic Definitions	325
14.2	Chapman-Kolmogorov Equation	332
14.3	Communicating Classes	336
14.4	* Gambler's Ruin	339
14.5	* First Passage, Recurrence and Transience	341
14.6	Long Run Evolution and Stationary Distributions	348
14.7	Exercises	356
14.8	References	365

15 Multidimensional Densities 367

15.1	Joint Density Function and Its Role	367
15.2	Expectation of Functions	377
15.3	Bivariate Normal	381
15.4	Conditional Densities and Expectations	385
15.5	Bivariate Normal Conditional Distributions	393

15.6	* Useful Formulas and Characterizations for Bivariate Normal . . .	395
15.6.1	Computing Bivariate Normal Probabilities	397
15.7	Conditional Expectation Given a Set	399
15.8	Exercises	401
15.9	References	405
16	Convolutions, Transformations and High Dimensions	406
16.1	Convolutions and Examples	406
16.2	Products and Quotients and the t and F Distribution	411
16.3	One to One Transformations	415
16.4	* n Dimensional Polar and Helmert's Transformation	421
16.4.1	Efficient Spherical Calculations with the Polar Coordinates	421
16.4.2	Independence of Mean and Variance in Normal Case . . .	424
16.5	* The Dirichlet Distribution	426
16.5.1	Picking a Point from the Surface of a Sphere	429
16.5.2	Poincaré's Lemma	429
16.6	* Ten Important High Dimensional Formulas for Easy Reference . .	430
16.7	Exercises	431
16.8	References	436
17	The Multivariate Normal and its Offsprings	437
17.1	Definition and Some Basic Properties	437
17.2	Conditional Distributions	440
17.3	Exchangeable Normal Variables	443
17.4	Sampling Distributions	446
17.4.1	* Wishart Expectation Identities	447
17.4.2	Hotelling's T^2 and Distribution of Quadratic Forms . . .	449
17.4.3	* Distribution of Correlation Coefficient	451
17.5	* Noncentral Distributions	452
17.6	* Some Important Inequalities for Easy Reference	453
17.7	Exercises	455
17.8	References	459
18	Order Statistics, Extremes and the Poisson Process	460
18.1	Basic Distribution Theory	460
18.2	Quantile Transformation and Existence of Moments	468

18.3	Spacings	473
18.3.1	Exponential Spacings and Rényi's Representation	473
18.3.2	Uniform Spacings	475
18.4	Conditional Distributions and Markov Property	476
18.5	Some Applications	479
18.5.1	Records	479
18.5.2	The Empirical CDF	481
18.6	The Poisson Process	484
18.6.1	Notation	485
18.6.2	Defining a Poisson Process	486
18.6.3	Important Properties of a Poisson Process	487
18.7	Some Useful Inequalities	497
18.8	Exercises	499
18.9	References	505