Skorohod Integration and Stochastic Calculus Beyond the Fractional Brownian Scale

by

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Abstract

We extend the Skorohod integral with respect to Gaussian processes that can be more irregular than any fractional Brownian motion. This is done by restricting the class of test random variables that are used to define Skorohod integrability. A detailed analysis of the resulting size of the space of test random variables is given, and it is determined that this space is non-empty even for some Gaussian processes which are not continuous on any closed interval. Despite the extreme irregularity of these stochastic integrators, the Skorohod integral is shown to be useful: an Ito formula is established, it is used to derive a Tanaka formula for a corresponding local time, linear additive and multiplicative stochastic differential equations are solved, and an analysis of existence and regularity of the stochastic heat equation is given.

1 Introduction

The purpose of this paper is to establish a stochastic calculus for processes that may have longer-range negative interaction than even fractional Brownian motions (fBm) with small Hurst parameters. This general class of stochastic processes will encompass standard Brownian motion, (fBm), with Hurst parameter less than 1/2, and more generally any class of Gaussian stochastic processes defined by any given scale of almost-sure uniform continuity that is bounded below by the modulus of continuity of Brownian motion. For processes that are more regular than Brownian motion, a different construction could be used, which we do not discuss here.

The topic of stochastic calculus, which was originated more than sixty years ago, with legendary associated names such as Levy, Ito, Stratonovich, saw a renewed interest in the late eighties, when, for example, a study of stochastic integration of non-adapted processes with respect to Brownian motion first appeared in [14] in the context of two-sided integration, and subsequently in [13] a general theory of anticipating stochastic integration was developed using the connection to Skorohod integrals. Our work inscribes itself in this context, which is now known as the Malliavin calculus (see for example Nualart's book [12] on the topic).

The most recent trend in Malliavin calculus has been in the study of the so-called *fractional Brownian motion* (abbreviated as fBm, see next section for a definition), anticipating stochastic integration being particularly well-suited for the study of this process whose increments are not independent, but can still be represented using standard Brownian motion. The theory of stochastic calculus for fBm is becoming relatively solid and mature, with main results such as Ito formulas which can be found for example in [1] and

[2]. Other approaches to stochastic integration w.r.t. fBm include the so-called Russo-Vallois integral, with recent stochastic calculus results in [11] and **??**. However both approaches have had difficulties in establishing the so-called Ito formula (the chain rule for non-random functions of fBm), which is the cornerstone of the stochastic calculus, particularly when the fBm's regularity, as measured by its *Hurst* parameter H, is in the range $H \in (0; 1/4)$. The Russo-Vallois integral appears to have a limit of $H > 1/6$, according to the results of **??**, a result which is confirmed in [5], despite some intriguing results for a special version of the Russo-Vallois integral in **??** in which no restriction on H is needed. On the other hand, Cheridito and Nualart propose in [5] a new, relaxed way of defining Skorohod integration which results in an Ito formula with no restriction on H. The idea is to restrict the space of test random variable needed to define the notion of Skorohod integrability. It is this idea which we adopt here, because of its success in establishing an unrestricted Ito formula.

We begin our study in Section 2 by proposing a basic and wide class of Gaussian processes which contains processes very close to fBm, as well as processes which may be much more irregular than fBm, including processes that are neither uniformly continuous nor bounded. Section 3 shows briefly how to define Wiener integration w.r.t. our processes. Before establishing the Malliavin calculus, Skorohod integral, and Ito formula for our processes in Section 5, we take some time and effort in Section 4 to evaluate the precise size of the spaces in which our test random variables will live; this is particularly important since there is no *a priori* guarantee that the spaces will not be empty (except for non-random variables), which would result in a failed definition of the Skorohod integral. As applications of Ito's formula (Theorem 27), we establish a Tanaka formula, and discuss related issues, for the local time of our processes in Section 6. Then we solve some simple finite and infinite-dimensional stochastic differential equations in Section 7. The last two sections of this article are meant as an illustration of our theory of Skorohod integration: we do not seek the most general results that may be readily available at this stage, and we discuss a certain number of open problems that our new class of processes raises, in the hope of encouraging further research on the topic.

2 Gaussian noise with arbitrary correlation

2.1 Definition

We begin by considering a class of Gaussian processes that may have arbitrary correlation between increments. First recall the scale of fBm is defined as the class of centered Gaussian processes B^H on $\mathbf{R}_+$ such that $B^H(0) = 0$ and

$$E|B^H(t) - B^H(s)|^2 = |t-s|^{2H}. \tag{1}$$

It is natural to generalize this class as follows. Let γ be a continuous increasing function on $\mathbf{R}_+$ or possibly only on a neighborhood of 0 in $\mathbf{R}_+$, such that $\lim_0 \gamma = 0$. The most naive idea is to attempt to define B^γ to be the Gaussian process such that

$$E|B^\gamma(t) - B^\gamma(s)|^2 = \gamma^2(|t-s|),$$
$$B^\gamma(0) = 0.$$

However, the corresponding process may not exist because, depending on the exact form of γ, its covariance function may not be of positive type (symmetric and non-negative definite). Therefore, for any fixed γ as above, we will be satisfied with finding a Gaussian process B such that the following hold:

(i) defining the canonical metric δ of B on $(\mathbf{R}_+)^2$ by

$$\delta^2(s,t) := E|B(t) - B(s)|^2$$

and denoting that two functions f and g are *commensurable* ($f \asymp g$) if there exist positive constants c, C such that $cg(x) \leq f(x) \leq Cg(x)$ for all values of a common variable x, we have

$$\delta(s,t) \asymp \gamma(|t-s|);$$

(ii) $B(0) = 0$.

If $\gamma(r) \gg r^H$ for all $H > 0$ then neighboring increments of B are negatively correlated, and the range of correlation is longer than for any fBm. Consider for example the following choice of γ:

$$\gamma^2(r) \asymp \frac{1}{\log \frac{1}{r}}. \tag{2}$$

It is worth noting that the corresponding process B is not almost-surely uniformly continuous. We will analyze this and other examples in more detail below. Stochastic integration with respect to the increments of such a process B can be achieved by means of the Malliavin calculus, as we will see below. The first step, however, is to understand the Wiener integral with respect to B. While our ultimate goal is to define a stochastic calculus with respect to B, much work can be achieved using only the Wiener integral, including linear additive stochastic evolution equations (see Section 7.2). It is most comforting to assume that B can also be taken to satisfy the following:

(iii) B is adapted to a Brownian filtration.

Proposition 1 *Let W be a standard Brownian motion on $\mathbf{R}_+$ with respect to the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Assume γ^2 is of class C^2 everywhere except at 0 and that $d\gamma^2/dr$ is non-increasing. The following centered Gaussian process satisfies conditions (i), (ii) and (iii) with respect to $\{\mathcal{F}_t\}_{t\geq 0}$: for any $t \geq 0$,*

$$B(t) = B^\gamma(t) := \int_0^t \varepsilon(t-s)\, dW(s) \tag{3}$$

where

$$\varepsilon(r) := \left(\frac{d(\gamma^2)}{dr}\right)^{1/2}.$$

In fact the constants c and C in (i) can be taken as 1 and $\sqrt{2}$ respectively.

Remark 2 *The C^2 assumption on γ is not restrictive in terms of the magnitude of the almost-sure modulus of continuity of B, and can be achieved without loss of generality given the possibility of multiplying γ by a factor that is bounded above and away from zero.*

Remark 3 *The modulus of continuity γ for a non-Lipshitz function $\gamma(r)$ satisfies $\lim_{r\to 0}(d\gamma/dr) = +\infty$. Thus, we can assume, again without loss of generality, that $d\gamma/dr$ is decreasing. Moreover, since we are aiming to study processes that are not more regular than Brownian motion (for which $\gamma^2(r) = r$), we can assume without loss of generality that $d\gamma^2/dr$ is non-increasing.*

Proof. (of the Proposition). Assume $t > s$. Then

$$\begin{aligned} E\left(B^\gamma(t) - B^\gamma(s)\right)^2 &= E\left(\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]\, dW(r) + \int_s^t \varepsilon(t-r)\, dW(r)\right)^2 \\ &= \int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2\, dr + \int_s^t dr \varepsilon^2(t-r)\, dr \\ &= \int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2\, dr + \gamma^2(t-s). \end{aligned} \tag{4}$$

because $\int \varepsilon^2 = \gamma^2$. Thus it is sufficient to show that

$$\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr \leq \gamma^2 (t-s) .$$

We calculate

$$\begin{aligned}
&\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr \\
&= \gamma^2(t) - \gamma^2(t-s) + \gamma^2(s) - 2\int_0^s \sqrt{\varepsilon^2(t-r)\,\varepsilon^2(s-r)} dr.
\end{aligned}$$

We assumed that ε^2 is non-increasing, so that

$$\begin{aligned}
&\int_0^s [\varepsilon(t-r) - \varepsilon(s-r)]^2 \, dr \\
&\leq \gamma^2(t) - \gamma^2(t-s) + \gamma^2(s) - 2\int_0^s \varepsilon^2(t-r)\, dr \\
&= \gamma^2(t) - \gamma^2(t-s) + \gamma^2(s) - 2\left(\gamma^2(t) - \gamma^2(t-s)\right) \\
&= \gamma^2(t-s) - \left(\gamma^2(t) - \gamma^2(s)\right) \\
&\leq \gamma^2(t-s) ,
\end{aligned}$$

which finishes the proof. ■

2.2 Relation to fractional Brownian motion

The fractional Brownian motion B^H, which is defined by $B^H(0) = 0$ and the formula (1), is a process that also satisfies properties (i), (ii), and (iii). However, it does not quite satisfy a representation formula (3). What we have shown above is that by letting $\gamma(r) = \gamma_H(r) = r^H$, we have constructed a process B^{γ_H} whose covariance structure is commensurate with that of fBm, which is to say that it shares the same regularity properties as fBm, and several other crucial properties ((i), (ii), (iii)); however, B^{γ_H} is arguably easier to work with in terms of stochastic calculus than fBm. This is an indication that for any other fixed γ, B^γ is a good choice for a process with covariance structure satisfying (i). A reader who is familiar with the technicalities inherent in the use of fractional integrals and derivatives needed to establish stochastic calculus for standard fBm, may appreciate the ease with which we establish the Ito formula in this article.

Our process B^{γ_H} shares another important property with standard fBm: that of *self-similarity*, a.k.a. the power scaling property. We recall the relevant concept before stating the result.

Definition 4 *A stochastic process X defined on $\mathbf{R}_+$ is said to be self-similar with parameter H if for any $a > 0$, the law of $\{X(at) : t \in \mathbf{R}_+\}$ and the law of $\{a^H X(t) : t \in \mathbf{R}_+\}$ are identical.*

Proposition 5 *B^{γ_H} is self-similar with parameter H.*

Proof. By construction, B^{γ_H} is a separable Gaussian process. Therefore, we only need to check the self-similarity property on the first two moments of the finite-dimensional distributions of B^{γ_H}. In other words, we need only to check that if f is a polynomial of degree 2 on $(\mathbf{R}_+)^m$ and $t_1, \cdots, t_m$ are fixed times, we have

$$\mathbf{E}\left[f\left(B^{\gamma_H}(at_1), \cdots, B^{\gamma_H}(at_m)\right)\right] = \mathbf{E}\left[f\left(a^H B^{\gamma_H}(t_1), \cdots, a^H B^{\gamma_H}(t_m)\right)\right].$$

Since B^{γ_H} is centered, it is thus sufficient to check this equality for monomials of degree 2. Equivalently, we can calculate, for $0 \leq s < t$ fixed, the following two second moments:

$$\sigma_t^2 := \mathbf{E}\left[B^{\gamma_H}(t)^2\right] \quad and \quad \delta(s,t)^2 := \mathbf{E}\left[\left(B^{\gamma_H}(t) - B^{\gamma_H}(s)\right)^2\right].$$

We get by definition of ε that $\sigma_{at}^2 = \int_0^{at} \varepsilon^2 (t-s)\, ds = a^{2H} s^{2H} = a^{2H}\sigma_t^2$, which is the self-similarity property for this moment, while for the other term, the calculation in the proof of the previous proposition yields immediately from (4):

$$
\begin{aligned}
\delta (as, at)^2 &= \int_0^{as} \left[\varepsilon (at-r) - \varepsilon (as-r)\right]^2 dr + \gamma^2 (at-as) \\
&= \int_0^{s} 2H \left[(a(t-r'))^{2H-1} - (a(s-r'))^{2H-1}\right]^2 a dr' + (a(t-s))^{2H} \\
&= a^{2H} \left[\int_0^{s} 2H \left[(t-r')^{2H-1} - (s-r')^{2H-1}\right]^2 dr' + (t-s)^{2H}\right] \\
&= a^{2H} \delta (s,t)^2 .
\end{aligned}
$$

This finishes the proof of the proposition. ■

With so many shared properties between standard fBm and our process B^{γ_H}, one may ask what difference there is between the two processes. It is well-known (see for example [15]) that fBm is the only stochastic process with finite variance that is self-similar with parameter H *and* has stationary increments. Therefore B^{γ_H} cannot have stationary increments. Of course, we can easily calculate by how much our increments fail to be stationary. We record this in the following

Remark 6 *Standard fBm B^H has stationary increments in the sense that for all $s,t,h \in \mathbf{R}_+$,*

$$
\mathbf{E}\left[\left(B^H(t+h) - B^H(t)\right)^2\right] = \mathbf{E}\left[\left(B^H(s+h) - B^H(s)\right)^2\right].
$$

However, the proof of Proposition 1 (line (4)) shows that

$$
\begin{aligned}
\frac{1}{2}\mathbf{E}\left[\left(B^{\gamma_H}(s+h) - B^{\gamma_H}(s)\right)^2\right] &\leq \mathbf{E}\left[\left(B^{\gamma_H}(t+h) - B^{\gamma_H}(t)\right)^2\right] \\
&\leq 2\mathbf{E}\left[\left(B^{\gamma_H}(s+h) - B^{\gamma_H}(s)\right)^2\right].
\end{aligned}
$$

In other words B^{γ_H} only fails to have stationary increments by factors no greater than 2.

3 Wiener integral with respect to B^γ

Let $(B^\gamma(t))_{t\in[0,T]}$ be the centered Gaussian process defined by its Wiener integral representation as in (3). We can formally take the differential of (3), to get

$$
dB^\gamma(t) = dt \int_0^t \varepsilon'(t-s)\, dW(s) + \varepsilon(t-t)\, dW(t).
$$

However, we are well aware of the fact that $\varepsilon(0) = +\infty$. Thus we formally perform the following transformation to properly define the Wiener integral: for a deterministic function f

$$
\begin{aligned}
&\int_0^t f(t)\, dB^\gamma(t) \\
&= \int_0^t ds f(s) \int_0^s \varepsilon'(s-r)\, dW(r) + \int_0^t f(s)\, \varepsilon(s-s)\, dW(s) \\
&= \int_0^t ds \int_0^s (f(s) - f(r))\, \varepsilon'(s-r)\, dW(r) \\
&+ \int_0^t ds \int_0^s f(r)\, \varepsilon'(s-r)\, dW(r) + \int_0^t f(s)\, \varepsilon(s-s)\, dW(s)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t ds \int_0^s (f(s) - f(r))\, \varepsilon'(s-r)\, dW(r) \\
&+ \int_0^t f(r)\, dW(r)\, (\varepsilon(t-r) - \varepsilon(r-r)) + \int_0^t f(s)\, \varepsilon(s-s)\, dW(s) \\
&= \int_0^t ds \int_0^s (f(s) - f(r))\, \varepsilon'(s-r)\, dW(r) + \int_0^t f(r)\, dW(r)\, \varepsilon(t-r).
\end{aligned}
$$

This justifies the following definition of the Wiener integral with respect to B^γ.

Definition 7 *Let B^γ be defined as in (3). Let f be a deterministic measurable function on $\mathbf{R}_+$. We define the operator $K^* = K^*_\gamma$ on f by*

$$
K^*_\gamma f(r) := \left[f(r)\, \varepsilon(T-r) + \int_r^T (f(s) - f(r))\, \varepsilon'(s-r)\, ds \right].
$$

If $K^*_\gamma f(\cdot)$ is in $L^2(dr)$ then we say that f belongs to the space $L^2_\gamma([0,T])$, and we denote

$$
\|f\|^2_\gamma = \left\|K^*_\gamma f\right\|^2_{L^2([0,T])} = \int_0^T \left| f(r)\, \varepsilon(T-r) + \int_r^t (f(s) - f(r))\, \varepsilon'(s-r)\, ds \right|^2 dr.
$$

This L^2_γ is the so-called canonical *Hilbert space of B^γ on $[0,T]$. For any f in $L^2_\gamma([0,T])$ we define the stochastic integral of f with respect to B^γ on $[0,T]$ by*

$$
\int_0^T f(t)\, dB^\gamma(t) = \int_0^T dW(r)\, K_\gamma f(r).
$$

4 The canonical spaces $\mathcal{H}$ and $\mathcal{H}_2$

The above notion of Wiener integral, apart from being defined only for non-random integrands, has the additional uninviting property that it is only defined for members of L^2_γ; we will see shortly that this canonical space is uncomfortably small, since it may not even contain $C^{1/2}$ when γ is very irregular. This implies that there should be no hope of defining stochastic integrals of standard processes such as semimartingales, or even standard Brownian motion, with respect to B^γ. However, when integrating a random function, the notion of Skorohod integral can be extended to include such processes, and many more, including B^γ itself. We therefore abandon the notion of Wiener integral for the time being, and introduce special spaces of test functions that will be crucial for our purposes.

Definition 8 *Let us fix the time interval $[0,T]$, with $T < 1$. We define the operator $K^{*,a}_\gamma$ to be the adjoint of the operator K^*_γ in $L^2([0,T])$. We denote by $\mathcal{H}$ the set*

$$
\mathcal{H} = (K^*_\gamma)^{-1}(L^2([0,T]))
$$

and by $\mathcal{H}_2$ the set:

$$
\mathcal{H}_2 = (K^*_\gamma)^{-1}((K^{*,a}_\gamma)^{-1}(L^2([0,T]))).
$$

We note that $\mathcal{H} = L^2_\gamma$, which was introduced previously as the domain of the Wiener integral.

Remark 9 *The space $\mathcal{H}$ endowed with the inner product*

$$
< f, g >_{\mathcal{H}} = < K^*_\gamma f, K^*_\gamma g >_{L^2([0,T])}
$$

is a Hilbert space.

Remark 10 *Observe that if we denote by* $\mathcal{H}' = \{f \in L^2([0,T]), K_\gamma^{*,a} f \in L^2([0,T])$ *then by definition*

$$< K_\gamma^{a,*} f, g >_{L^2([0,T])} = < f, K_\gamma^* g >_{L^2([0,T])},$$

for every $f \in \mathcal{H}'$ *and* $g \in \mathcal{H}$.

Next we study the richness of the spaces $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}_2$ by showing that these spaces contain sets of functions with specific moduli of continuity.

Definition 11 *Let* η *be a continuous increasing function on a neighborhood of* 0 *in* $\mathbf{R}_+$, *with* $\lim_{0+} \eta = 0$. *The space* C^η *is defined as the space of all functions defined on* $[0,T]$ *that admit* η *as a uniform modulus of continuity:*

$$C^\eta = \left\{ f \in L^2[0,T] : \sup_{0 \le r < s \le T} |f(s) - f(r)| / \eta(s-r) < \infty \right\}.$$

Proposition 12 *If* η *satisfies*

$$\int_0 \eta(s) |\varepsilon'(s)| \, ds < \infty \qquad ((\mathrm{C}_\eta))$$

then $\mathcal{H}$ *contains* C^η. *Moreover condition* (C_η) *is equivalent to the following:*

($\mathbf{D}_h$) *There exists a positive function* h *defined and decreasing on a neighborhood of* 0 *in* $\mathbf{R}_+ - \{0\}$ *such that* $\int_0 h < \infty$ *and for small* $r > 0$

$$\eta(r) = \int_0^r \frac{h(s)}{\varepsilon(s)} ds. \qquad (5)$$

Proof. For the first statement, for fixed η, we only need to show that if $f \in C^\eta$, then $K_\gamma^* f \in L^2([0,T])$. We treat the two terms in the sum defining $K_\gamma^* f$ separately. First observe that if $f \in C^\eta$ then f is bounded, so that by definition of γ,

$$\int_0^T f^2(r)\varepsilon^2(T-r)dr \le (\|f\|_\infty)^2 \int_0^T \varepsilon^2(T-r)dr = (\|f\|_\infty)^2 \gamma^2(T) < \infty.$$

For the $L^2[0,T]$-norm of the second term in $K_\gamma^* f$, since there exists a constant C_f such that $|f(s) - f(r)| \le C_f \eta(s-r)$, we have:

$$\begin{aligned}
&\int_0^T [\int_r^T (f(s) - f(r))\varepsilon'(s-r)ds]^2 dr \\
&\le C_f^2 \int_0^T \left[\int_r^T \eta(s-r) |\varepsilon'(s-r)| \, ds \right]^2 dr \\
&= C_f^2 \int_0^T \left[\int_0^{T-r} \eta(s) |\varepsilon'(s)| \, ds \right]^2 dr.
\end{aligned}$$

Note that if η is not defined up to T, we can simply extend η as an arbitrary constant by adjusting the constant C_f since f is bounded. Now $\eta|\varepsilon'|$ is integrable on all of $[0,T]$, including at 0, because of hypothesis (C_η) and the assumption that ε is differentiable except at 0 and η is continuous everywhere. This proves that $K_\gamma^* f \in L^2([0,T])$.

To prove the second statement, first note that we can assume that ε' is non-positive (see Remark 3). Now consider the following calculation, assuming (D$_h$):

$$\begin{aligned}\int_0^T \eta(r)\,|\varepsilon'(r)|\,dr &= \int_0^T \left(\int_0^r \frac{h(s)}{\varepsilon(s)}ds\right)(-\varepsilon'(r))\,dr \\ &= \int_0^T \left(\int_s^T -\varepsilon'(r)\,dr\right)\frac{h(s)}{\varepsilon(s)}ds \\ &= \int_0^T (\varepsilon(s)-\varepsilon(T))\frac{h(s)}{\varepsilon(s)}ds \\ &\le \int_0^T h(s)\,ds < \infty.\end{aligned}$$

This proves (D$_h$) implies (C$_\eta$). The proof of the converse implication is more technical. However, the result is less important since, with the first implication, we can already guarantee that C^η is contained in $\mathcal{H}$ as soon as η is of the form (5). Thus we leave the details of "(C$_\eta$) implies (D$_h$)" to the reader. ■

Proposition 13 *Let η and C^η be as in the previous definition and proposition. Then C^η is contained in $\mathcal{H}'$ and the adjoint operator $K_\gamma^{*,a}$ can be explicitly calculated for any $f \in C^\eta$ according to*

$$K_\gamma^{*,a} f(x) = f(x)\varepsilon(x) + \int_0^x [f(y)-f(x)]\varepsilon'(y-x)dy =: Gf(x)$$

Proof. We will show that for $f, g \in C^\eta$ we have

$$< Gf, g >_L^2 ([0,T]) =< f, K_\gamma^* g >_L^2 ([0,T]). \tag{6}$$

First observe that the left-had side of the equality is well defined, i.e. for $f \in C^\eta$ we have $Gf \in L^2([0,T])$. Indeed, if $f \in C^\eta$ then $f\varepsilon$ is bounded hence in $L^2([0,T])$. Moreover $|[f(y)-f(x)]\varepsilon'(y-x)|$ is bounded above by $\eta(|y-x|)\,|\varepsilon'(|y-x|)|$ which implies as in the proof of the previous proposition that the second term in the definition of Gf is in $L^2([0,T])$. Thus we have $C^\eta \subset \mathcal{H}'$. Now we only need to show equality (6). We denote by $P(T) =< Gf, g >_{L^2([0,T])}$ and by $Q(T) =< f, K_\gamma^* g >_{L^2([0,T])}$, then we have

$$P(T) = \int_0^T [f(x)g(x)\varepsilon(x) + g(x)\int_0^x (f(y)-f(x))\varepsilon'(y-x)dy]dx$$

and

$$Q(T) = \int_0^T [f(x)g(x)\varepsilon(T-x) + f(x)\int_x^T (g(y)-g(x))\varepsilon'(y-x)dy]dx$$

and we observe that $P(0) = Q(0)$. Hence it is enough to show that $P'(T) = Q'(T)$ in order to conclude the proposition. But

$$P'(T) = \frac{\delta}{\delta T}\int_0^T [f(x)g(x)\varepsilon(x) + g(x)\int_0^x (f(y)-f(x))\varepsilon'(y-x)dy]dx \tag{7}$$

$$= f(T)g(T)\varepsilon(T) + g(T)\int_0^T f(y) - f(T))\varepsilon'(T-y)dy \tag{8}$$

while

$$
\begin{aligned}
Q'(T) &= \frac{\delta}{\delta T}\int_0^T f(x)g(x)\varepsilon(T-x)dx + \frac{\delta}{\delta T}\int_0^T f(x)\int_x^T (g(y)-g(x))\varepsilon'(y-x)dy]dx \\
&= \frac{\delta}{\delta T}\int_0^T f(T-x)g(T-x)\varepsilon(x)dx + f(T)\cdot 0 + \int_0^T f(x)[g(T)-g(x)]\varepsilon'(t-x)dx \\
&= f(0)g(0)\varepsilon(T) + \int_0^T (fg)'(T-x)\varepsilon(x)dx + \int_0^T [f(x)g(T)-f(x)g(x)]\varepsilon'(T-x)dx \\
&= f(0)g(0)\varepsilon(T) + \int_0^T (fg)'(x)\varepsilon(T-x)dx + \int_0^T f(x)g(T)-f(x)g(x)]\varepsilon'(T-x)dx
\end{aligned}
$$

where the last equality was obtained by the change of variable $T-x \to x$. Therefore by denoting $k(x) = f(x)g(x)$, it is clear that $k \in C^{\eta}$, so we may write

$$
\begin{aligned}
P'(T) - Q'(T) &= h(T)\varepsilon(T) + \int_0^T [g(T)f(x) - g(T)f(T)]\varepsilon'(T-x)dx - k(0)\varepsilon(T) \\
&\quad - \int_0^T f(x)g(T) - f(x)g(x)]\varepsilon'(T-x)dx - \int_0^T (fg)'(x)\varepsilon(T-x)dx \\
&= (k(T)-k(0))\varepsilon(T) \\
&\qquad + \int_0^T [k(x)-k(T)]\varepsilon'(T-x)dx - \int_0^T k'(x)\varepsilon(T-x)dx \\
&= (k(T)-k(0))\varepsilon(T) + \int_0^T [(k(T)-k(x))\varepsilon(T-x)]'dx \\
&= (k(T)-k(0))\varepsilon(T) + [k(T)-k(x)]\varepsilon(T-x)|_0^T \\
&= (k(T)-k(0))\varepsilon(T) - (k(T)-k(0))\varepsilon(T) = 0.
\end{aligned}
$$

The last equality follows from the fact that $\lim_0 \eta\varepsilon = 0$. To prove this last statement, note that by statement (D_h) in the previous proposition, we have the existence of a positive decreasing integrable function h defined on a neighborhood of 0 in $\mathbf{R}_+ - \{0\}$ such that

$$
\eta(r) = \int_0^r h(s)/\varepsilon(s)\,ds.
$$

Therefore, since ε is also decreasing, we get

$$
\eta(r)\,\varepsilon(r) = \int_0^r \varepsilon(r)\,h(s)/\varepsilon(s)\,ds \le \int_0^r h(s)\,ds.
$$

Since h is integrable at 0, the function $r \mapsto \int_0^r h(s)\,ds$ tends to 0 when r tends to 0, which proves the claim that $\lim_0 \eta\varepsilon = 0$, and finishes the proof of the proposition. ■

For the next proposition on the size of $\mathcal{H}_2$, we need the following additional assumption on γ.

(A) Assume that near 0, ε is thrice continuously differentiable, ε' is non-decreasing, and ε'' is non-increasing. This can be assumed without loss of generality. Also recall that $\varepsilon^{(k)}$ has the sign of $(-1)^k$. Additionally, assume that $\varepsilon'''\varepsilon'(\varepsilon'')^{-2}$ is bounded near 0. This last condition will be satisfied in all the examples we will encounter below; it does not reduce the generality of the scale of processes that we may consider.

Proposition 14 *Assume Condition A. Let ζ and C^{ζ} be as in Definition 11. Assume moreover that ζ satisfies the following condition:*

(E$_h$) *There exists a positive function h defined, decreasing, and differentiable on a neighborhood of* 0 *in* $\mathbf{R}_+ - \{0\}$ *such that* $\int_0 h < \infty$ *and for small* $r > 0$,

$$\zeta(r) = -\frac{1}{\varepsilon''(r)} \frac{d(h/\varepsilon)}{dr}(r).$$

Then $\mathcal{H}_2$ *contains* C^ζ.

Proof.

Step 0. Setup. Observe that for any $f \in \mathcal{H}_2$, we have

$$K_\gamma^{*,a} K_\gamma^* f(x) = K_\gamma^* f(x)\varepsilon(x) + \int_0^x (K_\gamma^* f(x) - K_\gamma^* f(y))\varepsilon'(x-y)dy$$

In order to show that $K_\gamma^{*,a} K_\gamma^* f(x) \in L^2([0,T])$ it is enough to show that

$$\int_0^x (K_\gamma^* f(x) - K_\gamma^* f(y))\varepsilon'(x-y)dy \in L^2([0,T]). \tag{9}$$

Indeed,

$$\|K_\gamma^* f\varepsilon\|_{L^2([0,T])} \leq \|K_\gamma^* f\|_{L^2([0,T])}\|\varepsilon\|_{L^2([0,T])} = \|K_\gamma^* f\|_{L^2([0,T])}\gamma(T) < \infty.$$

From the proof of the previous proposition, we know that (9) holds as soon as $K_\gamma^* f$ is included in C^η where η satisfies Condition (C$_\eta$) or (D$_\eta$). Again, this clearly reduces to requiring that the function J defined by

$$J(r) := \int_r^T (f(s) - f(r))\varepsilon'(s-r)\,ds$$

belongs to C^η. We must consider, for a fixed pair (x,y), with say $x < y$, the quantity $L(x,y) = J(x) - J(y)$. We rewrite $L = L_1 + L_2 + L_3$ where

$$L_1(x,y) = \int_x^y \varepsilon'(s-x)(f(s) - f(x))\,ds,$$

$$L_2(x,y) = \int_y^T (\varepsilon'(s-x) - \varepsilon'(s-y))(f(s) - f(y))\,ds,$$

$$L_3(x,y) = (f(y) - f(x))(\varepsilon(T-x) - \varepsilon(y-x)).$$

Our assumption is that for some constant c, which we can take to be $c = 1$ to simplify the notation, we have for all $r, r' > 0$,

$$|f(r) - f(r')| \leq c\zeta(|r-r'|) = c\frac{1}{\varepsilon''(|r-r'|)} \frac{d(h/\varepsilon)}{dr}(|r-r'|). \tag{10}$$

We only need to show the three L_i's are bounded in absolute value by $\eta(y-x)$.

Step 1. ζ is acceptable. It would be well-advised to first check that the function ζ defined in Condition (E$_h$) is a bona-fide modulus of continuity function. Since h is integrable at 0, we can assume that $(-h')(r) \ll \left(r^2 \log\left(r^{-1}\right)\right)^{-1}$ near 0. Then since $(-h/\varepsilon)' = (-h')/\varepsilon + h\varepsilon'/\varepsilon^2 \leq (-h')/\varepsilon$ we get $(-h/\varepsilon)'(r) \ll \left(\varepsilon(r) r^2 \log\left(r^{-1}\right)\right)^{-1}$. If ϕ is a convex differentiable function that increases to $+\infty$ as r decreases to 0, by the mean value theorem, $\phi(r) - \phi(2r) \leq r \max_{(r,2r)} |\phi'| = |\phi'(r)|$, which implies $|\phi'(r)| > \phi(r)/r$. Applying this to both ε and $-\varepsilon'$ yields $\varepsilon''(r) \geq \varepsilon(r) r^{-2}$. This proves that $\lim_0 \zeta = 0$. The continuity of ζ is trivial given our hypotheses. The positivity of ζ near 0 can be obtained as follows. Note that we can choose h so that $h(r) \gg r^{-\alpha}$ for any $\alpha < 1$, while by the definition of ε (since ε^2 has to be integrable at 0), we must have $\varepsilon(r) \ll r^{-1/2}$; therefore as r tends to 0, h/ε tends to infinity faster than $r^{-\alpha+1/2}$. Because of the flexibility of choice for h, this limit can be attained in an increasing fashion, hence $(h/\varepsilon)' < 0$; the positivity of ε'', which is part of our hypothesis, now guarantees that ζ is positive. To guarantee that ζ is increasing, we can again invoke the flexibility on the choice of h, combined with the fact that $\zeta = o(1)$.

Step 2. Estimate for L_2. Using the notation $\delta = y - x$, the fact that ε' is non-decreasing, and the hypothesis (10) on f, we have

$$|L_2(x,y)| \leq \int_0^{T-y} (\varepsilon'(r+\delta) - \varepsilon'(r))\,\zeta(r)\,dr.$$

We can split up this estimate for L_2 into two pieces:

$$\begin{aligned}|L_2(x,y)| &\leq \int_0^{\delta} (\varepsilon'(r+\delta) - \varepsilon'(r))\,\zeta(r)\,dr + \int_\delta^{T} (\varepsilon'(r+\delta) - \varepsilon'(r))\,\zeta(r)\,dr \\ &:= L_{21}(\delta) + L_{22}(\delta).\end{aligned}$$

For the second piece we obtain, using the mean-value theorem and the hypothesis that ε'' is positive and decreasing,

$$\begin{aligned}L_{22}(\delta) &\leq \int_\delta^T \delta\varepsilon''(r)\,\zeta(r)\,dr = -\delta\int_\delta^T (h/\varepsilon)'(r)\,dr \\ &= \delta\left[(h/\varepsilon)(\delta) - (h/\varepsilon)(T)\right] \leq \delta\,(h/\varepsilon)(\delta) \leq \eta(\delta)\end{aligned}$$

where the last inequality follows from the definition of $\eta(\delta) = \int_0^\delta h/\varepsilon$ and the fact that h/ε is decreasing. To estimate the first piece we need an integration by parts. We write $L_{21}(\delta) = \int_0^\delta u dv$ where

$$\begin{aligned}u &= (\varepsilon'(r+\delta) - \varepsilon'(r))\left(-\frac{1}{\varepsilon''(r)}\right) \\ dv &= (h/\varepsilon)'(r)\,dr.\end{aligned}$$

so that

$$-\frac{du}{dr} = \frac{1}{\varepsilon''(r)}(\varepsilon''(r+\delta) - \varepsilon''(r)) - \frac{\varepsilon'''(r)}{\varepsilon''(r)^2}(\varepsilon'(r+\delta) - \varepsilon'(r)).$$

The first term in $-du/dr$ above is negative. The second term is positive and bounded above by $2\varepsilon'\varepsilon'''(\varepsilon'')^{-2}$, which is bounded by Assumption (A), say by a constant c. Therefore

$$-\int_0^\delta v du \leq c\int_0^\delta \frac{h}{\varepsilon} = c\eta(\delta),$$

and since u is negative,

$$\begin{aligned}[uv]_0^\delta &\leq \lim_{r\to 0}\frac{h}{\varepsilon}(r)\,(\varepsilon'(r+\delta) - \varepsilon'(r))\,\frac{1}{\varepsilon''(r)} \\ &\leq \lim_{r\to 0}\frac{h}{\varepsilon}(r)\,\frac{2\,|\varepsilon'(r)|}{\varepsilon''(r)}.\end{aligned}$$

We can show that this limit is 0. Indeed, we know $h \ll 1/r$. Moreover, using the argument with ϕ in Step 1, we have $|\varepsilon'|/\varepsilon'' < r$. The required limit is then obtained since by assumption, $1/\varepsilon$ tends to 0. Therefore we have proved

$$L_{21}(\delta) \leq c\eta(\delta),$$

which implies

$$|L_2(x,y)| \leq (c+1)\,\eta(y-x).$$

Step 3. Estimate for L_3. By assumption (10), using again $\delta = y - x$, and also using the estimate $(\varepsilon/\varepsilon'')(\delta) < \delta^2$ which was proved in Step 1, we have

$$|L_3(x,y)| \le -\left(\frac{h}{\varepsilon}\right)'(\delta)\,\frac{\varepsilon(\delta)}{\varepsilon''(\delta)} \le -\delta^2\left(\frac{h}{\varepsilon}\right)'(\delta).$$

By definition $\eta(\delta) = \int_0^\delta h/\varepsilon$. Therefore, $h/\varepsilon = \eta'$ and $(h/\varepsilon)' = \eta''$. First we can see that $\eta'(\delta) < \eta(\delta)/\delta$ because η is increasing and concave and $\eta(0) = 0$, the concavity coming from the fact that $\eta' = h/\varepsilon$ can be chosen to be decreasing by an appropriate choice of h since $\varepsilon \ll r^{-1/2}$ while $h \gg r^{-1/2}$. Next, since η' decreases from $+\infty$, we can also assume that $-\eta''$ is decreasing; then by the mean value theorem, we have $\eta'(\delta/2) - \eta'(\delta) \ge -\eta''(\delta)\,\delta/2$ which yields $-\eta''(\delta) \le \eta'(\delta/2)\,2/r$. Putting this together with the estimate on η' we get $-\eta''(\delta) \le 4\delta^{-2}\eta(\delta/2)$. Since η is increasing, we finally get

$$|L_3(x,y)| \le -\delta^2\eta''(\delta) \le 4\eta(\delta/2) \le 4\eta(\delta).$$

Step 4. Estimate for L_1**.** By assumption (10), still using $\delta = y - x$,

$$\begin{aligned}|L_1(x,y)| &= \left|\int_x^y \varepsilon'(s-x)\,(f(s) - f(x))\,ds\right| \\ &\le \int_0^\delta \varepsilon'(s)\left(\frac{h}{\varepsilon}\right)'(s)\,\frac{1}{\varepsilon''(s)}ds \\ &\le c\eta(\delta)\end{aligned}$$

where the last inequality is established by repeating the method of estimation of $\int_0^\delta u dv$ in Step 2.

This finishes the proof of the proposition. ∎

The previous proposition is crucial in showing that $\mathcal{H}_2$ is non-empty (modulo constant functions), which is a *sine qua non* condition for the validity of our stochastic calculus below. The next corollary aids in showing how sharp the previous proposition is, and the examples following show precisely how large we can expect $\mathcal{H}_2$ to be in specific cases of interest.

Corollary 15 *Let*

$$\tilde{\zeta}(r) = -\frac{1}{\varepsilon'(r)}\,\frac{d\left(\tilde{h}/\varepsilon'\right)}{dr}(r).$$

where $\tilde{h}$ *satisfies the same hypotheses as* h *except that* $\int_0 \tilde{h} = +\infty$. *Then* $\tilde{h}$ *can be chosen so that* $\tilde{\zeta}$ *is a bona-fide modulus of continuity, and* $\mathcal{H}_2$ *does not contain* $C^{\tilde{\zeta}}$.

Proof. The proof of the corollary uses the estimates in the proof of the previous propositions. We give only the main parts of the argument, leaving some of the details to the reader, since the corollary is not used in the remainder of the paper. First we can invoke the same argument as in Step 1 of the proof of Proposition 14 to justify that with $\tilde{h}(r) = r\log^{-1}\left(r^{-1}\right)$ we do have $\tilde{\zeta}$ non-negative, increasing and continuous at 0 with $\tilde{\zeta}(0) = 0$. Recall then that the dominant term in the calculation of the $\mathcal{H}_2$-norm of a function f is

$$\int_0^y (K_\gamma^* f(y) - K_\gamma^* f(x))\varepsilon'(y-x)dx. \tag{11}$$

We will show that f can be chosen in $C^{\tilde{\zeta}}$ so as to make the above term infinite for all x close to a fixed y. We treat the case $y = T$; smaller values of y are treated similarly, although the calculations are slightly more involved. The dominant term in $K_\gamma^* f(y) - K_\gamma^* f(x)$ is

$$\begin{aligned}
&\int_x^T (f(s) - f(x))\, \varepsilon'(s-x)\, ds - \int_y^T (f(s) - f(y))\, \varepsilon'(s-y)\, ds \\
&= \int_x^y (f(s) - f(x))\, \varepsilon'(s-x)\, ds.
\end{aligned}$$

Since the integral of the above expression, as a function of x in the space $L^1([0,y], \varepsilon'(y-x)\, dx)$, is required, by definition of $\mathcal{H}_2$, to be a member of $L^2([0,T], dy)$ after x-integration, we deduce that the expression must be absolutely integrable except possibly for a null set of values of (x,y). There exists a function f in $C^{\tilde{\zeta}}$ such that for all $0 \le x \le s \le y$, $|f(x) - f(s)| \ge \tilde{\zeta}(s-x)$. Thus

$$\begin{aligned}
\int_x^y |f(s) - f(x)|\, \varepsilon'(s-x)\, ds &\ge \int_x^y \left(\frac{\tilde{h}}{\varepsilon'}\right)'(s-x)\, ds = \int_0^{y-x} \left(\frac{\tilde{h}}{\varepsilon'}\right)' \\
&= \frac{\tilde{h}(y-x)}{\varepsilon'(y-x)}
\end{aligned}$$

where the last step is because of the fact that in all cases which we study, $\varepsilon'(r) \ge 1/r$, since we study only processes B^γ that are more irregular than Brownian motion. Integrating this last expression against $\varepsilon'(y-x)\, dx$ in $[0,y]$ yields infinity by definition of $\tilde{h}$. Since this holds for all x and y the expression in (11) cannot be in $L^2([0,T])$. ■

Definition 16 *Let $\beta > 0$ be fixed. Let γ be defined by*

$$\gamma^2(r) = [\log(1/r)]^{-\beta},$$

so that

$$\varepsilon^2(r) = \beta r [\log(1/r)]^{-\beta-1}.$$

We call the corresponding process B^γ, as defined in Proposition 1, the logarithmic Brownian motion (logBm) *with parameter β.*

Note that since γ has a singularity at $r = 1$, it is safe to define logBm only on closed intervals in $[0,1)$. *For larger intervals, simple scaling can be used; for infinite intervals, it is best to modify the behavior of γ for large r.*

From Proposition 1, we have that the canonical metric δ of B^γ is commensurate with γ. It is then well-known that if $\xi(r) := \gamma(r) \log^{1/2}(r)$ is continuous, it is almost-surely a uniform modulus of continuity for B^γ, i.e. $B^\gamma \in C^\xi$ a.s. In fact this property is sharp: if $B^\gamma \in C^\xi$ then γ is bounded below by $\xi \log^{-1/2}(\cdot)$; this property was established for homogeneous Gaussian processes in [17]; here a slightly modified argument, using the estimates in the proof of Proposition 1, can be invoked; we leave this to the reader, since the result is only tangential to our main results. What we can see immediately is that B^γ is a.s. uniformly continuous if and only if $\beta > 1$.

It can also be established that if $\beta \le 1$ then B^γ is unbounded. It is often quoted in the literature that a homogeneous Gaussian process is either a.s. uniformly continuous or is a.s. unbounded, from which it is sometimes inferred that in the unbounded case, the process is discontinuous. However, even though we know of no proof of this fact, we believe that even if $\beta \le 1$, the logBm is still pointwise continuous a.s., even if only at countably many points; this certainly does not contradict its unboundedness. But more importantly it would explain, heuristically, why we are able to define a stochastic calculus and a non-trivial local time with respect to it. We now summarize the above discussion, and give an indication of the size of $\mathcal{H}$ and $\mathcal{H}_2$. For these sizes, we simply used Propositions 12 and 14, with $h(r) = r^{-1} \log^{-\alpha}(1/r)$ for some $\alpha > 1$.

- *Fractional Brownian scale.* The process B^γ has a canonical metric that is commensurate with that of H-fBm if $\gamma(r) = r^H$, or more generally if $\gamma(r) \asymp r^H$. In this case, our Skorohod integral defined in Section (5.2) below has the same properties as that defined in [5]. It is interesting to note that our results above prove that $\mathcal{H}_2$ is indeed non-empty in this case, although this question did not seem concerning in [5]. According to our results, $\mathcal{H}_2$ contains $C^{1-2H'}$ for any $H' < H < 1/2$. The larger H is, the bigger $\mathcal{H}_2$ is. However, Corollary 15 shows that $\mathcal{H}_2$ does not contain the space $C^{\tilde{\zeta}}$ where $\tilde{\zeta}(r) = r^{1-2H}\log^{-1}(1/r)$. A slight historical digression on the Skorohod integral for fBm for $H \in (1/4; 1/2)$ might be relevant at this stage. However, we refer to the last paragraph in Subsection 5.2 below for such a development.

- *Regular logBm*: case $\beta > 1$. The logBm process B^γ is a.s. uniformly continuous with modulus of continuity $\xi(r) := \log^{(1-\beta)/2}(r)$. $\mathcal{H}$ contains the space C^η for
$$\eta(r) = r^{1/2}\log^{\beta-\alpha}(1/r)$$
for any $\alpha > 1$, so in particular it contains a space bigger than $C^{1/2}$. $\mathcal{H}_2$ contains the space C^ζ for
$$\zeta(r) = r\log^{\beta+1-\alpha}(1/r)$$
for any $\alpha > 1$, which is non-empty if $1 < \alpha \le \beta + 1$.

- *Irregular logBm*: case $\beta \in (0; 1]$. The logBm process B^γ is a.s. unbounded. $\mathcal{H}$ contains the space C^η for η as defined in the previous case for any $\alpha > 1$, which is never empty, but does not contains a space bigger than $C^{1/2}$. $\mathcal{H}_2$ contains the space C^ζ for ζ as defined in the previous case for any $\alpha > 1$, which is non-empty if $1 < \alpha \le \beta + 1$; therefore $\mathcal{H}_2$ is non-empty, and a stochastic calculus w.r.t. B^γ will be defined below.

- *Highly irregular processes.* One could study examples such as $\gamma(r) \asymp \log^{-1}(\log(1/r))$, or even using multiple iterations of the logarithm. One can check that $\mathcal{H}_2$ is non-empty in these cases, although the size of $\mathcal{H}_2$ decreases "dangerously". However, since the transition between the continuous and discontinuous processes occurs within the logBm scale, we have not yet found any compelling reasons to expand on these other examples.

5 Stochastic calculus

5.1 The derivative operator

We denote by $\mathcal{S}$ the set of smooth cylindrical random variables of the form
$$F = f(B^\gamma(\phi_1), \cdots B^\gamma(\phi_n)),\ n \ge 1,\ \ f \in C^\infty(R^n),\ \ \phi_i \in \mathcal{H}. \tag{12}$$
We define the differential operator D on $\mathcal{S}$ by
$$DF = \sum_{i=1}^n \frac{\delta f}{dx_i}(B^\gamma(\phi_1), \cdots B^\gamma(\phi_n))\phi_i.$$

Remark 17 *DF is an element of $L^2([0,T] \times \Omega) = L^2(\Omega; \mathcal{H})$*

Remark 18 *For all $p \ge 1$ $F \to DF$ is closable from $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H})$. The domain of D in $L^p(\Omega)$ is denoted by $D^{1,p}$, meaning that $D^{1,p}$ is the closure of the smooth random variables $\mathcal{S}$ with respect to the norm*
$$\|F\|_{1,p} = [E(|F|^p) + E(\|DF\|^p_{L^2(T)})]^{\frac{1}{p}}.$$

Remark 19 *For $p = 2$ the space $D^{1,2}$ is the Hilbert space with the scalar product*

$$< F, G >_{1,2} = E(FG) + E(< DF, DG >_{\mathcal{H}})$$

Theorem 20 *Let $H_m(x)$ be the m-th Hermite polynomial and let $\phi \in \mathcal{H}$ be an element of norm 1. Then it holds that*

$$m! H_m(B^\gamma(\phi)) = \int_{[0,T]^m} \phi(t_1)\phi(t_2)\cdots\phi(t_m) b(dt_1)\cdots B^\gamma(dt_m)$$

Note. The mth Hermite polynomial is given by

$$H_m(x) = \frac{(-1)^m}{m!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \;\; n \geq 1$$

and $H_0(x) = 1$.

This theorem is a direct application of the well-known result on multiple Wiener integrals, which can be found in Nualart's book [12], established for all isonormal Gaussian processes, and hence for our particular class of processes and their associated Hilbert spaces $\mathcal{H}$. Moreover using the same notation as we use, the proof of the following chain rule can also be found in [12].

Proposition 21 *Let $\phi : \mathbf{R}^m \to \mathbf{R}$ be a continuous differentiable function with bounded partial derivatives, and fix $p \geq 1$. Suppose that $F = (F^1, \cdots F^m)$ is a random vector whose components belong to the space $D^{1,p}$. Then*

$$D(\phi(F)) = \sum_{i=1}^{m} \frac{\delta\phi}{\delta x_i}(F) DF^i.$$

5.2 The divergence operator and its extension

Definition 22 *The divergence operator δ is defined to be the adjoint of the derivative operator D viewed as an operator from $L^2(\Omega) \to L^2(\Omega, \mathcal{H})$.*

Remark 23 *δ is an operator from $L^2(\Omega, \mathcal{H})$ into $L^2(\Omega)$, and its domain denoted by $Dom\ \delta$ is the space of processes $u \in L^2(\Omega, \mathcal{H}$ such that $F \to E(< DF, u >_{\mathcal{H}})$ is a bounded linear functional on $(\mathcal{S}, \|\cdot\|_2)$. Since for such u the functional $F \to E(< DF, u >_{\mathcal{H}})$ is linear and bounded, we can write it as an inner product, hence there is an unique element $\delta(u)$ in $L^2(\Omega)$ such that*

$$E(< DF, u >_{\mathcal{H}}) = E(\delta_p(u) F),$$

for all $F \in \mathcal{S}_{\mathcal{H}}$.

Notation. We use the notation:

$$\delta(u) = \int_{[0,T]} u_t dB_t^\gamma.$$

This integral will also be referred to as the Skorohod integral below.

It is now understood that for fBm B^H with parameter $H \leq 1/4$, B^H is not in the domain $Dom\delta$ of its own Skorohod integral: see [5]. The same argument used therein can be applied to our process B^γ with $\gamma(r) \geq r^{1/4}$. However, an extension of the Skorohod integral was developed in [5] for which B^H is integrable. We are about to see how to extend the Skorohod integral and the Ito formula in general to allow for all our processes B^γ to be integrable. Our proofs use the same algebraic method based on Gaussian chaos and Hermite polynomials as in [5], but they are both simpler, since they do not require the explicit use of fractional calculus, and wider-ranging, since they are not restricted to the Hölder scale for fBm.

Definition 24 *We denote by $\mathcal{S}_{\mathcal{H}_2}$ the set of smooth cylindrical random variables of the form*

$$F = f(B^{\gamma}(\phi_1), \cdots B^{\gamma}(\phi_n)),\ n \geq 1,\ f \in C^{\infty}(R^n),\ \phi_i \in \mathcal{H}_2.$$

Definition 25 *Let $\{u_t, t \in [0,T]\}$ be such that $E\int_0^T u_t^2 dt < \infty$. We say that $u \in Dom^*\delta$ if there exists $\delta(u) \in L^2(\Omega)$ such that for all $F \in \mathcal{S}_{\mathcal{H}_2}$ we have*

$$\int_0^T E(u_t K_{\gamma}^{*,a} K_{\gamma}^{*} D_t F)dt = E(\delta(u)F).$$

The divergence operator defined for $u \in Dom^*\delta$ will also be called the Skorohod integral of u with respect to B^{γ}.

Remark 26 *The following properties are easily seen.*

1) *$Dom\delta \subset Dom^*\delta$.*

2) *If $u \in Dom^*\delta$ then $E(u) \in \mathcal{H}$.*

3) *If u is deterministic then $u \in Dom^*\delta$ iff $u \in \mathcal{H}$ iff $u \in Dom\delta$.*

Before presenting the Ito formula, we finish this subsection with some remarks and questions on the relation with standard Skorohod integration. Before the article [5] appeared, it was commonly thought that Skorohod integration for fBm had a lower limit, and the threshold $H > 1/4$ was often quoted as the most irregular level for which a Skorohod integration could be defined. Skorohod integration in [5] is pushed beyond this level by modifying the size of the space of test functions needed to assert integrability. Our results here show that in the range $H \in (1/4; 1/2)$, the size of the space of test functions in [5] or in this article is smaller than the original test space for Skorohod integration; indeed (compare line (12) and Definition (24) below), the latter is based on $\mathcal{H}$ while the former is based on $\mathcal{H}_2$. In this sense, our Skorohod integral, or that of [5], only generalizes the standard Skorohod integral for $H \in (1/4, 1/2)$ insofar as the Skorohod integrals themselves are equal as linear forms on $\mathcal{S}_{\mathcal{H}_2} \subsetneq L^2(\Omega)$. The fact that the Ito formulas coincide in the standard case and in the case of [5] (Lemma 9 therein) shows that a lot can be deduced from the coincidence of the two ways of defining Skorohod integrals. As such it is not surprising that a study of local time may be possible, for example. On the other hand, our Ito formula (Theorem 27 below) has exactly the same form again, even though our process B^{γ_H} is not identical to fBm (see Remark 6); in other words the Ito formula for deterministic functions of B^{γ} is clearly not an appropriate test for thoroughly comparing Skorohod integrals. The precise extent to which standard Skorohod integration and the integration defined in this article or in [5] actually coincide is not clear to us at this stage.

5.3 The Itô Formula

Following the arguments of Cheridito and Nualart in [5], in this section we will prove the basic result of stochastic calculus. Our proof does not require the use of fractional derivatives – in fact we had to find a way to do without them, since we do not work in the power scale. Some other aspects of the proof have presumably well-known structures, and are very similar to some arguments in [5], such as the proof of the algebraic identities using Hermite polynomials (19), (20), and (21). We have included brief proofs of all such claims, for the sake of readability.

Theorem 27 *Let $f \in C^{\infty}(\mathbf{R})$ be a function such that for all $n \geq 0$ there exist constants C_n and D_n with $D_n < \frac{1}{2}\log\frac{1}{T}$ such that*

$$|f^{(n)}(y) \leq C_n e^{D_n y^2},\ \ y \in \mathbf{R}.$$

Then for all $t \leq T$ the process $f'(B^\gamma_s)1_{(0,t]}(s) \in Dom^*\delta$ and we have

$$\delta(f'(B^\gamma_s)1_{(0,t]}) = f(B^\gamma_t) - f(0) - \int_0^t f''(B^\gamma_s)\gamma(s)\gamma'(s)ds.$$

Proof. The process $f'(B^\gamma_s)1_{(0,t]}(s) \in Dom^*\delta$ and the formula in the above theorem is true iff for all $F \in \mathcal{S}_{\mathcal{H}_2}$ we have

$$\int_0^T E(f'(B^\gamma_s)1_{(0,t]}(s)K^{*,a}_\gamma K^*_\gamma D_s F)ds$$
$$= E((f(B^\gamma_t) - f(0) - \int_0^t f''(B^\gamma_s)\gamma(s)\gamma'(s)ds)F). \quad (13)$$

Since $H_n(B^\gamma(\phi)), n \geq 1$, with H_n being the n-th Hermitian polynomial, are dense in $\mathcal{S}_{\mathcal{H}_2}$ it is enough to show (13) for F of this type.

However $D_s H_n(B^\gamma(\phi)) = H_{n-1}(B^\gamma(\phi))\phi(s)$; hence (13) is equivalent to

$$\int_0^T E(f'(B^\gamma_s)1_{[0,t]}(s)K^{*,a}_\gamma K^*_\gamma H_{n-1}(B^\gamma(\phi))\phi(s)))ds$$
$$= E((f(B^\gamma_t) - f(0) - \int_0^t f''(B^\gamma_s)\gamma(s)\gamma'(s)ds H_n(B^\gamma(\phi))). \quad (14)$$

Since $H_{n-1}(B^\gamma(\phi))$ does not depend on s we can rewrite (14) as

$$\int_0^T E(f'(B^\gamma_s)H_{n-1}(B^\gamma(\phi))(K^{*,a}_\gamma K^*_\gamma H_{n-1}\phi)(s))ds$$
$$= E((f(B^\gamma_t) - f(0) - \int_0^T f''(B^\gamma_s)\gamma(s)\gamma'(s)ds H_n(B^\gamma(\phi))). \quad (15)$$

Let us compute $E(f^{(n)}(B^\gamma_t))$.

If $p(\sigma, y) := (2\pi\sigma)^{-1/2}\exp(-\frac{1}{2}\frac{y^2}{\sigma})$, $\sigma > 0$, $y \in \mathbf{R}$. Observe that $\frac{\partial p}{\partial \sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}$. Then

$$\begin{aligned}
\frac{d}{dt}E(f^{(n)}(B^\gamma_t)) &= \frac{d}{dt}\int_{\mathbf{R}} p(\gamma^2(t), y)f^{(n)}(y)dy \\
&= \int_{\mathbf{R}} \frac{\partial p}{\partial \sigma}(\gamma^2(t), y)2\gamma(t)\gamma'(t)f^{(n)}(y)dy \\
&= \int_{\mathbf{R}} \frac{\partial^2 p}{\partial y^2}(\gamma^2(t), y)f^{(n)}(y)\gamma(t)\gamma'(t)dy \\
&= \int_{\mathbf{R}} \gamma(t)\gamma'(t)p(\gamma^2(t), y)f^{(n+2)}(y)dy \\
&= \gamma(t)\gamma'(t)E(f^{(n+2)}(B^\gamma_t)) \quad (16)
\end{aligned}$$

The fourth equality is obtained using the properties of f and integration by parts applied twice. Indeed,

$$\int_{\mathbf{R}} \frac{\partial^2 p}{\partial y^2}(\gamma^2(t), y)f^{(n)}(y)dy = \int_{\mathbf{R}} p(\gamma^2(t), y)f^{(n+2)}(y)dy$$
$$+\frac{\partial p}{\partial y}(\gamma^2(t), y)f^{(n)}(y)\big|_{-\infty}^{\infty} - p(\gamma^2(t), y)f^{(n+1)}(y)\big|_{-\infty}^{\infty} \quad (17)$$

But $\frac{\partial p}{\partial y}(\gamma^2(t), y)f^{(n)}(y) \leq -\frac{C_n}{2\gamma(t)}e^{y^2(-\frac{1}{\gamma(t)}+D_n)}$. Since $D_n < -\frac{1}{\gamma(t)}$ for all $t \leq T$ we conclude that the term $\frac{\partial p}{\partial y}(\gamma^2(t), y)f^{(n)}(y)\big|_{-\infty}^{\infty} = 0$. Similarly $p(\gamma^2(t), y)f^{(n+1)}(y)\big|_{-\infty}^{\infty} = 0$. Hence the 4th equality.

Now we proceed to verify equality (15). For $n = 0$, the left hand side of equality (15) is 0, and the right hand side is

$$E((f(B_t^\gamma) - f(0) - \int_0^t f''(B_s^\gamma)\gamma(s)\gamma'(s)ds) \cdot 1) = 0,$$

so the equality is verified. For $n \geq 1$, for all $s \in (0, t]$ we have

$$\begin{aligned}
\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}} &= \langle K_\gamma^\alpha 1_{(0,s]}, K_\gamma^\alpha \phi\rangle_{L^2([0,T])} \\
&= \langle 1_{(0,s]}, K_\gamma^{\alpha,a} K_\gamma^\alpha \phi\rangle_{L^2([0,T])} \\
&= \int_0^s K_\gamma^{\alpha,a} K_\gamma^\alpha \phi(\mu) d\mu
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{ds}(E[f^{(n)}(B_s^\gamma)]\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}}^n) = \\
&\qquad \gamma(s)\gamma'(s)E[f^{(n+2)}(B_s^\gamma)]\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}}^n + nE[f^{(n)}(B_s^\gamma)]\langle 1_{(0,s]}, \phi\rangle^{n-1} K_\gamma^{*,a} K_\gamma^* \phi(s).
\end{aligned}$$

Hence,

$$\begin{aligned}
&E[f^{(n)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^n \\
&\quad = \int_0^t \gamma(s)\gamma'(s)E[f^{(n+2)}(B_s^\gamma)]\langle 1_{(0,s]}, \phi\rangle_{\mathcal{H}}^n ds \\
&\quad + n\int_0^t E[f^{(n)}(B_s^\gamma)]\langle 1_{(0,s]}, \phi\rangle^{n-1} K_\gamma^{*,a} K_\gamma^* \phi(s) ds. \qquad (18)
\end{aligned}$$

Now let us show that

$$E[f^{(n)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^n = n!E[f(B_t^\gamma)H_n(B^\gamma(\phi))], \qquad (19)$$

and

$$E[f^{(n)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^{n-1} = (n-1)!E[f'(B_t^\gamma)H_{n-1}(B^\gamma(\phi))], \qquad (20)$$

and also

$$E[f^{(n+2)}(B_t^\gamma)]\langle 1_{(0,t]}, \phi\rangle_{\mathcal{H}}^n = n!E[f''(B_t^\gamma)H_n(B^\gamma(\phi))]. \qquad (21)$$

We know $E < u, DF >_{\mathcal{H}} = E[\delta(u)F]$. Also, by Theorem 1.1.2 in [12],

$$u = H_{k-1}(B^\gamma(\phi))\phi(t) = \frac{1}{(k-1)!}\int_{[0,T]^{k-1}} \phi(t_1)\cdots\phi(t_{k-1})\phi(t)B^\gamma(dt_1)\cdots B^\gamma(dt_{k-1}))$$

and

$$\begin{aligned}
\delta(u) &= \frac{1}{(k-1)!}\int_{[0,T]^k} \phi(t_1)\cdots\phi(t_{k-1})\phi(t)B^\gamma(dt_1)\cdots B^\gamma(dt_{k-1}B^\gamma(dt))) \\
&= \frac{k!}{(k-1)!}H_k(B^\gamma(\phi)) = kH_k(B^\gamma(\phi))
\end{aligned}$$

and $u \in Dom(\delta)$. Also observe that

$$\begin{aligned}
E(f(B_t^\gamma)) &< 1_{[0,T]}, \phi >_{\mathcal{H}} = E(f(B_t^\gamma)) < K_\gamma^* 1_{[0,T]}, K_\gamma^* \phi >_{L^2(),T)} \\
&= < E(f(B_t^\gamma))1_{[0,T]}, K_\gamma^{*,a} K_\gamma^* \phi >_{L^2(),T)} \\
&= \int_0^t E(f(B_t^\gamma))1_{[0,T]} K_\gamma^{*,a} K_\gamma^* \phi = E(\delta(\phi)f(B_t^\gamma))
\end{aligned}$$

We prove the first equality by induction. The other two have the same proof. For $n=1$ we have

$$E(f'(B_t^\gamma)) < 1_{[0,T]}, \phi >_{\mathcal{H}} = E(\delta(\phi) f(B_t^\gamma)) = E[H_1(B^\gamma(\phi)) f(B_t^\gamma))$$

Now assume the equality is true for $n=k$ and prove it is true for $n=k+1$.

$$\begin{aligned}(k+1)! E[f(B_t^\gamma) H_{k+1}(B^\gamma(\phi))] &= k! E[f(B_t^\gamma)\delta(H_k(B^\gamma(\phi))\phi(t))] \\ k! \int_0^t & E(f(B_s^\gamma)) H_k(B^\gamma(\phi)) K_\gamma^{*,a} K_\gamma^* \phi(s) ds \\ = \int_0^t & E(f^{(k)}(B_s^\gamma)) < 1_{[0,t]}, \phi >_{\mathcal{H}}^k K_\gamma^{*,a} K_\gamma^* \phi(s) ds \\ = \int_0^t & E(f^{(k)}(B_s^\gamma)) K_\gamma^{*,a} K_\gamma^* \phi(s) ds < 1_{[0,t]}, \phi >_{\mathcal{H}}^k \\ = & E(f^{(k+1)}(B_s^\gamma)) < 1_{[0,t]}, \phi >_{\mathcal{H}}^{k+1},\end{aligned}$$

where the last equality is deduce by the induction step.

Using (19), (20), and (21) into (18) we obtain:

$$\begin{aligned} n! E[f(B_t^\gamma) H_n(B^\gamma(\phi))] = & \int_0^t \gamma(s)\gamma'(s) n! E[f''(B_s^\gamma)] H_n(B^\gamma(\phi)) ds \\ & + n \int_0^t (n-1)! E[f'(B_t^\gamma) H_{n-1}(B^\gamma(\phi))] K_\gamma^{*,a} K_\gamma^* \phi(s) ds, \end{aligned}$$

which is equivalent to (15). ■

6 Local time

6.1 Existence

There are two distinct "natural" ways of defining the local time of a Gaussian process. If one attempts to keep the highest possible analogy with the standard Brownian case, one can define λ_t as the occupation measure $\lambda_t(A) = \int_0^t 1_A(B_s^H) ds$ and use the same notation abusively to define its density with respect to Lebesgue measure. This was done for example originally in Berman's paper [3]. On the other hand, and more recently, several stochastic analysts working on fractional Brownian motion have chosen to consider a different occupation measure because it yields a connection to stochastic calculus via the Itô-Tanaka formula: see for example [7]; also see the summary on local time for fBm-based processes in [18].

We use herein the same type of definition, since our motivations are of the same nature. Specifically, we let L_t^a be the density at point a of the occupation measure

$$A \mapsto \int_0^t \mathbf{1}_A(B_s^\gamma) d\left(\gamma^2\right)(s) = \int_0^t \mathbf{1}_A(B_s^\gamma) 2\gamma(s)\gamma'(s) ds.$$

This is the same definition as for fBm in articles such as [7], since then $\gamma^2(s) = s^{2H}$; the existence of this occupation measure can be established in the same way. In particular, we state the following result, leaving its proof to the reader:

$$L_t^a = \int_0^t 2\gamma(s)\gamma'(s)\lambda^a(ds). \tag{22}$$

While L_t can be interpreted as the density of an "occupation time measure", it is important to note that the word "time" cannot have the same interpretation as for λ_t since for L_t time is heavily weighted at the origin. Note also that from Proposition 29 below, L has a version that is jointly continuous in t and a on any set bounded away from the line $t=0$.

6.2 Tanaka formula

Theorem 28 *Let $x, y \le T$. Then $1_{(y,\infty)}B_t^\gamma 1_{(0,x]}(t) \in Dom^*\delta$ and*

$$\delta(1_{(y,\infty)}(B_\cdot^\gamma)1_{(0,x]}(\cdot)) = (B_x^\gamma - y)^+ - \frac{1}{2}L_x^y.$$

Proof. For $\varepsilon > 0$, denote by $p_{\varepsilon(x)} = (2\pi\varepsilon)^{-1/2}\exp\left(-x^2/2\varepsilon\right)$ and by

$$f_\varepsilon(\alpha) = \int_{-\infty}^{\alpha}\int_{-\infty}^{v} p_\varepsilon(z-y)dzdv, \alpha \in \mathbf{R}.$$

Observe now that $f_\varepsilon(\alpha) \to (\alpha - y)^+$ and $f'_\varepsilon(\alpha) = \int_{-\infty}^{\alpha} p_\varepsilon(z-y)dz \to \frac{1}{2}1_{\{0\}}(\alpha) + 1_{(y,\infty)}(\alpha)$. Hence $f_\varepsilon(B_x^\gamma) \to (B_x^\gamma - y)^+$ in $L^2(\Omega)$ and $f'_\varepsilon(B_t^\gamma)1_{(0,x]}(t) \to 1_{(y,\infty)}(B_t^\gamma)1_{(0,x]}(t)$ in $L^2(\Omega \times \mathbf{R})$.

Moreover, since the functions f_ε satisfy the conditions of Ito formula we deduce that $f'_\varepsilon(B_t^\gamma)1_{(0,x]}(t) \in Dom^*\delta$ and

$$\delta[f'_\varepsilon(B_\cdot^\gamma)1_{(0,x]}(\cdot)] = f_\varepsilon(B_x^\gamma) - f_\varepsilon(B_0^\gamma) - \int_0^x f''_\varepsilon(B_s^\gamma)\gamma(s)\gamma'(s)ds. \tag{23}$$

Therefore if we show that

$$\lim_{\varepsilon\to 0}\delta[f'_\varepsilon(B_\cdot^\gamma)1_{(0,x]}(\cdot)] = \delta(1_{(y,\infty)}(B_\cdot^\gamma)1_{(0,x]}(\cdot)) \tag{24}$$

and that

$$\lim_{\varepsilon\to 0}\int_0^x f''_\varepsilon(B_s^\gamma)2\gamma(s)\gamma'(s)ds = L_x^y \tag{25}$$

the theorem will be proved.

The convergence (24) follows from the fact that δ is a closed operator on Dom^*, i.e. if $u_n, u \in Dom^*\delta \cap L^2(\Omega, L^2(\mathbf{R}_+))$ are such that $\lim_{n\to\infty} u_n = u$ in $L^2(\Omega, L^2(\mathbf{R}_+))$ and if there is $U \in L^2(\Omega)$ such that $\lim_{n\to\infty}\delta(u_n) = U$ in $L^2(\Omega)$ then $u \in Dom^*\delta$ and $\delta(u) = U$.In our case $u_\epsilon = f'_\varepsilon(B_\cdot^\gamma)1_{(0,x]}(\cdot)$ and using Cauchy convergence in (23) we obtain the convergence of $\delta(u_n)$ hence (24).

In order to prove (25) we remark first that for continuous functions g with compact support we have:

$$\int_0^t g(B_s^\gamma)ds = \int_{\mathbf{R}} g(y)\lambda_t^y dy.$$

Indeed, this is true for simple functions. Then by writing a continuous function with compact support g as limit of simple functions we obtain the desired equality. Now using the definition of L_t^a we obtain in the same fashion:

$$\int_{\mathbf{R}} g(y)L_t^y dy = \int_0^t g(B_s)^\gamma d\gamma^2(s),$$

for any function g with compact support.

The validity of formula (22) can be used to establish the following convergence, whose proof is also left to the reader: $\int_0^t p_\varepsilon(B_s - y)d\gamma^2(s)$ converges uniformly in y. We can write now, for each continuous function with compact support:

$$\begin{aligned}\int_{\mathbf{R}}\left(\lim_{\epsilon\to 0}\int_0^t p_\varepsilon(B_s - y)d\gamma^2(s)\right)g(y)dy &= \lim_{\epsilon\to 0}\int_{\mathbf{R}}\left(\int_0^t p_\varepsilon(B_s - y)d\gamma^2(s)\right)g(y)dy \\ &= \lim_{\epsilon\to 0}\int_0^t\left(\int_{\mathbf{R}} p_\varepsilon(B_s - y)g(y)dy\right)d\gamma^2(s)\end{aligned}$$

Since g is continuous with compact support it will have a maximum on $[0,t]$ and applying dominated convergence we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^t\left(\int_{\mathbf{R}} p_\varepsilon(B_s - y)g(y)dy\right)d\gamma^2(s) &= \int_0^t\left(\lim_{\epsilon\to 0}\int_{\mathbf{R}} p_\varepsilon(B_s - y)g(y)dy\right)d\gamma^2(s) \\ &= \int_0^t g(B_s)d\gamma^2(s)\end{aligned}$$

Therefore for each continuous function g with compact support we have

$$\int_{\mathbf{R}} \left(\lim_{\epsilon \to 0} \int_0^t p_\varepsilon(B_s - y) d\gamma^2(s) \right) g(y) dy = \int_{\mathbf{R}} g(y) L_t^y dy,$$

which implies (25). ■

6.3 Regularity

It is often quoted that the more a stochastic process is irregular, the more its local time will be regular in the time parameter. This was noticed already in Berman's original article [3]. A slightly more careful analysis shows that in fact the oft quoted idea above needs to be taken with a small grain of salt in the case of L because of the singularity at the origin in the definition of L when B^γ is more irregular than standard Brownian motion.

Proposition 29 *Let $t_0 > 0$ be fixed. Then L has a jointly continuous version in (t, a) on the set $[t_0, 1] \times \mathbf{R}$. In particular, uniformly in x, this version is θ-Hölder-continuous in $t \geq t_0$ for any parameter $\theta < 1/2$; however, in the case of logBm, we can increase this to any $\theta < 1$.*

Proof. We have the following limit: for some $\delta > 0$ and for some $\theta' < 1$

$$\lim_{h \to 0} h^{-\theta'} \int_0^h (\gamma(t))^{-1-\delta} dt = 0.$$

Indeed, we can assume that $\gamma(t) \geq t^{1/2}$ (see Remark 3), so that we can choose and $\theta' < 1/2$. In the case of logBm, we have of course that for all $\alpha > 0$, $\gamma(t) \geq t^\alpha$, so that we can choose any $\theta' < 1$. By Theorem 8.1 in [3] we deduce that here is a continuous version, in the two parameters (a, t), for the density λ_t^a of the occupation measure λ_t, and that uniformly in a, the version is θ-Hölder-continuous for any $\theta < 1/2$ in the general case, and for any $\theta < 1$ in the logBm case. Since the local time, as defined above, is related with the density λ_t^a by (22) we can use integration by parts to express L as follows:

$$L_t^a = \varepsilon^2(t) \lambda^a(t) - \lim_{0+} \varepsilon^2 \lambda^a - \int_0^t \frac{d\varepsilon^2}{ds}(s) \lambda^a(s) ds. \tag{26}$$

Since the above limit is 0 (follows from 22 and integration by parts) it is true that away from 0 L^a and λ^a share, up to a constant, the same modulus of continuity, since ε^2 is smooth and bounded there. ■

The modulus of continuity at 0 is not the same as in the above proposition, however. Indeed, one can quote, for example, Yimin Xiao's result in [19], which implies that in the case of fBm for some constant K, a.s.

$$|\lambda^a(t)| \leq K t^{1-H} (\log\log(1/t))^H.$$

One then sees that we can only guarantee the following behavior for the local time for fBm: close to 0, for some constant K, almost surely,

$$|L_t^a| \leq K\varepsilon^2(t) \lambda^a(t) = K t^{2H-1} \lambda^a(t) \leq K t^H.$$

When $H < 1/2$, we see that the uniform Hölder behavior of L^a on any interval containing the origin deteriorates to the exponent H instead of $1 - H$.

In any non-Hölder case, for example with the scale of logarithmic Brownian motion (logBm) defined by $\gamma^2(r) = \log^{-\beta}(1/r)$, we cannot exploit any existing results. Even Yimin Xiao's sharp uniform continuity result which states that in the fBm case in [19], almost surely, for all t and all small r

$$|\lambda^a(t+r) - \lambda^a(t)| \leq K r^{1-H} (\log(1/r))^H,$$

(or her pointwise continuity result where t is fixed and the log is replaced by $\log\log$) cannot be extended beyond the Hölder scale, since her proofs rely heavily on the fact that γ should be regularly varying. In accordance with her results, however, we state the following conjecture, which we will investigate in another publication.

Conjecture 30 *There exists a constant K such that almost surely, for all $t \in [0,1]$, for all small $r > 0$*

$$|\lambda^a(t+r) - \lambda^a(t)| \leq Kr/\gamma(r/\log(1/r)).$$

With this conjecture, the integration-by-parts formula (26) will yield the same modulus of continuity for L^a (away from 0) if $\lim_{0+} \varepsilon^2 \lambda^a = 0$. This limit is trivial in the Hölder scale, and is easily seen otherwise: indeed when γ is larger than any power function the relation $\gamma(r/\log(1/r)) > \gamma(t)\gamma(1/\log(1/t))$ holds trivially; coupled with the convexity relation $t\gamma'(t) \leq \gamma(t)$, we get

$$\varepsilon^2(t)\lambda^a(t) \leq \gamma(t)/\gamma(1/\log(1/t))$$

hence the required limit, and the following additional conjecture.

Corollary 31 *Let I be a closed interval in $(0,1]$. There exists a constant K such that almost surely, for all $t \in I$, for all small $r > 0$*

$$\left|L^a_{t+r} - L^a_t\right| \leq Kr/\gamma(r/\log(1/r)).$$

To establish this conjecture, we cannot expect to be able to use the Tanaka formula:

$$L^0_t = (B^\gamma_t)_+ - \int_0^t \mathbf{1}_{[0,\infty)}(B^\gamma_s)\,\delta B^\gamma(s). \tag{27}$$

Indeed, neither of the two terms in this formula should be more regular than B^γ itself. Specifically, in the case where γ is bigger than any power, it is well-known that B^γ has $\gamma(r)\log^{1/2}(1/r)$ as a uniform modulus of continuity, and it has been established in [17] that this modulus is sharp, so that the same holds for $(B^\gamma_t)_+$. We expect that the Skorohod integral above should not be more regular than B^γ. In fact this has been established in [1], in the Hölder scale at least, in the case of fBm. Thus no soft argument can be used to see why L^0 would be so much more regular than either of its two components.

However, if the above conjecture and corollary can be established, then we could get an important result by exploiting the Tanaka formula: we could prove that B^γ and the Skorohod integral process $\int_0^t \mathbf{1}_{[0,\infty)}(B^\gamma_s)\,\delta B^\gamma(s)$ share the same modulus of continuity, since the local time L^0, being much more regular than B^γ, cannot effect the latter's via the additive relation (27). More generally, working with approximations, we would have the following.

Corollary 32 *For appropriately smooth and/or bounded functions f on $\mathbf{R}$, the Skorohod integral process*

$$t \mapsto \int_0^t f(B^\gamma_s)\,\delta B^\gamma(s)$$

admits the function $\gamma(r)\log^{1/2}(1/r)$ as an almost-sure uniform modulus of continuity.

7 Finite and infinite-dimensional stochastic differential equations

A well-known difficulty with Skorohod stochastic integration w.r.t. fBm is that solving even the simplest non-linear differential equation is yet an open problem. There are two notable exceptions, however: the linear additive and the linear multiplicative equations, yielding the so-called fractional Ornstein-Uhlenbeck and Geometric fractional Brownian motion processes respectively. In this section, we show that this can be done for integration with respect to our processes B^γ. We keep our formulations to a minimal level of complexity. Additional non-linear terms in the drift parts can be considered using variants of the arguments given in some of the references cited in the introduction; we will not investigate these details here.

7.1 Finite-dimensional equations

Proposition 33 *Let γ and B^{γ} be fixed as in Proposition 1. Consider the stochastic differential equation*

$$X(t) = X_0 + \int_0^t bX(s)\,ds + \int_0^t \sigma X(s)\,\delta B^{\gamma}(s), \quad t \geq 0, \tag{28}$$

where X_0, b, σ are fixed non-random constants, and where $\int_0^t \sigma X(s)\,\delta B^{\gamma}(s)$ represents the Skorohod integral $\delta\left(\mathbf{1}_{[0,t]}(\cdot)\,\sigma X\right)$ as in Definition 25.

This linear multiplicative stochastic differential equation (28) has a solution given by the following geometric γ-Brownian motion *(GγBm):*

$$X(t) = X_0 \exp\left(\sigma B^{\gamma}(t) + bt - \frac{1}{2}\sigma^2\gamma^2(t)\right). \tag{29}$$

This solution is unique in the class of processes Z such that $Z(t) = g(t, B^{\gamma}(t))$ where g is a deterministic function in $C^{1,2}$ satisfying the conditions of Theorem 27 uniformly in t.

Proof. Ito's formula (Theorem 27) can be extended to include functions that depend also on time. This can be proven by approximation of such functions with respect to the time parameter. We omit the details. We thus have for any function f of class $C^{1,2}$ on $\mathbf{R}_+ \times \mathbf{R}$ satisfying the hypotheses of Theorem 27 with respect to the second parameter uniformly in the first parameter, that $\frac{\partial f}{\partial x}(\cdot, B^{\gamma}(\cdot))\,\mathbf{1}_{[0,t]}$ is in $Dom^*\delta$ and for all $t \geq 0$:

$$\begin{aligned} f(t, B^{\gamma}(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^{\gamma}(s))\,ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, B^{\gamma}(s))\,\delta B^{\gamma}(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{\gamma}(s))\,\gamma\gamma'(s)\,ds. \end{aligned} \tag{30}$$

We apply this with $f(t,x) = X_0 \exp\left(\sigma x + bt - (1/2)\,\sigma^2\gamma^2(t)\right)$ which immediately yields equation 28.

For the uniqueness, let Y be another solution to (28). Since $Y(t) = g(t, B^{\gamma}(t))$ for some g, we can use Ito's formula to show that for any function h such that $h \circ g$ is of class $C^{1,2}$ and satisfies the conditions of Theorem 27 uniformly in t, the following version of Ito holds for Y:

$$h(Y(t)) = f(Y(0)) + \int_0^t h'(Y(s))\,\delta Y(s) + \frac{1}{2}\int_0^t h''(Y(s))\,d[Y,Y]_{\gamma}(s),$$

where the notations $\delta Y(s)$ and $d[Y,Y]_{\gamma}(s)$ are defined as follows:

$$\begin{aligned} \delta Y(s) &:= g'(B^{\gamma}(s))\,\delta B^{\gamma}(s) + g''(B^{\gamma}(s))\,\gamma\gamma'(s)\,ds, \\ d[Y,Y]_{\gamma}(s) &:= \left|g'(B^{\gamma}(s))\right|^2 2\gamma\gamma'(s)\,ds. \end{aligned}$$

With this Ito formula in hand, we can now consider the processes $U = \log X$ and $V = \log Y$. A trivial calculation then yields that both U and V are solutions of the following equation in Z:

$$Z(t) = \log X_0 + \sigma B^{\gamma}(t) + bt - \sigma^2 \int_0^t \gamma\gamma'(s)\,ds.$$

Obviously, this is a trivial equation since the right-hand side is explicit, which proves uniqueness. ■

Remark 34 *We are not able to find a simple proof of uniqueness for the above equation in a wider space, because of the restrictive range of our Ito formula, valid only for deterministic functions of B^{γ}. Extending the validity of the Ito formula will be the subject of another article.*

Proposition 35 *Let γ and B^{γ} be fixed as in Proposition 1. Consider the stochastic differential equation*

$$X(t) = X_0 + \int_0^t aX(s)\,ds + B^{\gamma}(t), \quad t \geq 0, \tag{31}$$

where X_0 and a are fixed non-random constants.

*This linear additive stochastic differential equation (28) has a solution given by the following γ-*Ornstein-Uhlenbeck *process (γOU):*

$$X(t) = X_0 e^{at} + \int_0^t e^{a(t-s)}\delta B^{\gamma}(s). \tag{32}$$

where the last integral is in the Wiener sense (section 3). The solution is unique, up to indistinguishability, in the class of all separable processes in $L^2(\Omega)$.

Proof. Although this proposition can be considered as a consequence of the results presented in the next section, we include a quick proof for completeness. First note that the Wiener integral in (32) is well-defined since the smooth function $\exp(-as)$ is obviously in $\mathcal{H}$ which contains any C^{α}, $\alpha < 1/2$. Assume that X exists satisfying (31). Then we define Y by $Y(t) = \exp(-at)X(t)$. Then Y is the sum of a differentiable process $ae^{-at}\int_0^t X(s)\,ds$ and of $e^{-at}B^{\gamma}(t)$ which is of the form $g(t, B^{\gamma}(t))$ where g is deterministic. By Ito's formula (30) and equation (31) we see that

$$\begin{aligned} Y(t) = X_0 - \int_0^t ae^{-as}a\left[\int_0^s X(r)\,dr\right]ds - a\int_0^t e^{-as}B^{\gamma}(s)\,ds \\ + \int_0^t e^{-as}\left(aX(s)\,ds + \delta B^{\gamma}(s)\right). \end{aligned}$$

Now from (31) again, we may replace $a\left[\int_0^s X(r)\,dr\right]$ by $X(s) - B^{\gamma}(s)$, yielding simply

$$Y(t) = X_0 + \int_0^t e^{-as}\delta B^{\gamma}(s),$$

which proves uniqueness. This same calculation also shows that X given by (32) solves (31). ■

7.2 Stochastic heat equations

An equation is commonly called the stochastic heat equation on $\mathbf{R}$ if it is of the form

$$u(t,x) = u_0(x) + \int_0^t \Delta_x u(s,x)\,ds + \int_0^t \sigma(u(s,x))\,dW(s) \tag{33}$$

for some Gaussian noise term W and some possibly non-linear function σ. As announced above, because of the difficulties inherent in Skorohod integration, we restrict ourselves to $\sigma = Id$ or $\sigma = 1$. For the case $\sigma = Id$, we present our results as conjectures.

The additive stochastic heat equation

$$u(t,x) = u_0(x) + \int_0^t \Delta_x u(s,x)\,ds + B^{\gamma}(t,x) \tag{34}$$

can be interpreted in its *evolution* form, as is often done, in the manner of Da Prato and Zabczyk [8], as

$$u(t,x) = P_t u_0(x) + \int_0^t P_{t-s}B^{\gamma}(\delta s, \cdot)(x). \tag{35}$$

where $B^{\gamma}(t,\cdot)$ is an infinite-dimensional version of our $B^{\gamma}(t)$. Obviously here, since the right-hand side of the equation does not contain u, this u given by (35) *is* the unique (evolution) solution to (34), when it exists. It is well-known however that u may exist even if a strong-sense solution of (34) fails to exist, hence the use of the terminology *evolution solution*.

To be specific, let us assume B^γ is a centered Gaussian random field on $\mathbf{R}_+ \times S^1$ where S^1 is the circle (parametrized by $[0, 2\pi)$) with a given covariance structure Q in space and the same behavior as our one-dimensional B^γ defined in Proposition 1. In other words it can be written as

$$B^\gamma (t, x) = \int_0^t \varepsilon (t - s) W (ds, x)$$

where W has covariance $\mathbf{E} [W (t, x) W (s, y)] = Q (x, y) \min (s, t)$. The operator P is the semigroup generated by Δ. In other words, for any test function f in $L^2 (\mathbf{R})$

$$P_t f (x) = \int_{\mathbf{R}} (2\pi t)^{-1/2} \exp \left(- (x - y)^2 / (2t) \right) f (y) \, dy.$$

The notation $\int_0^t P_{t-s} B^\gamma (\delta s, \cdot) (x)$ is best understood if W (and consequently B^γ) can be expanded in a basis of a convenient space of functions. We use the trigonometric basis for $L^2 (\mathbf{R})$, which are also the set of eigenfunctions of the Laplacian Δ. To make matters as simple as possible, we assume that $u_0 = 0$ and that W is spatially homogeneous. In this case, we know W can be expanded along the trigonometric basis, with identical coefficients for like sine and cosine terms. Consequently we have

$$B^\gamma (t, x) = \sqrt{q_0} B_0^\gamma (t) + \sum_{n=1}^{\infty} \sqrt{q_n} \bar{B}_n^\gamma (t) \sin (nx) + \sum_{n=1}^{\infty} \sqrt{q_n} B_n^\gamma (t) \cos (nx)$$

where $(B_n^\gamma)_n$ and $\left(\bar{B}_n^\gamma \right)_n$ are independent families of independent copies of the B^γ in Proposition 1, and $(q_n)_n$ is a sequence of non-negative numbers. Since $\sin nx$ and $\cos nx$ share the eigenvalue n^2 with respect to Δ, they have the eigenvalue $\exp \left(-n^2 t \right)$ for P_t. Consequently, we can rewrite (35) as

$$\begin{aligned} u (t, x) = & \sqrt{q_0} \int_0^t e^{-(t-s)n^2} B_0^\gamma (\delta s) \\ & + \sum_{n=1}^{\infty} \sqrt{q_n} \cos (nx) \int_0^t B_n^\gamma (\delta s) e^{-(t-s)n^2} \\ & + \sum_{n=1}^{\infty} \sqrt{q_n} \sin (nx) \int_0^t \bar{B}_n^\gamma (\delta s) e^{-(t-s)n^2}. \end{aligned}$$

This shows in particular that for fixed t, $u (t, \cdot)$ is a homogeneous Gaussian process. It is worth noting that the above solution u may exist as a bona-fide Gaussian process even if $B^\gamma (t, \cdot)$ is not a bona-fide process in the space variable. The random element $B^\gamma (t, \cdot)$ may be generalized-function-valued (Schwartz-distribution-valued). For a precise description of such an object, the reader is referred to Section 3, and in particular Paragraph 3.1., in [16]. However, it is enough to notice that if $\sum q_n = \infty$ then $B^\gamma (t, \cdot)$ is a random generalized function. The next theorem gives a precise result in this direction.

Theorem 36 *Let γ and ε be as in Proposition 1. Assume moreover that ε' satisfies*

$$-\varepsilon' (r) = |\varepsilon' (r)| \asymp r^{-3/2} f (r)$$

where f is an increasing differentiable function. Also define

$$F (x) = \int_0^x \left(\frac{1}{s} \int_0^s \varepsilon (r) \, dr \right)^2 ds.$$

Assume also that f satisfies the following technical assumptions:

1. *f'/f is bounded by λ on the interval $[1/(4\lambda), 1]$;*

2. *for some $a > 0$, for all $r \leq a$, $f'(r) \leq f(r)/(2r)$;*

3. *with $g(x) = (xf'(x) f(x))^{1/2}$, $g'(r) \leq g(r)/(2r)$.*

Then the evolution solution $u(t,x)$ to equation (34) exists and is unique as a random field in $L^2(\Omega \times [0,t] \times S^1)$ as soon as

$$\sum_{n=1}^{\infty} q_n \left(F\left(n^{-2}\right) + f^2\left(n^{-2}\right)\right) < \infty. \tag{36}$$

The second statement in the following corollary shows that the above theorem is sharp, since it reproduces the sufficient condition of [16] which was also shown therein to be necessary in the case of fBm itself. It also shows that in the two basic scales of regularity, the functions F and f^2 are commensurate.

Corollary 37 *The functions f and F can be estimated in the cases of logBm and fBm scales. Specifically we have that in the following two cases, the technical conditions on γ and ε all hold, and the theorem translates as follows:*

- logarithmic Brownian scale. *If $\varepsilon'(r) \asymp r^{-3/2} \log^{-(\beta+1)/2}(r)$ for some $\beta > 1/2$, so that $\gamma(r) \asymp \log^{-\beta/2}(r)$, then (36) can be replaced by*

$$\sum_{n=1}^{\infty} q_n \log^{-(\beta+1)}(n) < \infty.$$

- fractional Brownian scale. *If $\varepsilon'(r) \asymp r^{H-3/2}$ for $H \in (0, 1/2]$, so that $\gamma(r) \asymp r^H$, then (36) can be replaced by*

$$\sum_{n=1}^{\infty} q_n n^{-4H} < \infty.$$

Proof. (Theorem 36). The proof's structure is identical to the general theorem relative to infinite-dimensional fBm in [16], proved in Section 3.3 therein. We give only the main difference in the calculation. It is regarding, in the notation of [16], the bounding of the term $I_2(\lambda, t)$. In our context, one can check that the only relevant values of λ are $\lambda = n^2, n \in \mathbf{N}$, and that we have

$$I_2(\lambda, t) = \int_0^t e^{-2\lambda s} \left(\int_0^s \left(e^{\lambda r} - 1\right) \varepsilon'(r)\, dr\right)^2 ds.$$

We will assume in this proof that $t \leq 1$, and we will indeed replace t by this value for all upper bounds below. More generally, to be able to consider arbitrary bounded intervals $[0, T]$, cases such as logBm must be modified in consequence to ensure that γ is defined and bounded on such intervals, e.g. in the case of logBm by replacing $\gamma(r)$ by $\gamma(r/2T)$ say. The results in the theorem hold for the existence of u for all $t \in [0, \infty)$ as long as one begins with a locally bounded γ. We omit the details.

We rewrite

$$\begin{aligned} I_2(\lambda, t) \leq I_2(\lambda, 1) &= \int_0^{1/\lambda} e^{-2\lambda s} \left(\int_0^s \left(e^{\lambda r} - 1\right) \varepsilon'(r)\, dr\right)^2 ds \\ &+ \int_{1/\lambda}^1 e^{-2\lambda s} \left(\int_0^{1/\lambda} \left(e^{\lambda r} - 1\right) \varepsilon'(r)\, dr\right)^2 ds \\ &+ \int_{1/\lambda}^1 e^{-2\lambda s} \left(\int_{1/\lambda}^s \left(e^{\lambda r} - 1\right) \varepsilon'(r)\, dr\right)^2 ds \\ &:= I_{2,0}(\lambda) + I_{2,1}(\lambda) + I_{2,2}(\lambda). \end{aligned}$$

The first term $I_{2,0}(\lambda)$ is controlled as follows. Up to universal constants, we bound $e^{\lambda r}-1$ above by λr, and $e^{-2\lambda s}$ by 1, yielding:

$$I_{2,0}(\lambda) \leq \int_0^{1/\lambda} \lambda^2 \left(\int_0^s r \left| \varepsilon'(r) \right| dr \right)^2 ds.$$

By integration by parts we get that

$$\int_0^s r \left| \varepsilon'(r) \right| dr = \int_0^s \varepsilon(r)\, dr - s\varepsilon(s) + \lim_{h \to 0} h\varepsilon(h).$$

The limit above is 0 since $\varepsilon(r) \ll r^{-1/2}$. Since ε is decreasing, $\int_0^s \varepsilon(r)\, dr$ exceeds $s\varepsilon(s)$, and thus we decide to ignore the smaller of the two, yielding an upper bound. It follows that

$$\begin{aligned} I_{2,0}(\lambda) &\leq \int_0^{1/\lambda} \lambda^2 \left(\int_0^s \varepsilon(r)\, dr \right)^2 ds \\ &\leq \int_0^{1/\lambda} \left(\frac{1}{s} \int_0^s \varepsilon(r)\, dr \right)^2 ds, \end{aligned}$$

which is the required estimate.

For the second term, using the integration by parts calculation above,

$$\begin{aligned} I_{2,1}(\lambda) &\leq \int_{1/\lambda}^1 e^{-2s\lambda} \left(\int_0^{1/\lambda} \lambda r \left| \varepsilon'(r) \right| dr \right)^2 ds \\ &\leq \int_{1/\lambda}^1 e^{-2s\lambda} \left(\lambda \int_0^{1/\lambda} \varepsilon(r)\, dr \right)^2 ds \\ &= \frac{1}{2\lambda} \left(e^{-2} - e^{-2\lambda} \right) \left(\lambda \int_0^{1/\lambda} \varepsilon(r)\, dr \right)^2 \\ &\leq \int_0^{1/\lambda} \left(\frac{1}{s} \int_0^s \varepsilon(r)\, dr \right)^2 ds, \end{aligned}$$

where the last inequality comes from the fact that the function $h(s) = s^{-1} \int_0^s \varepsilon(r)\, dr$ is decreasing on $[0, 1/\lambda]$. This fact can be see as follows: $h'(s) = s^{-2} \left(s\varepsilon(s) - \int_0^s \varepsilon(r)\, dr \right) < 0$ since ε itself is decreasing.

The last term can be rewritten using a scalar change of variables, and then integration by parts, as follows:

$$I_{2,2}(\lambda) \leq \lambda^{-3} \int_1^{\lambda} e^{-2s} \left(\int_1^s e^r \left| \varepsilon'\left(\frac{r}{\lambda} \right) \right| dr \right)^2 ds$$

Now we use the representation $\left| \varepsilon'(r) \right| \asymp r^{-3/2} f(r)$ with f differentiable and increasing, and $|\varepsilon'|$ decreasing:

$$I_{2,2}(\lambda) \leq \int_1^{\lambda} e^{-2s} \left(\int_1^s e^r r^{-3/2} f\left(\frac{r}{\lambda} \right) dr \right)^2 ds.$$

We decompose the inside integral into three parts, and exploit the monotonicity of the integrands in each corresponding interval:

$$\begin{aligned} &\int_1^s e^r r^{-3/2} f\left(\frac{r}{\lambda} \right) dr \\ &= \int_1^{s/4} e^r r^{-3/2} f\left(\frac{r}{\lambda} \right) dr + \int_{s/4}^{s/2} e^r r^{-3/2} f\left(\frac{r}{\lambda} \right) dr + \int_{s/2}^{s} e^r r^{-3/2} f\left(\frac{r}{\lambda} \right) dr \\ &\leq e^{s/2} f\left(\frac{s}{4\lambda} \right) + e^{s/2} \left(\frac{s}{4} \right)^{-3/2} f\left(\frac{s}{2\lambda} \right) + e^s \left(\frac{s}{2} \right)^{-3/2} f\left(\frac{s}{2\lambda} \right) \\ &= I_{2,2,0}(s,\lambda) + I_{2,2,1}(s,\lambda) + I_{2,2,2}(s,\lambda). \end{aligned}$$

To deal with $I_{2,2,0}(s,\lambda)$, we first integrating by parts, using the condition that f'/f is bounded by λ on the interval $[1/(4\lambda), 1]$:

$$\begin{aligned}
I_{2,2,0}(\lambda) &:= \int_1^\lambda I_{2,2,0}(s,\lambda)^2 e^{-2s} ds = \int_1^\lambda e^{-s} f^2\left(\frac{s}{4\lambda}\right) ds \\
&= e^{-1} f^2\left(\frac{1}{4\lambda}\right) - e^{-\lambda} f^2(1/4) + \frac{1}{4\lambda}\int_1^\lambda e^{-s} 2f\left(\frac{s}{4\lambda}\right) f'\left(\frac{s}{4\lambda}\right) ds \\
&\leq e^{-1} f^2\left(\frac{1}{4\lambda}\right) + \frac{1}{2}\int_1^\lambda e^{-2} f^2\left(\frac{s}{4\lambda}\right) ds \\
&= e^{-1} f^2\left(\frac{1}{4\lambda}\right) + \frac{1}{2} I_{2,2,0}(\lambda).
\end{aligned}$$

This implies that

$$I_{2,2,0}(\lambda) \leq f^2\left(\frac{1}{4\lambda}\right).$$

For the next term we can see that can be dealt with exactly as $I_{2,2,0}(\lambda)$; in fact, it is of smaller order because of the factor s^{-3}.

The last term is more delicate. We begin by noting that

$$\begin{aligned}
I_{2,2,2}(\lambda) &:= \int_1^\lambda I_{2,2,0}(s,\lambda)^2 e^{-2s} ds = 8\int_1^\lambda s^{-3} f^2\left(\frac{s}{2\lambda}\right) ds \\
&= 8\lambda^{-2}\int_{1/\lambda}^1 r^{-3} f^2(r/2)\, dr.
\end{aligned}$$

To bound this quantity, we introduce a modified version of it: for fixed $a > 0$, we let

$$I_g^a(x) := x^2 \int_x^1 r^{-3} g^2(r)\, dr$$

where the function g will be chosen to be equal to $g(x) = (xf'(x) f(x))^{1/2}$. With this choice we do see that according to the hypotheses of the theorem, for some $a > 0$, assuming $\lambda > a^{-1}$, for all $r \leq a$, $g'(r) \leq g(r)/(2r)$. We calculate I_g^a by integration by parts, using the parts $v = r^{-2}$ and $du = r^{-1}g^2(r)$ so that $u(r) = \int_0^r y^{-1} g^2(y)$ and $dv = -2r^{-3}dr$:

$$I_g^a(x) = x^2 u(1) - u(x) + x^2 \int_x^1 r^{-3} 2u(r)\, dr. \tag{37}$$

Now let $J_g^a(x) := x^2 \int_x^1 r^{-3} 2u(r)\, dr$. By hypothesis, $2g(r)g'(r) \leq g^2(r)/r$ for all $r \in [0,a]$, which implies for all such r that

$$g^2(r) - g(0) = \int_0^r 2g(y) g'(y)\, dy \leq u(r).$$

We have that $g(0) = \lim_0 g = 0$. Indeed, by hypothesis, $g^2(x) = 2xf'(x) f(x) \leq f^2(x)$ which tends to 0 at 0. This implies

$$I_g^a(x) \leq x^2 \int_x^1 r^{-3} u(r)\, dr = \frac{1}{2} J_g^a(x). \tag{38}$$

Combining (37) and (38) we obtain

$$\begin{aligned}
J_g^a(x) &= I_g^a(x) - x^2 u(1) + u(x) \\
&\leq \frac{1}{2} J_g^a(x) - x^2 u(1) + u(x)
\end{aligned}$$

which implies

$$J_g^a(x) \leq 2u(x). \tag{39}$$

Returning now to the definition of J_g^a and u we have

$$\begin{aligned} u(x) &= \int_0^x r^{-1} g^2(r)\, dr \\ &= \int_0^x r^{-1} r f'(r) f(r)\, dr \\ &= \frac{1}{2} f^2(x) \end{aligned} \tag{40}$$

and

$$J_g^a(x) = x^2 \int_x^1 r^{-3} f^2(r)\, dr.$$

With $x = \lambda^{-1}$, we recognize a piece of the integral defining $I_{2,2,2}(\lambda)$. In fact we have by (39) and (40) that

$$\begin{aligned} I_{2,2,2}(\lambda) &= 8\lambda^{-2} \int_{1/\lambda}^1 r^{-3} f^2(r/2)\, dr \\ &\leq 8\lambda^{-2} \int_{1/\lambda}^1 r^{-3} f^2(r)\, dr \\ &= 8J_g^a\left(\lambda^{-1}\right) + 8\lambda^{-2} \int_a^1 r^{-3} f^2(r)\, dr \\ &\leq 4f^2\left(\frac{1}{\lambda}\right) + \frac{1}{\lambda^2} K_f. \end{aligned}$$

where K_f is a constant depending only on f. The second term above is negligible compared to the first, since we know that $f(r) \gg r^{-1/2}$.

In conclusion we have for large λ and for all $t \leq 1$, with $F(x) = \int_0^x \left(\frac{1}{s}\int_0^s \varepsilon(r)\, dr\right)^2 ds$,

$$I_2(\lambda, t) \leq F\left(\frac{1}{\lambda}\right) + 48f^2\left(\frac{1}{\lambda}\right),$$

which is the result required to obtain the first statement of the theorem.

(Proof of the Corollary). The statements in the corollary regarding the fBm and logBm scales are readily verified by trivial estimations of f and F in these cases. ■

We finish this article by mentioning a conjecture on the multiplicative stochastic heat equation. This is the case $\sigma(u) = u$.

Conjecture 38 *The evolution for of equation (33) with $\sigma(u) = u$, namely*

$$u(t,x) = P_t u_0(x) + \int_0^t P_{t-s}\left[B^\gamma(\delta s, \cdot)\, u(s,\cdot)\right](x),$$

has a unique solution in $L^2\left(\Omega \times [0,t] \times S^1\right)$ as soon as $\sum q_n < \infty$, and it is given by the following Feynman-Kac formula:

$$u(t,x) = \mathbf{E}^b\left[u_0(x + b_t) \exp\left(\int_0^t B^\gamma(\delta r, x + b_t - b_r) - Q(0)\gamma^2(t)/2\right)\right],$$

where b is a standard Brownian motion independent of B^γ and $\mathbf{E}^b$ is the expectation with respect to b.

A joint paper in preparation by one of the two authors of this paper establishes this Feynman-Kac formula for fBm in the case of $H > 1/2$. It uses a Wiener chaos decomposition and some associated estimates. We do not believe that these estimates are yet available for $H < 1/2$, making proving the above conjecture non-trivial, although the result is readily believable.

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