Asymptotic properties of functionals of increments of a continuous semimartingale with stochastic sampling times

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Abstract

This paper is concerned with asymptotic behavior of a variety of functionals of increments of continuous semimartingales. Sampling times are assumed to follow a rather general discretization scheme. If an underlying semimartingale is thought of as a financial asset price process, a general sampling scheme like the one employed in this paper is capable of reflecting what happens whenever the financial trading data are recorded in a tick-by-tick fashion. A law of large numbers and a major central limit theorem are proved after an appropriate normalization. Applications of our results include statistical estimation and inference for high-frequency financial data models.

Keywords: central limit theorem, continuous semimartingale, law of large numbers, stochastic sampling times.

1 Introduction

One of the common tasks in stochastic processes theory is to estimate the parameters of the process. As an alternative, nonparametric estimation, such as that of spot or integrated quadratic volatility, may need to be performed as well. Over the past decade, the field of volatility modeling and analysis for high-frequency financial data has developed optimistically. A plethora of methodologies were introduced to estimate the quadratic variation of a price process from high-frequency data. Estimation methods for univariate volatilities include realized volatility [2, 3, 6], bi-power realized variation [7], two-time scale realized volatility [27], multi-scale realized volatility [25], wavelet realized volatility [14], realized kernel volatility [4, 5], pre-averaging realized volatility [10, 18], and Fourier realized volatility [12, 19]. For multiple assets, popular co-volatility estimators are Hayashi and Yoshida (HY) estimator based on overlap intervals [16], the previous-tick and multi-scale approach [26], refresh-time scheme and realized kernel volatility [5], generalized synchronization scheme and quasi-maximum likelihood estimation [1], pre-averaging approach [10], large volatility matrix estimators based on regularization

[20, 21, 22, 23]. These methods have been shown to be successful in applications; moreover, they have significantly improved our understanding of time-varying volatility of stochastic processes as well as the ability to predict future volatility. A comprehensive review in this literature was given in, e.g., [24].

Most of the time, a stochastic process X_t is observed at discrete times that are commonly non-equispaced. Moreover, in many cases, such as that of various asset price processes in financial mathematics, the frequency of sampling is extremely high and occurs on a tick-by-tick basis. That results, in turn, in a random high-frequency sampling. We consider the so-called finite horizon case, where the observation window is a fixed time interval $[0, T]$ for some $T > 0$. The sampling times are $t(n, i)$, $i = 1, \ldots, n$ and the average duration time between the two consecutive sampling times $\tau(n, i) = t(n, i) - t(n, i-1)$ goes to zero as the sample size goes to infinity. In order to conduct any nonparametric inference, one typically needs, as a first step, the consistency of various functionals of increments of the the process X_t. Usually just consistency is not enough, and one also need rates of convergence and an associated central limit theorem. To obtain these results, certain restrictions on the nature of sampling process have to be imposed. These assumptions imply that both expectation and the variance of a duration time between two successive sampling times go to zero as the sample size $n \to \infty$ at rates of $O(n^{-1})$ and $O(n^{-2})$, respectively. Such an assumption is rather mild in the sense that it includes, for example, a well known Poisson model that implies exponentially distributed duration times.

The need to take the random high-frequency sampling into account when performing non-parametric estimation and inference has been noted earlier. Barndorff-Nielsen et al. [4] noted, for example, that the regular realized kernel estimator of quadratic volatility becomes inconsistent under a typical random high-frequency sampling scheme. Hayashi et al. [15] considered irregular sampling schemes while posing conditions on the variance of the sampling durations. The main contribution of this paper is that we obtain both a law of large numbers and a major central limit theorem under very broad assumptions on the nature of the sampling process. No specific distribution for the duration times is assumed as well. Our results can be rather easily generalized to the case where the duration times are not independent. The paper is structured as follows. Section (2) is concerned with the detailed model set-up. Section (3) discusses the law of large numbers while section (4) covers a major important central limit theorem.

2 Model Set-up

1. Price model:
Assume that we have a probability space $(\Omega, P, \mathcal{F})$ and an assigned filtration $\{\mathcal{F}_t\}_{t\geq 0}$ containing all the price process related information up to time t; also, let $\{W_t\}$ be a Brownian Motion defined on this space. Let $X_t = ln(S_t)$ be the log price process such that $dX_t = b_t dt + \sigma_t dW_t$ with a drift process b_t and the volatility process σ_t. We assume that the drift process b_t and the volatility process σ_t are adapted to $\mathcal{F}_t$. For brevity, we denote the integrated volatility $\mathrm{IV} = \int_0^T \sigma_t^2 \, dt$.

Throughout this paper, we will use several important assumptions on the nature of the process X_t. For convenience, we start with enumerating all of them in one location.

1. **Assumption A:**
   Given any finite $T > 0$, we assume that the spot volatility σ_t^2, $0 \leq t \leq T$ can be bounded with probability 1:
   $$P\{\sigma_t^2 \leq M_T,\ 0 \leq t \leq T\} = 1$$
   where M_T is a random variable with finite fourth moment:
   $$E(M_T^4) < \infty$$

2. **Assumption B:**
   We also assume that the drift b_t, $0 \leq t \leq T$ can be bounded with probability 1:
   $$P\{|b_t| \leq A_T, 0 \leq t \leq T\} = 1$$
   for any fixed $T > 0$ where A_T is a random variable with finite fourth moment:
   $$E(A_T^4) < \infty$$

   Assumption H:
   Let X_t be a continuous Itô semimartingale with the representation
   $$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$
   where W_t is a standard Wiener process and b_t, σ_t are locally bounded. Moreover, the volatility process σ_t is also an Itô semimartingale of the form
   $$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma} dW_s + \tilde{\kappa}(\tilde{\delta}) \star (\underline{\mu} - \underline{\nu})_t + \tilde{\kappa}'(\tilde{\delta}) \star \underline{\mu}_t$$
   where $\underline{\mu}$ is a Poisson random measure on $(0, \infty) \times E$ with intensity measure $\underline{\nu}(dt, dx) = dt \otimes \lambda(dx)$, where λ is a σ-finite and infinite measure without atom on an auxiliary measurable set $(E, \mathcal{E})$. $\tilde{\kappa}$ is a truncation function and $\tilde{\kappa}'(x) = x - \tilde{\kappa}(x)$. $\tilde{\delta}(\omega, t, x)$ is a predictable function on $\Omega \times R_+ \times E$. Moreover, we assume that

   (a) Let $\tilde{\gamma}$ be a (non-random) nonnegative function such that $\int_E (\tilde{\gamma}(x)^2 \wedge 1)\lambda(dx) < \infty$. Then, the processes $\tilde{b}_t(\omega)$ and $\sup_{x\in E} \frac{\|\tilde{\delta}(\omega,t,x)\|}{\tilde{\gamma}(x)}$ are locally bounded, and

   (b) All paths $t \to b_t(\omega)$, $t \to \tilde{\sigma}_t(\omega)$, $t \to \tilde{\delta}(\omega, t, x)$ are right-continuous with left limits (càdlàg).

The following assumption is fairly strong and is only used as a starting point for the classical localization procedure. All of our results are proved under the much more relaxed local boundedness assumption as stated in Assumption H.

Assumption SH:
In addition to the assumption (H) we have, for some constant Λ and all (ω, t, x):

$$\|b_t(\omega)\| \leq \Lambda, \|\sigma_t(\omega)\| \leq \Lambda, \|X_t(\omega)\| \leq \Lambda$$
$$\|\tilde{b}_t(\omega)\| \leq \Lambda, \|\tilde{\sigma}_t(\omega)\| \leq \Lambda, \|\tilde{\delta}(\omega, t, x)\| \leq \Lambda(\tilde{\gamma}(x) \wedge 1).$$

3. **Trading time model: Assumption T**
To study asymptotic properties, we will allow the frequency of observations increases to infinity. Hence at each stage n, we have strictly increasing observation times $(t(n,i) : i \geq 0)$, and without restriction we may assume $t(n,0) = 0$. We further denote

$$\tau(n,i) = t(n,i) - t(n,i-1)$$
$$N_t^n = \sup(i : t(n,i) \leq t)$$
$$E[\tau(n,i)] = \Delta_n$$
$$\pi_t^n = \sup_{i=1,\cdots,N_t^n} \tau(n,i), \quad \delta_t^n = \inf_{i=1,\cdots,N_t^n} \tau(n,i)$$

We assume that with T the time horizon,

$$\pi_T^n = O_p(n^{-1}), \quad \delta_T^n = O_p(n^{-1}) \quad as \quad n \to \infty$$

Remark 2.1. *The Assumption T implies the following useful results:*

$$\Delta_n = O(n^{-1}), \quad Var(\tau(n,i)) = O(n^{-2})$$
$$\sum_{i=1}^{N_t^n} (t(n,i+1) - t(n,i))^2 = O_p(n^{-1})$$

and

$$\sum_{i=1}^{N_t^n} 1 = N_t^n = O_p(n)$$

which can be very useful in our proofs of LLN and CLT.

Remark 2.2. *Note that this assumption includes, for example, the Poisson model in which the exponential distribution is commonly used to model duration times. Historically, the assumption of exponential distribution for duration times was quite popular. As an example, a well known model of [11] models the trading times as a simple Poisson process which means that the trading durations are i.i.d. exponentially distributed with some parameter λ. Other alternative models of trading times* *may assume that the trading durations are correlated over time as in, for example, the autoregressive conditional duration (ACD) model introduced by [13]. Moreover, Bouchaud et al. [8] offers a comprehensive study on the empirical properties of the whole order book. Since our main interest lies in estimation of realized volatility of the data, we are going to start with a simple assumption of independent duration times first. We will consider possible generalization to the ACD model as a next step in our research.*

Remark 2.3. *From now on, for convenience purposes, we use t_i^n instead of $t(n,i)$, especially when it is a subscript itself. On occasion, whenever it does not cause any confusion, the index n is omitted and t_i^n is simply denoted t_i. All of the above also applies to $\tau(n,i)$.*

Finally, the last assumption concerns the relationship between transaction times t_i^n and the price process X_t.

4. **Independence Assumption C:**
   Let $\{\mathcal{N}_t^n\}_{t\geq 0}$ be the filtration generated by transaction times $0 \leq t_1^n, \ldots, t_{N_t^n}^n \leq t$ for some $0 \leq t \leq T$. We assume that $\mathcal{N}_t^n$ is independent of $\mathcal{F}_t$.

3 Laws of large numbers (LLNs) for increments of functions of semimartingales

Our first goal is to obtain a uniform law of large numbers for normalized increments of the semimartingale process $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$ when all of the durations $\{\tau_i^n\}_{i=2}^{N_t^n}$ satisfy Assumption T. We denote $\Delta_i^n X = X_{t_i} - X_{t_{i-1}}$ the increments of this process. For an arbitrary function f, functions of the increments of X_t are $V(f)_t = \Sigma_{i=1}^{N_t^n} f(\Delta_i^n X)$ and, in the normalized form, $V'(f)_t = \Sigma_{i=1}^{N_t^n} f(\Delta_i^n X/\sqrt{\tau_i})$. Finally, we also define the so-called approximate variation of the pth order for the process X_t as $\tilde{X}_t$ as $B(p)_t = \Sigma_{i=1}^{N_t^n} |\Delta_i^n X|^p$.

Before formulating our LLN, we need to define the idea of uniform convergence in probability.

Definition 3.1. *A sequence of jointly measurable stochastic processes ξ_t^n is said to converge locally uniformly in probability to a process ξ_t if $\lim_{n\to\infty} P\left(\sup_{s\leq t} |\xi_t^n - \xi_t| > K\right) = 0$ for any $K > 0$ and any finite t. This convergence is commonly denoted $\xi_t^n \overset{u.c.p.}{\to} \xi_t$.*

Now we can state the following uniform law of large numbers.

Theorem 3.2. *Assume (H) and (T). Let f be a continuous function on R^k for some $k \geq 1$, which satisfies*

$$|f(x_1, \ldots, x_k)| \leq K_0 \prod_{j=1}^{k} (1 + \|x_j\|^p)$$

for some $p > 0$ and K_0. Define

$$V^{'}(f,k)_t = \sum_{i=1}^{N_t^n} f\left(\Delta_i^n X/\sqrt{\tau_i}, \cdots, \Delta_{i+k-1}^n X/\sqrt{\tau_{i+k-1}}\right).$$

Also, let $\rho_\sigma^{\otimes k}(f) = E[f(X)]$ where $X = (x_1, x_2, \cdots, x_k) \sim N(0, \sigma^2 I)$ and I is a $k \times k$ identity matrix. Then,

$$\Delta_n V^{'}(f,k)_t \xrightarrow{u.c.p.} \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du.$$

Proof. To prove this theorem, we will use the so-called localization procedure described in detail in [17]. Essentially, we prove the statement we need under the weaker assumption (SH) and then extend it to a more general situation through the use of a Lemma 3.14 in [17], p. 218. In what follows we only prove that the statement of the Theorem (3.2) is true when X satisfies (SH). For convenience purposes, from now on we denote t_i^n the time of the ith transaction within the interval $[0, T]$; the superscript n refers to the total number of transactions in this interval. We approximate $\Delta_{i+l}^n X$ by substituting the value $\sigma_{t_{i-1}^n}$ for the whole function σ_t on each of the intervals $[t_{i+l-1}^n, t_{i+l}^n]$ for $1 \le l \le n-i$. To make the notation more precise, we also define $\Delta_{i+l}^n W := W_{t_{i+l}^n} - W_{t_{i+l-1}^n}$, $\beta_{i,l}^n := \sigma_{t_{i-1}^n} \Delta_{i+l}^n W/\sqrt{\tau_{i+l}}$, and $x_{i,l}^n := \frac{1}{\sqrt{\tau_{i+l}}} \int_{t_{i+l-1}^n}^{t_{i+l}^n} \left(b_s ds + (\sigma_s - \sigma_{t_{i-1}^n}) dW_s\right)$. Therefore, we can now write

$$\Delta_{i+l}^n X = \sqrt{\tau_{i+l}}(x_{i,l}^n + \beta_{i,l}^n).$$

Define $E_{i+l-1}^n(\cdot) = E\left(\cdot | \mathcal{F}_{t_{i+l-1}^n} \bigvee \mathcal{N}_{t_{i+l-1}^n}\right)$; then, it's easy to check that, for any $q > 0$, there exists a constant K_q such that

$$E_{i+l-1}^n(\|\beta_{i,l}^n\|^q) \le K_q.$$

A repeated use of Doob's and Burkholder-Davis-Gundy inequalities (see [9]) results in, first, $E\left(\|\sigma_{t+s} - \sigma_t\|^q | \mathcal{F}_t\right) \le K_q s^{1 \wedge (q/2)}$; this, in turn, lets us claim that $E_{i+l-1}^n\left(|x_{i,l}^n|\right) \le K_q E_{i+l-1}^n(\sqrt{\tau_{i+l}}) \to 0$. Next, for any function f satisfying the assumptions in Theorem (3.2), and any $A > 0$, we have $G_A(\epsilon) = \sup_{\{x_j, y_j : \|x_j\| \le A, \|y_j\| \le \epsilon\}} \|f(x_1 + y_1, \cdots, x_k + y_k) - f(x_1, \cdots, x_k)\| \xrightarrow{\epsilon \to 0} 0$. Let us introduce an auxiliary functional $V^{''}(f,k)_t = \Sigma_{i=1}^{N_t^n} f(\beta_{i,0}^n, \cdots, \beta_{i,k-1}^n)$. We can conclude, then, that

$$\Delta_n \left(V'(f,k)_t - V^{''}(f,k)_t\right) \xrightarrow{u.c.p.} 0$$

and so it is sufficient to show the uniform convergence to zero for the functional $\Delta_n V^{''}(f,k)$.

First, denote $\eta_i^n = \Delta_n f(\beta_{i,0}^n, \cdots, \beta_{i,k-1}^n)$. Then we have $E_{i-1}^n(\eta_i^n) = \Delta_n \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) = (\tau_i^n + \Delta_n - \tau_i^n)\rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) = \tau_i^n \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) + (\Delta_n - \tau_i^n)\rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f)$. As a first step note that

$$\sum_{i=1}^{N_t^n} (\Delta_n - \tau_i^n)\rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) \xrightarrow{u.c.p.} 0$$

because we have, due to Lemma 3.4 in [17], that $\sum_{i=1}^{N_t^n} E_{i-1}^n\left((\Delta_n - \tau_i^n)\rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f)\right) = 0$ and $\sum_{i=1}^{N_t^n} E_{i-1}^n\left(|(\Delta_n - \tau_i^n)\rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f)|^2\right) = \sum_{i=1}^{N_t^n}\left(\rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f)\right)^2 Var(\tau_i^n) \leq K\sum_{i=1}^{N_t^n} Var(\tau_i^n) \xrightarrow{P} 0$. Note that we also have $E_{i-1}^n(|\eta_i^n|^2) \leq K\Delta_n^2$, thus by Riemann integration, we have

$$\Sigma_{i=1}^{N_t^n} E_{i-1}^n(\eta_i^n) = \sum_{i=1}^{N_t^n} \Delta_n \rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f) \xrightarrow{u.c.p.} \sum_{i=1}^{N_t^n} \tau_i \rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f) \xrightarrow{u.c.p.} \int_0^t \rho^{\otimes k}_{\sigma_v}(f_v)dv$$

which concludes our proof. □

4 Main central limit theorem

Now, we have to obtain the CLT for the increments of Y_t. A major problem in doing so is to be able to characterize the limit, and, more specifically, the quadratic variation of the limiting process. As usual, we start with the necessary notation. Consider a sequence $(U_i)_{i\geq 1}$ of independent $\mathcal{N}(0,1)$ variables. Recall that ρ_σ, defined before, is actually the distribution law of σU_1, and so $\rho_\sigma(g) = E(g(\sigma U_1))$. Also recall that a function of k-dimensional argument $f(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ exhibits polynomial growth if $|f(x_1, \ldots, x_k)| \leq K_0 \prod_{j=1}^k (1 + |x_j|)^p$ for a positive constant K_0 and some positive p. For such a function f on $\mathbb{R}^k$ we set

$$R_\sigma(f,k) = \sum_{l=-k+1}^{k-1} E\left[f(\sigma U_k, \cdots, \sigma U_{2k-1}) f(\sigma U_{l+k}, \cdots, \sigma U_{l+2k-1})\right] - (2k-1)E^2\left[f(\sigma U_1, \cdots, \sigma U_k)\right]$$

$$= \sum_{l=-k+1}^{k-1} E\left[f(\sigma U_k, \cdots, \sigma U_{2k-1}) f(\sigma U_{l+k}, \cdots, \sigma U_{l+2k-1})\right] - (2k-1)\left[\rho_\sigma^{\otimes k}(f)\right]^2$$

Our main result is as follows.

Theorem 4.1. *Assume (H) and (T). Let f satisfy either one of the two assumptions stated below.*

- *(a) f is a polynomial function on $\mathbb{R}^k$ for some $k \geq 1$, which is globally even, that is*

$$f(-x_1, \cdots, -x_l, \cdots, -x_k) = f(x_1, \cdots, x_l, \cdots, x_k)$$

- *(b) f is a continuous and once differentiable function with all derivatives exhibiting polynomial growth on $\mathbb{R}^k$ for some $k \geq 1$, which is even in each argument, i.e.*

$$f(x_1, \cdots, -x_l, \cdots, x_k) = f(x_1, \cdots, x_l, \cdots, x_k), \quad \forall\ 1 \leq l \leq k$$

If X is continuous, then the process

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'(f,k)_t - \int_0^t \rho_{\sigma_u}^{\otimes k}(f)du\right)$$

converge stably in law to a continuous process $U'(f,k)$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the space $(\Omega, \mathcal{F}, P)$. Such a process $U'(f,k)$ is a centered Gaussian $\mathbb{R}^1$-valued process with independent increments that, conditionally on the σ-field $\mathcal{F}$, satisfies

$$\tilde{E}(U'(f,k)_t U'(f,k)_t) = \int_0^t R_{\sigma_u}(f,k)du + M\int_0^t \left[\rho_{\sigma_u}^{\otimes k}(f)\right]^2 du$$

$$= \sum_{l=-k+1}^{k-1}\int_0^t E\left[f(\sigma_u U_k, \cdots, \sigma_u U_{2k-1})f(\sigma_u U_{l+k}, \cdots, \sigma_u U_{l+2k-1})\right]du - (2k-1-M)\int_0^t \left[\rho_{\sigma_u}^{\otimes k}(f)\right]^2 du$$

$$= \int_0^t R'_{\sigma_u}(f,k)du$$

where

$$R'_{\sigma_u}(f,k) = \sum_{l=-k+1}^{k-1} E\left[f(\sigma_u U_k, \cdots, \sigma_u U_{2k-1})f(\sigma_u U_{l+k}, \cdots, \sigma_u U_{l+2k-1})\right] - (2k-1-M)\left[\rho_{\sigma_u}^{\otimes k}(f)\right]^2,$$

M is a constant defined as $M = Var(\tau_i^n)/\Delta_n^2$, and $\tilde{\mathbb{E}}$ refers to the expectation defined on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. If $S_\sigma(f,k)$ is the square root of $R'_\sigma(f,k)$, then there exists a 1-dimensional Brownian motion B on an extension of the space $(\Omega, \mathcal{F}, P)$, independent of $\mathcal{F}$, such that $U'(f,k)$ is given by

$$U'(f,k)_t = \int_0^t S_{\sigma_u}(f,k)dB_u$$

Proof:

First, we define the following convenient notation:

$$\zeta_i^n = f(\Delta_i^n X/\sqrt{\tau_i}, \cdots, \Delta_{i+k-1}^n X/\sqrt{\tau_{i+k-1}}),$$

$$\zeta_i'^n = f(\beta_{i,0}^n, \cdots, \beta_{i,k-1}^n),$$

$$\zeta_i''^n = \zeta_i^n - \zeta_i'^n$$

The basic idea of the proof is to replace each normalized increment $\Delta_{i+l}^n X/\sqrt{\tau_i}$ by $\beta_{i,l}^n$, and show that CLT is true for that simpler process, then justify this replacement by showing that the simpler process converges to the original process we are really interested in. Since the proof is rather long and technical, we separate it into a sequence of lemmas are proved separately. Then, they are combined to produce a proof of the general result.

Lemma 4.2.

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\left(\zeta_i''^n - E_{i-1}^n(\zeta_i''^n)\right) \xrightarrow{u.c.p} 0$$

Lemma 4.3.

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n\sum_{i=1}^{N_t^n}\rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f) - \int_0^t \rho^{\otimes k}_{\sigma_u}(f)du\right) \xrightarrow{s} Z_t$$

where Z_t is a Gaussian random variable $N\left(0, M\int_0^t \left[\rho^{\otimes k}_{\sigma_s}(f)\right]^2 ds\right)$, and $M = Var(\tau_i)/\Delta_n^2$

Lemma 4.4. *The processes*

$$\bar{U}_t^n = \sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\left(\zeta_i'^n - \rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f)\right)$$

converge stably in law to the process $U(f,k)$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the space $(\Omega, \mathcal{F}, P)$, which is a centered Gaussian $\mathbb{R}^1$-valued process with independent increments that, conditionally on the σ-field $\mathcal{F}$, satisfies

$$\tilde{E}(U(f,k)_t U(f,k)_t) = \int_0^t R_{\sigma_u}(f,k)du$$

Lemma 4.5.

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n(\zeta_i''^n) \xrightarrow{u.c.p} 0$$

Once we prove these four lemmas, then our Theorem (4.1) follows rather easily. As long as the limiting terms in (4.3) and (4.4) are independent (and we establish that independence as part of the proof),

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'(f,k)_t - \int_0^t \rho^{\otimes k}_{\sigma_u}(f)du\right) = \sqrt{\Delta_n}V'(f,k)_t - \frac{1}{\sqrt{\Delta_n}}\int_0^t \rho^{\otimes k}_{\sigma_u}(f)du$$

$$= \sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\zeta_i^n - \frac{1}{\sqrt{\Delta_n}}\int_0^t \rho^{\otimes k}_{\sigma_u}(f)du$$

$$= \sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\left(\zeta_i'^n + \zeta_i''^n\right) - \frac{1}{\sqrt{\Delta_n}}\int_0^t \rho^{\otimes k}_{\sigma_u}(f)du$$

$$= \sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\left(\zeta_i'^n + \zeta_i''^n\right) - \sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f)du + \sqrt{\Delta_n}\sum_{i=1}^{N_t^n}\rho^{\otimes k}_{\sigma_{t_{i-1}^n}}(f)du - \frac{1}{\sqrt{\Delta_n}}\int_0^t \rho^{\otimes k}_{\sigma_u}(f)du$$

$$= \bar{U}_t^n + \sqrt{\Delta_n} \sum_{i=1}^{N_t^n} \left(\zeta_i''^n - E_{i-1}^n(\zeta_i''^n) + E_{i-1}^n(\zeta_i''^n) \right) + \frac{1}{\sqrt{\Delta_n}} \left(\Delta_n \sum_{i=1}^{N_t^n} \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) du - \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du \right)$$

$$= \bar{U}_t^n + M_t^n + Z_t$$

where M_t^n represents all the terms in the above equation besides $\bar{U}_t^n$ and Z_t. Due to Lemmas (4.2) and (4.5), M_t^n converge to 0 uniformly in probability.

Proof of Lemma (4.2) In order to prove (4.2), we need to prove the following proposition

Proposition 4.6. *Assume (SH). Let $k \geq 1$ and let $q > 0$. Let f be a continuous function on $\mathbb{R}^k$, that exhibits polynomial growth as in* (3.2) *for some $p \geq 0$ and $K_0 \geq 0$. If we further assume that X is continuous, then as $n \to \infty$:*

$$\sup_{i \geq 0, \omega \in \Omega} E_{i-1}^n \left(\left| f \left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+k-1}^n X}{\sqrt{\tau_{i+k-1}}} \right) - f(\beta_{i,0}^n, \cdots, \beta_{i,k-1}^n) \right|^q \right) \to 0$$

The proof of (4.6) relies on Cauchy-Schwarz inequality and Lemma (3.17) in [17]. It is rather straightforward and we omit it here to make the presentation more concise. The combination of this proposition and Lemma (3.4) from [17] results in the needed conclusion.

Proof of Lemma (4.3)

As before, we prove this result under the assumption (SH). Recall that under (SH) $||\sigma_t|| \leq \Lambda$ and denote by $\mathcal{M}'$ the interval $(0, \Lambda]$. Define the function $g(\sigma) = \rho_\sigma^{\otimes k}(f)$ on the set $\mathcal{M}'$ and introduce $c_i = \frac{\tau_i}{\Delta_n}$. As a first step, define $\eta_i^n = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} \left(g(\sigma_u) - g(\sigma_{t_{i-1}}^n) \right) du$ and $\epsilon_i^n = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} g(\sigma_{t_{i-1}}^n) \left(1 - \frac{1}{c_i} \right) du$. Simple algebra suggests that

$$\frac{1}{\sqrt{\Delta_n}} \left(\Delta_n \sum_{i=1}^{N_t^n} \rho_{\sigma_{t_{i-1}^n}}^{\otimes k}(f) du - \int_0^t \rho_{\sigma_u}^{\otimes k}(f) du \right) = -\sum_{i=1}^{N_t^n} \eta_i^n - \sum_{i=1}^{N_t^n} \epsilon_i^n$$

and so it is enough to show that

$$\sum_{i=1}^{N_t^n} \eta_i^n \xrightarrow{u.c.p.} 0, \quad \sum_{i=1}^{N_t^n} \epsilon_i^n \xrightarrow{u.c.p.} 0$$

To prove that $\sum_{i=1}^{N_t^n} \eta_i^n \xrightarrow{u.c.p.} 0$ we expand it first as $\eta_i^n = \eta_i'^n + \eta_i''^n$, where

$$\eta_i'^n = \frac{1}{\sqrt{\Delta_n}} g'(\sigma_{t_{i-1}}^n) \int_{t_{i-1}}^{t_i} (\sigma_u - \sigma_{t_{i-1}}^n) du$$

$$\eta_i''^n = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} \left[g(\sigma_u) - g(\sigma_{t_{i-1}}^n) - g'(\sigma_{t_{i-1}}^n)(\sigma_u - \sigma_{t_{i-1}}^n) \right] du$$

Moreover, we further expand $\eta_i^{'n}$ as $\eta_i^{'n} = \mu_i^n + \mu_i^{'n}$ where $\mu_i^n = \frac{1}{\sqrt{\Delta_n}} g'(\sigma_{t_{i-1}}^n) \int_{t_{i-1}}^{t_i} du \int_{t_{i-1}}^{u} \tilde{b}_s ds$ and $\mu_i^{'n} = \frac{1}{\sqrt{\Delta_n}} g'(\sigma_{t_{i-1}}^n) \int_{t_{i-1}}^{t_i} du \left(\int_{t_{i-1}}^{u} \tilde{\sigma}_s dW_s + \int_{t_{i-1}}^{u} \int \tilde{\delta}(s,x)(\underline{\mu} - \underline{\nu})(ds, dx) \right)$. Because g is C_b^1 and $\tilde{b}$ is bounded, we have $|\mu_i^n| \leq \Lambda \frac{\tau_i^2}{\sqrt{\Delta_n}}$, and so $\sum_{i=1}^{N_t^n} |\mu_i^n| \leq \Lambda \frac{\sum \tau_i^2}{\sqrt{\Delta_n}}$. Based on the Assumption T, we have $E(\tau_i^2) = O(\Delta_n^2)$, and so $\sum_{i=1}^{N_t^n} E_{i-1}^n(\|\mu_i^n\|) \xrightarrow{P} 0$. Then, by Lemma (3.4) in [17], we have $\sum_{i=1}^{N_t^n} \mu_i^n \xrightarrow{u.c.p.} 0$. Using a similar argument plus Doob's and Cauchy-Schwarz inequalities, we also have $E_{i-1}^n(\mu_i^{'n}) = 0$ and $E_{i-1}^n((\mu_i^{'n})^2) \leq \Lambda E_{i-1}^n(\tau_i^2)$. Thus we have, yet again, $\sum_{i=1}^{N_t^n} \mu_i^{'n} \xrightarrow{u.c.p.} 0$ and $\sum_{i=1}^{N_t^n} \eta_i^{'n} \xrightarrow{u.c.p.} 0$.

As for $\eta_i^{''n}$, since X is continuous and f is assumed to have polynomial growth, we further know that g is C_b^2 on the compact set $\mathcal{M}$. Then by Taylor expansion, we have $|g(\sigma') - g(\sigma) - g'(\sigma)(\sigma' - \sigma)| \leq \Lambda \|\sigma' - \sigma\|^2$ for all $\sigma, \sigma' \in \mathcal{M}$. Therefore, $\eta_i^{''n} \leq \frac{K}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} |\sigma_u - \sigma_{t_{i-1}}^n|^2 du$. Due to the inequality (3.73) from [17] we have

$$E(\|\sigma_{s+t} - \sigma_t\|^q | \mathcal{F}_t) \leq K_q s^{1 \wedge (q/2)}$$

for a constant K_q that may depend on q. Therefore, for some $K > 0$, we have $E_{i-1}^n(|\eta_i^{''n}|) \leq \frac{\Lambda \cdot E(\tau_i^2)}{\sqrt{\Delta_n}} \leq K \Delta_n^{3/2}$ and $\sum_{i=1}^{[t/\Delta_n]} E_{i-1}^n(|\eta_i^{''n}|) \to 0$. Then we have shown that $\sum_{i=1}^{N_t^n} \eta_i^n \xrightarrow{u.c.p.} 0$.

Now we prove that

$$\sum_{i=1}^{N_t^n} \epsilon_i^n \xrightarrow{u.c.p.} 0$$

First of all, we have $\epsilon_i^n = \frac{1}{\sqrt{\Delta_n}} \int_{t_{i-1}}^{t_i} g(\sigma_{t_{i-1}}^n)(1 - \frac{1}{c_i}) du = \frac{1}{\sqrt{\Delta_n}} g(\sigma_{t_{i-1}}^n)(\tau_i - \Delta_n)$ and so $E(\epsilon_i^n) = 0$ because $E(\tau_i) = \Delta_n$. Moreover, by Assumption T we have the conditional variance $Var_{i-1}^n(\epsilon_i^n) = E_{i-1}^n(\|\epsilon_i^n\|^2) = g^2(\sigma_{t_{i-1}^n}) \frac{Var(\tau_i)}{\Delta_n} = M g^2(\sigma_{t_{i-1}^n}) \Delta_n$ for some $\epsilon > 0$ where $M = Var(\tau_i)/\Delta_n^2$ is a constant. Thus

$$s_n^2 = \sum_{i=1}^{N_t^n} Var_{i-1}^n(\epsilon_i^n) = M \sum_{i=1}^{N_t^n} g^2(\sigma_{t_{i-1}^n}) \Delta_n \to M \sum_{i=1}^{N_t^n} g^2(\sigma_{t_{i-1}^n}) \tau_i \to M \int_0^t g^2(\sigma_s) ds$$

The above implies that the Lindeberg condition is satisfied for $\sum_{i=1}^{N_t^n} \epsilon_i^n$, and so we have the stable convergence to Z_t:

$$\sum_{i=1}^{N_t^n} \epsilon_i^n \xrightarrow{s} Z_t$$

where Z_t is a Gaussian distributed random variable with mean zero and variance $M \int_0^t g^2(\sigma_s) ds$:

$$Z \sim N\left(0, M \int_0^t g^2(\sigma_s) ds\right)$$

Proof of Lemma (4.4)

To make this proof simpler, we only consider the case $k=2$. For the case of $k\geq 3$ no new ideas are needed but the derivations are much more involved and tedious.

Let $g_t(x)=\int\rho_{\sigma_t}(dy)f(x,y)$, we have

$$\bar{U}_t^n=\sum_{i=2}^{N_t^n+1}\eta_i^n+\gamma_1^{\prime n}-\gamma_{N_t^n+1}^{\prime n}$$

where $\eta_i^n=\gamma_i^n+\gamma_i^{\prime n}$ and $\gamma_i^n=\sqrt{\Delta_n}\left(f(\beta_{i-1,0}^n,\beta_{i-1,1}^n)-\int\rho_{\sigma_{t_{i-2}^n}}(dx)f(\beta_{i-1,0}^n,x)\right)$ and $\gamma_i^{\prime n}=\sqrt{\Delta_n}\left(\int\rho_{\sigma_{t_{i-1}^n}}(dx)f(\beta_{i,0}^n,x)-\rho_{\sigma_{t_{i-1}^n}}^{\otimes 2}(f)\right)$. As before, we use the localization procedure to establish the result we need. First, based on earlier results, we can easily show that $E_{i+l-1}^n(|\beta_{i,l}^n|^q)\leq K_q$ for some constant K_q that depends on q; this implies, in turn, that $[E(|\gamma_i^{\prime n}|)\leq K\sqrt{\Delta_n}$. For brevity, define $\bar{U}_t^{\prime n}=\sum_{i=2}^{N_t^n+1}\eta_i^n$; now, it is enough to show that $\bar{U}_t^{\prime n}$ converges stably in law to the process $U(f,2)_t$. Note that η_i^n is $\mathcal{F}_{t_i^n}$ measurable. Combining the conclusion of Theorem (3.2) and Lemma (4.3), we show that $E_{i-1}^n(\eta_i^n)=0$; moreover, due to localization and the polynomial growth of function f, it is also easy to check that $E_{i-1}^n(|\eta_i^n|^4)\leq K\Delta_n^2$, Before calculating $E_{i-1}^n((\eta_i^n)^2)$, we first list several simple facts that can be used later:

$$E_{i-1}^n(\beta_{i-1,0}^n)=\beta_{i-1,0}^n$$

$$\beta_{i-1,1}^n|\mathcal{F}_{t_{i-1}^n}\bigvee\mathcal{N}_{t_{i-1}^n}\sim N(0,\sigma_{t_{i-2}^n}^2)=\rho_{\sigma_{t_{i-2}^n}}$$

$$\beta_{i,0}^n|\mathcal{F}_{t_{i-1}^n}\bigvee\mathcal{N}_{t_{i-1}^n}\sim N(0,\sigma_{t_{i-1}^n}^2)=\rho_{\sigma_{t_{i-1}^n}}$$

Our main goal at this stage is to calculate $\sum_{i=2}^{N_t^n+1}E_{i-1}^n((\eta_i^n)^2)$ for the variance term. As a first step, expand $E_{i-1}^n\left((\eta_i^n)^2\right)=\Delta_n\phi_i^n$ where $\phi_i^n=g(t_{i-2}^n,t_{i-1}^n,\beta_{i-1,0}^n)$, and

$$\begin{aligned}g(s,t,x)=&\int\rho_{\sigma_s}(dy)f^2(x,y)-\left(\int\rho_{\sigma_s}(dy)f(x,y)\right)^2\\&+\int\rho_{\sigma_t}(dy)\left(\rho_{\sigma_t}(dz)f(y,z)\right)^2-\left(\rho_{\sigma_t}^{\otimes 2}(f)\right)^2-2\rho_{\sigma_t}^{\otimes 2}(f)\int\rho_{\sigma_s}(dy)f(x,y)\\&+2\int\rho(dy)\rho(dz)f(x,\sigma_s y)f(\sigma_t y,\sigma_t z)\end{aligned}$$

Clearly, if we can establish that

$$\sum_{i=2}^{N_t^n+1}E_{i-1}^n(\Delta_i^n N\eta_i^n)\xrightarrow{P}0 \tag{4.1}$$

for any N which is a component of W (in the 1-dimensional case the W itself) or is a bounded martingale orthogonal to W, and

$$\Delta_n\sum_{i=2}^{N_t^n+1}\phi_i^n\xrightarrow{P}\int_0^t R_{\sigma_u}(f,2)du \tag{4.2}$$

then the Lemma (3.7) from [17] will yield the stable convergence in law of $\bar{U}_t^{'n}$ to $U'(f,2)$. First, (4.1) follows from the following

Proposition 4.7. *Under (SH), for any function $(\omega, x) \mapsto g(\omega, x)$ on $\Omega \times R$ which is $\left(\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right) \otimes \mathcal{R}$-measurable and even, and with polynomial growth in x, we have*

$$E_{i-1}^n(\Delta_i^n N g(., \beta_i^n)) = 0$$

where N can be either the process W itself or any bounded martingale orthogonal to both W and $\{\tau_i\}_{i\geq 1}$.

The proposition (4.7) is a rather standard statement and its proof is omitted for brevity. To prove the (4.1), we just need to show that $E_{i-1}^n(\Delta_i^n N \gamma_i^n) = 0$ and $E_{i-1}^n(\Delta_i^n N \gamma_i^{'n}) = 0$. The part involving $\gamma_i^{'n}$ is a direct consequence of Proposition (4.7). Furthermore, while N is a martingale orthogonal to W, we can derive $E_{i-1}^n(\Delta_i^n N \gamma_i^n) = 0$ following similar arguments as in the proof of Proposition (4.7). So it only remains to prove that while N is W itself, $\sum_{i=2}^{N_t^n+1} \xi_i^n \xrightarrow{P} 0$, where $\xi_i^n = E_{i-1}^n(\gamma_i^n \Delta_i^n N) = E_{i-1}^n(\gamma_i^n \Delta_i^n W)$. Since f is globally even and ρ_s is a measure symmetric about the origin, it is not hard to see that $h(\sigma, x, y)$ is globally even in (x, y), and thus $\int \rho_\sigma(dy) h(\sigma, x, y) y$ is odd in x. Further note that $\sigma_{t_{i-2}^n} \in \mathcal{F}_{t_{i-1}^n}$ and $\Delta_{i-1}^n W \in \mathcal{F}_{t_{i-1}^n}$, then it is obvious that $\xi_i^n = E_{i-1}^n(\gamma_i^n \Delta_i^n W) = E_{i-1}^n(\sqrt{\Delta_n} h(\sigma_{t_{i-2}^n}, \Delta_{i-1}^n W/\sqrt{\tau_{i-1}}, \Delta_i^n W/\sqrt{\tau_i}) \Delta_i^n W) = 0$. Thus we finish the proof of (4.1). In order to finish the proof of Lemma (4.4) we only need to verify the property (4.2). Recall that $E_{i-1}^n\left((\eta_i^n)^2\right) = \Delta_n \phi_i^n$. We have

$$\phi_i^n = g(t_{i-2}^n, t_{i-1}^n, \beta_{i-1}^n)$$

where function $g(s, t, x)$ is as defined before. Observe that ϕ_i^n is $\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}$-measurable and

$$E_{i-2,i-1}^n(\phi_i^n) = E\left(\phi_i^n | \mathcal{F}_{t_{i-2}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right) = h(t_{i-2}^n, t_{i-1}^n), \quad E_{i-2,i-1}^n(|\phi_i^n|^2) \leq K$$

where $h(s,t) = \int \rho_{\sigma_s}(dx) g(s,t,x)$.
Then by Lemma (3.4) from [17], the property (B) would follow if we can show that

$$\Delta_n \sum_{i=1}^{N_t^n} h(t_{i-1}^n, t_i^n) \xrightarrow{P} \int_0^t R_{\sigma_u}(f, 2) du.$$

Since, due to Lemma (3.4) from [17], $\Delta_n \sum_{i=1}^{N_t^n} h(t_{i-1}^n, t_i^n) - \sum_{i=1}^{N_t^n} \tau_i h(t_{i-1}^n, t_i^n) \xrightarrow{P} 0$, we only need to verify that

$$\sum_{i=1}^{N_t^n} \tau_i h(t_{i-1}^n, t_i^n) \xrightarrow{P} \int_0^t R_{\sigma_u}(f, 2) du. \tag{4.3}$$

(4.3) follows immediately from the Taylor expansion of the function $h(t_{i-1}^n, y)$ at the point $y = t_{i-1}^n$ and the use of Riemann sum approximation.

Proof of Independence:

Before moving forward, we need to check whether or not the limiting term in Lemma (4.3) is conditionally independent of the limiting term in Lemma (4.4). If they are not independent, then a covariance term needs to be added into the variance of the limiting distribution in our CLT. For simplicity, we still work on the case $k = 2$ since this will not change the fact we want to prove. To do this, it is only necessary to check the correlation between $\sum_{i=1}^{N_t^n} \epsilon_i^n$ and $\sum_{i=2}^{N_t^n+1} \eta_i^n$, because other terms converge in probability uniformly as $n \to \infty$. In this proof, the key point is to show that τ_i^n is conditionally independent of $f(\beta_{i-1,0}^n, \beta_{i-1,1}^n)$ given $\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}$, which is true since

$$\beta_{i-1,0}^n | \mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n} = \beta_{i-1,0}^n$$

$$\beta_{i-1,1}^n | \mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n} \sim N(0, \sigma_{t_{i-2}^n}^2)$$

Thus we have

$$E\left((\tau_i^n - \Delta_n) f(\beta_{i-1,0}^n, \beta_{i-1,1}^n)\right) = E\left[E\left((\tau_i^n - \Delta_n) f(\beta_{i-1,0}^n, \beta_{i-1,1}^n) | \mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right)\right] = 0$$

Similarly we have

$$E\left((\tau_i^n - \Delta_n) \int \rho_{\sigma_{t_{i-2}^n}}(dx) f(\beta_{i-1,0}^n, x)\right) = 0$$

It is also easy to check that

$$E\left((\tau_i - \Delta_n) \int \rho_{\sigma_{t_{i-1}^n}}(dx) f(\beta_{i,0}^n, x)\right) = 0$$

Thus we have

$$E(\epsilon_i^n \eta_i^n) = E\left[\rho_{\sigma_{t_{i-1}}}^{\otimes 2}(f)(\tau_i^n - \Delta_n)\left(f(\beta_{i-1,0}^n, \beta_{i-1,1}^n) - \int \rho_{\sigma_{t_{i-2}}}(dx) f(\beta_{i-1,0}^n, x)\right)\right]$$

$$+E\left[\rho_{\sigma_{t_{i-1}}}^{\otimes 2}(f)(\tau_i^n - \Delta_n)\left(\int \rho_{\sigma_{t_{i-1}}}(dx) f(\beta_{i,0}^n, x) - \rho_{\sigma_{t_{i-1}^n}}^{\otimes 2}(f)\right)\right] = 0$$

And it is trivial to check

$$E(\epsilon_i^n \eta_j^n) = 0, \quad \text{if} \quad i > j$$

To calculate $E(\epsilon_{i-1}^n \eta_i^n)$, just conditional on $\mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}$ and notice that $\epsilon_{i-1}^n \in \mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}$, then

$$E(\epsilon_{i-1}^n \eta_i^n) = E\left[E\left(\epsilon_{i-1}^n \eta_i^n | \mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right)\right]$$

$$= E\left[\epsilon_{i-1}^n E\left(\eta_i^n | \mathcal{F}_{t_{i-1}^n} \bigvee \mathcal{N}_{t_{i-1}^n}\right)\right] = E\left[\epsilon_{i-1}^n \times 0\right] = 0$$

And similarly we have

$$E(\epsilon_i^n \eta_j^n) = 0, \quad \text{if} \quad i+1 < j$$

Thus we have shown that for any $i, j > 0$,

$$E(\epsilon_i^n \eta_j^n) = 0$$

Thus

$$E\left[\left(\sum_{i=1}^{N_t^n} \epsilon_i^n\right)\left(\sum_{i=2}^{N_t^n+1} \eta_i^n\right)\right] = 0$$

and this confirms the independence of the two limiting terms.
Proof of Lemma (4.5)

We start with defining, for $l = 0, \cdots, k-1$, the following functional sequence:

$$g_{i,l}^n(x) = \int f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+l-1}^n X}{\sqrt{\tau_{i+l-1}}}, x, x_{l+1}, \cdots, x_{k-1}\right) \rho_{\sigma_{t_{i-1}^n}}^{\otimes(k-l-1)}(dx_{l+1}, \cdots, dx_{k-1})$$

Note that, as a function of ω, $g_{i,l}^n(x)$ is $\mathcal{F}_{t_{i+l-1}^n} \bigvee \mathcal{H}_{t_{i+l-1}^n}$-measurable, while as a function of x it is once continuously differentiable. Also, for some random variable $Z_{i,l}$ that is $\mathcal{F}_{t_{i+l-2}^n} \bigvee \mathcal{H}_{t_{i+l-2}^n}$-measurable, we easily obtain that, based on the assumptions about the price process X_t and the assumption (SH), $|g_{i,l}^n(x)| + |\bigtriangledown g_{i,l}^n(x)| \leq K Z_{i,l}^n(1 + |x|^r)$ where $r \geq 0$ and $E_{i-1}^n(|Z_{i,l}^n|^p) \leq K_p \quad \forall p > 0$. For all $A \geq 1$ there is also a positive function $G_A(\epsilon)$ converging to 0 as $\epsilon \to 0$, such that with $Z_{i,l}^n$ as above:

$$|x| \leq A, Z_{i,l}^n \leq A, |y| \leq \epsilon \Longrightarrow |\bigtriangledown g_{i,l}^n(x+y) - \bigtriangledown g_{i,l}^n(x)| \leq G_A(\epsilon)$$

If we define

$$\zeta_i''^n = \sum_{l=0}^{k-1} f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+l}^n X}{\sqrt{\tau_{i+l}}}, \beta_{i,l+1}^n, \cdots, \beta_{i,k-1}^n\right) - f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}}, \cdots, \frac{\Delta_{i+l-1}^n X}{\sqrt{\tau_{i+l-1}}}, \beta_{i,l}^n, \cdots, \beta_{i,k-1}^n\right),$$

we can immediately verify that

$$E_{i-1}^n(\zeta_i''^n) = \sum_{l=0}^{k-1} E_{i-1}^n\left(g_{i,l}^n(\Delta_{i+l}^n X/\sqrt{\tau_{i+l}}) - g_{i,l}^n(\beta_{i,l}^n)\right)$$

Now, it is enough to prove that for any $l \geq 0$ we have $\sqrt{\Delta_n} \sum_{i=1}^{N_t^n} E_{i-1}^n\left(g_{i,l}^n(\Delta_{i+l}^n X/\sqrt{\tau_{i+l}}) - g_{i,l}^n(\beta_{i,l}^n)\right) \xrightarrow{u.c.p.}$ 0, or, defining $\xi_{i,l}^n = \Delta_{i+l}^n X/\sqrt{\tau_{i+l}} - \beta_{i,l}^n$,

$$\sqrt{\Delta_n} \sum_{i=1}^{N_t^n} E_{i-1}^n\left(g_{i,l}^n(\beta_{i,l}^n + \xi_{i,l}^n) - g_{i,l}^n(\beta_{i,l}^n)\right) \xrightarrow{u.c.p.} 0.$$

Using Taylor expansion, the left side of the above can be further written as

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(g_{i,l}^n(\beta_{i,l}^n+\xi_{i,l}^n)-g_{i,l}^n(\beta_{i,l}^n)\right)$$

$$=\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left[\triangledown g_{i,l}^n(\beta_{i,l}^n)\xi_{i,l}^n+\left(\triangledown g_{i,l}^n(\beta_{i,l}'^n)-\triangledown g_{i,l}^n(\beta_{i,l}^n)\right)\xi_{i,l}^n\right]$$

$$=\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(\triangledown g_{i,l}^n(\beta_{i,l}^n)\xi_{i,l}^n\right)+\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(\left(\triangledown g_{i,l}^n(\beta_{i,l}'^n)-\triangledown g_{i,l}^n(\beta_{i,l}^n)\right)\xi_{i,l}^n\right)$$

Thus, in order to prove the Lemma, we only needs to show

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(\triangledown g_{i,l}^n(\beta_{i,l}^n)\xi_{i,l}^n\right)\xrightarrow{u.c.p.}0 \tag{4.4}$$

and

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(\left(\triangledown g_{i,l}^n(\beta_{i,l}'^n)-\triangledown g_{i,l}^n(\beta_{i,l}^n)\right)\xi_{i,l}^n\right)\xrightarrow{u.c.p.}0 \tag{4.5}$$

separately. The proof of (4.5) is a straightforward application of multivariate calculus and is therefore omitted for brevity.

Proof of (4.4)

To prove (4.4), following the same scheme as in [17], we first further decompose $\xi_{i,l}^n$ into two parts as below:

$$\xi_{i,l}^n=\left(\hat{\xi}_{i,l}^n+\tilde{\xi}_{i,l}^n\right)/\sqrt{\tau_{i+l}}$$

where

$$\hat{\xi}_{i,l}^n=\int_{t_{i+l-1}^n}^{t_{i+l}^n}(b_s-b_{t_{i+l-1}^n})ds$$

$$+\int_{t_{i+l-1}^n}^{t_{i+l}^n}\left[\int_{t_{i+l-1}^n}^{s}\left(\tilde{b}_u du+(\tilde{\sigma}_u-\tilde{\sigma}_{t_{i+l-1}^n})dW_u\right)+\int_{t_{i+l-1}}^{s}\int_E\left(\tilde{\delta}(u,x)-\tilde{\delta}(t_{i+l-1}^n,x)\right)(\underline{\mu}-\underline{\nu})(du,dx)\right]dW_s$$

$$\tilde{\xi}_{i,l}^n=b_{t_{i+l-1}^n}\tau_{i+l}+\int_{t_{i+l-1}^n}^{t_{i+l}^n}\left[\tilde{\sigma}_{t_{i+l-1}^n}\int_{t_{i+l-1}^n}^{s}dW_u+\int_{t_{i+l-1}^n}^{s}\int\tilde{\delta}(t_{i+l-1},x)(\underline{\mu}-\underline{\nu})(du,dx)\right]dW_s$$

Then (4.4) amounts to the following two claims:

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left[\frac{1}{\sqrt{\tau_{i+l}}}\triangledown g_{i,l}^n(\beta_{i,l}^n)\tilde{\xi}_{i,l}^n\right]\xrightarrow{u.c.p.}0 \tag{4.6}$$

and

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left[\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown g_{i,l}^n(\beta_{i,l}^n)\hat{\xi}_{i,l}^n\right] \xrightarrow{u.c.p.} 0. \tag{4.7}$$

Proof of (4.6)

Note that the restriction of $\underline{\mu}$ to $(t_{i+l-1},\infty)\times E$ and the increments of W after time t_{i+l-1} are independent, then conditional on $\mathcal{M}_{t^n_{i+l-1}} = \mathcal{F}_{t^n_{i+l-1}} \bigvee \sigma(W_t : t \geq 0) \bigvee \sigma(\tau_i : i \geq 0)$, we get

$$E(\tilde{\xi}_{i,l}^n|\mathcal{M}_{t_{i+l-1}}) = b_{t_{i+l-1}}\tau_{i+l} + \tilde{\sigma}_{t^n_{i+l-1}}\int_{t_{i+l-1}}^{t_{i+l}}\left(\int_{t^n_{i+l-1}}^{s} dW_u\right)dW_s$$

which is even in W. Thus for a function h which is odd with polynomial growth, we deduce

$$E_{i+l-1}^n(\tilde{\xi}_{i,l}^n h(\beta_{i,l}^n)) = 0$$

At this point, it is easy to see that, if function f is even in each argument, then $g_{i,l}^n(\beta_{i,l}^n)$ is even and $\bigtriangledown g_{i,l}^n(\beta_{i,l}^n)$ is odd. This implies immediately that (4.6) is true and we are only left with proving (4.6) in case of the globally even polynomial function f. To achieve that, first, define

$$h\left(\Delta_{i,l}X, x\right) = g_{i,l}^n(x) = \int f\left(\frac{\Delta_i^n X}{\sqrt{\tau_i}},\cdots,\frac{\Delta_{i+l-1}^n X}{\sqrt{\tau_{i+l-1}}}, x, x_{l+1},\cdots,x_{k-1}\right)\rho_{\sigma_{t^n_{i-1}}}^{\otimes(k-l-1)}(dx_{l+1},\cdots,dx_{k-1})$$

where $\Delta_{i,l}X = \left(\frac{\Delta_i^n X}{\sqrt{\tau_i}},\cdots,\frac{\Delta_{i+l-1}^n X}{\sqrt{\tau_{i+l-1}}}\right)$

Clearly, the function h is globally even in $(\Delta_{i,l}X, x)$ since f is globally even and the Gaussian law is symmetric. Since f is a continuous function with at most polynomial growth, we can expand function h as $h(\Delta_{i,l}X, x) = a(\Delta_{i,l}X) + b_{i,l}(x) + c(\Delta_{i,l}X, x)$ where function a only contains constant and terms with no x involved, $b_{i,l}$ only contains terms with x (but not any part of $\Delta_{i,l}X$) involved, and function c contains the rest, i.e. those terms with both x and part of $\Delta_{i,l}X$ involved. Denote the partial differential w.r.t x $\bigtriangledown_x$, then obviously we have $\bigtriangledown_x a(\Delta_{i,l}X) = 0$. Since h is globally even, the terms that only contain x must be even in x, i.e. $b_{i,l}(x)$ is even is x. Thus $\bigtriangledown_x b_{i,l}(x)$ is odd in x and we have $E_{i-1}^n\left(\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown_x b_{i,l}(x)\tilde{\xi}_{i,l}^n\right) = 0$ while $x = \beta_{i,l}^n$ from the arguments above. Since f is a polynomial function, we can write function c as $c(\Delta_{i,l}X, x) = \sum_{j=0}^{l-1}\left(\frac{\Delta_{i+j}^n X}{\sqrt{\tau_{i+j}}}\right)^{p_j} x^{q_j}$. Since function c should still be globally even in $(\Delta_{i,l}X, x)$ (because function h is globally even), for any j, $p_j + q_j$ must be an even number. Thus $\bigtriangledown_x c(\Delta_{i,l}X, x)$ is globally odd in $(\Delta_{i,l}X, x)$. Let τ represent the vector $(\tau_i,\cdots,\tau_{i+l-1})$ and dW represent any terms that contain an integral w.r.t the Brownian motion. Now we treat function $\bigtriangledown_x c(\Delta_{i,l}X, \beta_{i,l}^n)$ as a function of dW and τ_{i+j} so that $\bigtriangledown_x c(\Delta_{i,l}X, \beta_{i,l}^n) = c_1(\tau, dW) + c_2(dW)$ where c_1 is the term that depends on both τ and dW while c_2 depends on dW onlye. Recall that

$$E(\tilde{\xi}_{i,l}^n|\mathcal{M}_{t_{i+l-1}}) = b_{t_{i+l-1}}\tau_{i+l} + \tilde{\sigma}_{t^n_{i+l-1}}\int_{t_{i+l-1}}^{t_{i+l}}\left(\int_{t^n_{i+l-1}}^{s} dW_u\right)dW_s$$

and so

$$E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown_x c(\Delta_{i,l}X,\beta^n_{i,l})\tilde{\xi}^n_{i,l}\right) = E^n_{i-1}\left(\left(\frac{1}{\sqrt{\tau_{i+l}}}c_1(\tau,dW)+\frac{1}{\sqrt{\tau_{i+l}}}c_2(dW)\right)\tilde{\xi}^n_{i,l}\right)$$

Since function $\bigtriangledown_x c(\Delta_{i,l}X,\beta^n_{i,l})$ is globally odd, then it is easy to check that c_2 is of odd power of dW and $E^n_{i-1}\left(c_2(dW)\tilde{\xi}^n_{i,l}\right) = 0$. As for the term $E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}}c_1(\tau,dW)\tilde{\xi}^n_{i,l}\right)$, after simple calculations it is easy to check that those terms are all, at least, of order $O(n^{-\frac{3}{2}})$. (since $E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}}\tau^2_{i+j}\right) = O(n^{-\frac{3}{2}})$).
Thus we still have

$$\sqrt{\Delta_n}\sum_{i=1}^{N^n_t} E^n_{i-1}\left(\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown_x c(\Delta_{i,l}X,\beta^n_{i,l})\tilde{\xi}^n_{i,l}\right) \xrightarrow{u.c.p.} 0$$

Finally recall that

$$g^n_{i,l}(x) = h(\Delta_{i,l}X,x) = a(\Delta_{i,l}X,x) + b_{i,l}(x) + c(\Delta_{i,l}X,x)$$

and so (4.6) has been established by showing that it converges to zero for each of the functions above in the decomposition.
Proof of (4.7)

To make our notation more convenient, denote $E\left(\cdot|\mathcal{F}_{i+l-1}\vee\mathcal{N}_T\right) = E^*_{i+l-1}$. It is convenient to split the statement of (4.7) into two separate Lemmas that, taken together, suffice to establish it. We state these Lemmas first.

Lemma A2A:
Assuming (SH), we have

$$E^*_{i+l-1}\left(|\hat{\xi}^n_{i,l}|^2\right) \le K\tau_{i+l}\left(\tau^2_{i+l}+\alpha^n_{i,l}\right)$$

where

$$\alpha^n_{i,l} = E^*_{i+l-1}\left(\int_{t^n_{i+l-1}}^{t^n_{i+l}}\left(|b_s - b_{t_{i+l-1}}|^2 + |\tilde{\sigma}_s - \tilde{\sigma}_{t^n_{i+l-1}}|^2 + \int|\tilde{\delta}(s,x)-\tilde{\delta}(t^n_{i+l-1},x)|^2\lambda(dx)\right)ds\right)$$

Lemma A2B: Under the same assumptions as the previous Lemma,

$$\sqrt{\Delta_n}\sum_{i=1}^{N^n_t}\sqrt{E(\alpha^n_{i,l})} \to 0$$

Both Lemmas have rather simple proofs. The proof of Lemma A2B is rather elementary and we omit it altogether. The proof of Lemma A2A relies extensively on the use of Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities and we omit it as well in the interest of brevity. Now we can finally prove the statement (4.7). Combining Lemma A2A and A2B, we can show that

$$\sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left[\frac{1}{\sqrt{\tau_{i+l}}}\bigtriangledown g_{i,l}^n(\beta_{i,l}^n)\hat{\xi}_{i,l}^n\right] \leq \sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(\frac{K}{\sqrt{\tau_{i+l}}}Z_{i,l}^n(1+|\beta_{i,l}^n|^r)|\hat{\xi}_{i,l}^n|\right)$$

$$\leq \sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(\frac{KZ_{i,l}^n}{\sqrt{\tau_{i+l}}}\sqrt{E_{i+l-1}^n(1+|\beta_{i,l}^n|^r)^2}\cdot\sqrt{E_{i+l-1}^*|\hat{\xi}_{i,l}^n|^2}\right)$$

$$\leq \sqrt{\Delta_n}\sum_{i=1}^{N_t^n} E_{i-1}^n\left(KZ_{i,l}^n(\tau_{i+l}+\sqrt{\alpha_{i,l}^n})\right)$$

$$\leq \sqrt{\Delta_n}\sum_{i=1}^{N_t^n} K\left(\Delta_n+\sqrt{E_{i-1}^n(\alpha_{i,l}^n)}\right)\xrightarrow{u.c.p.} 0$$

Thus we finished the proof of (4.7). Combining this with the proof of (4.6) and (4.5), the result of Lemma (4.5) is immediately obtained.

Note that in Theorem (4.1) the function f is a 1-dimensional function on $\mathbb{R}^k$. However, it is easy to check that the CLT should still be true even when f is a q-dimensional function on $\mathbb{R}^k$ as long as every assumption in Theorem (4.1) still holds true. Such a version may be more useful in many applications since it offers us more flexibility when constructing function f. We will state such a q-dimensional version as a Corollary here. Since its proof is almost the same as that of (4.1) but with an added layer of technical complexity, it will be omitted here.

Corollary 4.8. *Assume (H) and (T). Let $f=(f_1,\cdots,f_q)$ be a q-dimensional function on $\mathbb{R}^k$ satisfying any one of the two cases below*

- *(a) a polynomial function which is globally even, that is*

$$f(-x_1,\cdots,-x_l,\cdots,-x_k)=f(x_1,\cdots,x_l,\cdots,x_k)$$

- *(b) a C^1 function with derivatives having polynomial growth on R^k, which is even in each argument, i.e.*

$$f(x_1,\cdots,-x_l,\cdots,x_k)=f(x_1,\cdots,x_l,\cdots,x_k),\quad \forall\ 1\leq l\leq k$$

If X is continuous, then the process

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'(f,k)_t-\int_0^t \rho_{\sigma_u}^{\otimes k}(f)du\right)$$

converge stably in law to a continuous process $U'(f,k)$ defined on an extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ of the space $(\Omega,\mathcal{F},P)$, which conditionally on the σ-field $\mathcal{F}$ is a centered Gaussian $\mathbb{R}^q$-valued process with independent increments, satisfying

$$\tilde{E}(U'(f_i,k)_tU'(f_j,k)_t)=\int_0^t R_{\sigma_u}^{i,j}(f,k)du$$

where for $i, j = 1, \cdots, q$ we set:

$$R_{\sigma}^{i,j}(f,k) = \sum_{l=-k+1}^{k-1} E\left(f_i(\sigma U_k, \cdots, \sigma U_{2k-1}) f_j(\sigma U_{l+k}, \cdots, \sigma U_{l+2k-1})\right)$$
$$-(2k-1-M) E\left(f_i(\sigma U_1, \cdots, \sigma U_k)\right) E\left(f_j(\sigma U_1, \cdots, \sigma U_k)\right)$$

5 Discussion

In this paper, we obtain some general asymptotic results for normalized functionals of increments of a continuous semimartingale process under a broad ranging random sampling scheme. In our approach, the random duration times τ_i between the two successive trading times t_{i-1} and t_i are not specified down to a specific distribution but. Rather, we only impose a general restriction on how the largest and smallest duration time behaves in large samples; this assumption implies, in turn, the rate at which both the expected value and the variance of a duration time goes to zero as the sample size $n \to \infty$. Such a broad random discretization scheme includes, as a special case, the classical Poisson arrival scheme. Through delicate treatment of the functionals of the increments of the stochastic process for asset returns and duration times, we proved some important asymptotic results for the new estimator including the law of large numbers and the central limit theorem. This work builds the theoretical foundation for statistical estimation and inference on continuous semimartingales under wide ranging selection of random discretization schemes.

There is a number of possible extensions that could be considered as part of the future research. As an example, in this paper it is assumed that the stochastic trading times t_i are independent of the log price process Y_t. This is somewhat restrictive from the application viewpoint; thus, another step ahead would be to obtain a similar law of large numbers and the central limit theorem under a reasonable dependence assumption between the two. Another interesting extension that could be considered is the possibility of dependence between duration times in our random discretization scheme.

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