Random variables from an “unusual” distribution

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Technical Report #12-01

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May 2012

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There is a distribution which arises in connection with generating the logarithm of stable random variables, which seems quite unusual. We give here a probabilistic interpretation of this distribution, which, however, I am not sure is optimal for generating random variables from it. For one thing, it uses two random variables. For another, even after this, it still uses a transcendental function all the time. However, it is still useful to illustrate what can sometimes be done.

This distribution on the positive real line has cdf $\coth(x) - x\operatorname{csch}(x)^2$ and density $2\operatorname{csch}(x)^2(x\coth(x) - 1)$.

Let us modify this last expression to be

$$2\operatorname{csch}(x)^2\coth(x)(x - \tanh(x),$$

and observe that $x - \tanh(x) = \int_0^x \tanh(t)^2 dt$.

This can be represented as a bivariate distribution with density

$$2\operatorname{csch}(x)^2\coth(x)\tanh(t)^2$$

on the part of the first quadrant with $0 < t < x$. Now $2\operatorname{csch}(x)^2\coth(x)$ is the derivative of $-\operatorname{csch}(x)^2$, so the marginal density of T is

$$\operatorname{csch}(t)^2\tanh(t)^2 = \operatorname{sech}(t)^2,$$

from which we see that the cdf of T is $\tanh(t)$, and the tail cdf of X given $T = t$ is $\operatorname{csch}(x)^2/\operatorname{csch}(t)^2$. Thus we can take T to be the arctanh of a uniform random variable U and $\sinh(X)^2 = \sinh(T)^2/V$, where V is a uniform random variable independent of U.

However, $\sinh(z)^2 = \tanh(z)^2/(1 - \tanh(z)^2)$, and we are interested in X, so finally

$$\sinh(X) = \sqrt{\frac{U^2}{V(1-U^2)}}.$$

Letting A be the argument of the radical, an easy calculation gives

$$X = \ln(\sqrt{A} + \sqrt{1+A}),$$

or if preferred,

$$X = \ln(1 + 2A + 2\sqrt{A(1+A)})/2.$$