Local Risk Minimization with Jumps: An Asymptotic Approach

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Abstract

We study the local risk minimization hedging strategy when the model has only jumps with Poisson arrival rate. Although models with jumps can be nicely justified in economic point of view, the actual computation of an appropriate hedging strategy is not easy in general. In this paper, we find a closed form hedging strategy that is easily computed, using an asymptotic approach. We call it as the asymptotic local risk minimization strategy. We also compare its hedging error to those of other quadratic hedging methods. By numerical examples, we see that the asymptotic local risk minimization strategy has smaller error in quadratic sense than Black-Scholes hedging strategy.

1 Introduction

Although the Black-Scholes model [1] has been standard in mathematical finance literature for decades, its normality assumption on return distributions has been noticed as a primary drawback. Empirical studies (e.g. Mandelbrot [11] and Fama [5]) showed that the actual return distributions had heavier tails than those of the normal distribution. Asymmetry of the return distribution was also often observed in empirical studies, (e.g. Morgan [13] and Richardson and Smith [16]) which give even more reasons to investigate alternatives. One possible approach to an alternative is adding jumps to price processes, as in Merton [12]. Unfortunately, adding jumps, in general, leads a market to an incomplete world, where the standard hedging method in a complete market becomes inapplicable.

On the hedging problem in incomplete markets, the local risk minimization and the mean-variance hedging have been two major quadratic hedging approaches. The local risk minimization sacrifies the self-financing property, but its terminal value is the same as the payoff of a contingent claim. On the other hand, the mean-variance hedging focuses on the self-financing property. Föllmer and Sondermann [3] studied the risk minimization when the asset price process is a martingale under the original measure,

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and Föllmer and Schweizer [4] and Schweizer [14] studied the local risk minimization for a general semimartingale case. Schweizer [15] provided the solution to the mean-variance hedging for general claims with continuous price processes. While mean-variance hedging gives a control over the total risk, the local risk minimization often gives a simpler hedging strategy. (See Heath, Platen, and Schweizer [7], for example.) There have been many studies on the two above quadratic criteria in an incomplete market since they had been proposed. To name a few, Frey [6] studied a risk minimizing strategy when the price process is a pure jump process with a stochastic jump rate and a martingale under the original measure. Chan [2] found a local risk minimizing strategy when the price process is driven by general Lévy processes. Lee [10] found the local risk minimizing strategy when the price process has jumps with instantaneous feedback from the current price.

The purpose of this paper is to provide a closed form of an asymptotic local risk minimization strategy when the price process has jumps with possibly asymmetric return distribution. We assume that the underlying price process is a compound Poisson process. Jumps occur at random times and the jump size can follow an asymmetric or a fat-tailed distribution. In particular, we consider a sequence of compound Poisson processes whose limit goes to the Black-Scholes model as the jump intensity goes to infinity. Under this model, we try to find the local risk minimization hedging strategy. A presence of jumps makes it difficult to get a closed form of a hedging strategy, even in this relatively simple case. Thus, instead of finding an exact form, we calculate an asymptotic local risk minimization hedging strategy. The main result is in Section 3. We also show some numerical comparisons to other hedging strategies. As discussed in Heath, Platen, and Schweizer [7], the local risk minimization is not usually the best choice in terms of the quadratic hedging error, but it gives a rather simple and easy form of calculation.

In Section 2, we explain our model. We show some theoretical results of an asymptotic local risk minimization in Section 3. Section 4 provides results of a simulation experiment. We see that the asymptotic local risk minimization strategy improves the mean square hedging error from the traditional Black-Scholes hedging strategy. Although its hedging error is slightly bigger than that of the compound Poisson hedging strategy by Song and Mykland [17], the asymptotic local risk minimization strategy has a simpler form and involves less complicated asymptotics. See Section 4 for more details on the compound Poisson hedging strategy.

2 Model

We consider a sequence of price processes $S^{(n)}$ defined on probability spaces $(\Omega, \mathcal{F}^{(n)}, \mathbf{P}^{(n)})$ with filtrations $\{\mathcal{F}_t^{(n)}, t \geq 0\}$ generated by $S^{(n)}$. We suppose that for each n, the log stock price process follows a compound Poisson process under $\mathbf{P}^{(n)}$ such that

$$\log S_t^{(n)} = \log S_0^{(n)} + \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}, \tag{1}$$

where $N^{(n)}$ is a Poisson process with rate λ_n, and $Z_i^{(n)}$'s are i.i.d. random variables that are distributed as $Z^{(n)}$ and are independent of $N^{(n)}$. The initial stock price $S_0^{(n)}$ is the same as S_0 for all n. We assume that as n goes to ∞, λ_n goes to ∞ and $Z_i^{(n)}$ converges to 0 in distribution. $N^{(n)}$ models the occurrence of jumps and $Z^{(n)}$ models the size of jumps. In other words, when we have a stock on which a contingent claim is written, we model the stock price process as a compound Poisson process with a specific jump intensity, which is to be determined based on the frequency of trading activities. Then we construct a sequence of processes, indexed by n, including the orginal process. This is the sequence of compound Poisson processes that we introduced in (1). And we consider the asymptotics as n goes to infinity.

We define the jump size distribution more precisely as

$$Z^{(n)} = \frac{1}{\sqrt{\lambda_n}} Q + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2), \tag{2}$$

where Q is a random variable with mean 0, second moment σ^2 and the third moment k_3, under $\mathbf{P}^{(n)}$, for all n. Q has a distribution that does not depend on n and it has finite moments of all orders. We can interpret that μ and σ are the leading terms of the expected rate of return and the volatility, respectively, and $k_3/\sqrt{\lambda_n}$ is the leading term of the skewness of the underlying price process, as, for instance,

$$E(\log S_t^{(n)} - E(\log S_t^{(n)}))^3 = \frac{k_3 t}{\sqrt{\lambda_n}} + \frac{3\sigma^4 t}{\lambda_n}(\mu - \frac{1}{2}\sigma^2) + \frac{(\mu - \frac{1}{2}\sigma^2)^3 t}{\lambda_n^2}.$$

λ_n is related to the level of the trading activity of an individual stock. A heavily traded stock is modeled with a large value of λ_n, and a less heavily traded stock is modeled with a smaller value of λ_n.

We note that the allowance of nonzero k_3 enables us to consider asymmetric distributions of return processes, which is an advantage over models considering only symmetric return distributions. If we consider a second order asymptotics in Section 3, the kurtosis of the return distribution would be included in the proposed hedging strategy through $E(Q^4)$.

Notice that as n goes to ∞, $\log S^{(n)}$ converges in distribution to $\log S$ which is

$$\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t,$$

where B is a Brownian motion under the limiting measure $\mathbf{P}$.(Proposition 2.1 in Song and Mykland [17])

We can also write the model (1) as a form of a stochastic differential equation

$$dS_t^{(n)} = S_{t-}^{(n)} dR_t^{(n)}, \tag{3}$$

where $R_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} (\exp(Z_i^{(n)}) - 1)$. It is useful to find the canonical decomposition of $S^{(n)}$ for calculation. Let us define $\widetilde{R}_t^{(n)} = R_t^{(n)} - \lambda_n t E(e^{Z^{(n)}} - 1)$. Then we can easily show that $\widetilde{R}^{(n)}$ is a martingale under $\mathbf{P}^{(n)}$ and $S^{(n)}$ has the canonical decomposition

$$\begin{aligned} S_t^{(n)} &= S_0 + \int_0^t S_{s-}^{(n)} d\widetilde{R}_s^{(n)} + \int_0^t S_{s-}^{(n)} \lambda_n E(e^{Z^{(n)}} - 1) ds \\ &= S_0 + M_t^{(n)} + A_t^{(n)}, \end{aligned}$$

where $M_t^{(n)} = \int_0^t S_{s-}^{(n)} d\widetilde{R}_s^{(n)}$ is the martingale part and $A_t^{(n)} = \int_0^t S_{s-}^{(n)} \lambda_n E(e^{Z^{(n)}} - 1) ds$ is the predictable part.

3 The Local Risk Minimization

3.1 The Minimal Martingale Measure

In finding local risk minimization strategies, the minimal martingale measure plays a key role. For the definition and examples in a continuous semimartingale case, see Föllmer and Schweizer [4]. Chan [2] found the minimal martingale measure when the process is driven by Lévy processes. Lee [10] developed a method to find it when the process allows non-Lévy type jumps as well.

The following theorem is a version of Lee [10], modified to fit in our model. We need an additional condition

$$c^*(\exp(Z_i^{(n)}) - 1) < 1 \ a.s. \tag{4}$$

where $c^* = \frac{E(e^{Z^{(n)}}-1)}{E(e^{Z^{(n)}}-1)^2}$ to prevent a possibility of a signed measure. We remark that condition (4) is equivalent to $Z_i^{(n)} < \log(1 + \frac{1}{c^*})$ if $c^* > 0$, and $Z_i^{(n)} > \log(1 + \frac{1}{c^*})$ if $c^* < -1$. If $-1 \leq c^* \leq 0$, (4) is always satisfied.

Example 1 Suppose that Q in (2) is a binary random variable such as

$$Q = \begin{cases} \sigma\sqrt{\frac{1-p}{p}}, & \text{w.p. } p, \\ -\sigma\sqrt{\frac{p}{1-p}}, & \text{w.p. } 1-p. \end{cases}$$

Then $EQ = 0$, $EQ^2 = \sigma^2$, and $EQ^3 = \frac{\sigma^3(1-2p)}{\sqrt{(1-p)p}}$. If $p \neq \frac{1}{2}$, Q has a nonzero skewness. If $\mu = 0.08$, $\sigma = 0.4$, $p = \frac{1}{3}$, and $\lambda_n = 10$, then $c^*(\exp(Z^{(n)}) - 1)$ is either 0.0915 or

-0.0400 so that the condition (4) is satisfied. It is easy to check numerically that (4) holds for larger λ_n's.[1]

Theorem 1 *Suppose that $Z^{(n)}$ satisfies the condition (4). Define $Y^{(n)}$ such that*

$$Y_t^{(n)} = 1 - \int_0^t Y_{s-}^{(n)} c^* d\widetilde{R}_s^{(n)},$$

where $c^* = \frac{E(e^{Z^{(n)}}-1)}{E(e^{Z^{(n)}}-1)^2}$. Then, $Y_t^{(n)} > 0$and $E(Y_t) = 1$ for all $t \in (0,T]$. Furthermore, $\mathbf{Q}^{(n)}$ defined by $\frac{d\mathbf{Q}^{(n)}}{d\mathbf{P}^{(n)}} = Y_T^{(n)}$ is the unique minimal martingale measure of $S^{(n)}$.

Proof: Recall $\widetilde{R}_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} (e^{Z_i^{(n)}}-1) - \lambda_n t E(e^{Z^{(n)}}-1)$. $Y_t^{(n)}$ has a form of $1+\int_0^t Y_{s-}^{(n)} dX_s$, where $X_t = -c^* \widetilde{R}_t^{(n)}$. Then $Y^{(n)}$ is the stochastic exponential of X, and thus, we have

$$Y_t^{(n)} = \exp(c^* \lambda_n t E(e^{Z^{(n)}} - 1)) \prod_{i=1}^{N_t^{(n)}} (1 - c^* E(e^{Z_i^{(n)}} - 1)).$$

$Y_t^{(n)} > 0$ is obvious from the condition (4). Also, for all $t \in (0,T]$,

$$\begin{aligned} E(Y_t^{(n)}) &= \exp(c^* \lambda_n t E(e^{Z^{(n)}} - 1)) E(\prod_{i=1}^{N_t^{(n)}} (1 - c^* E(e^{Z_i^{(n)}} - 1))) \\ &= \exp(c^* \lambda_n t E(e^{Z^{(n)}} - 1)) E(E(\prod_{i=1}^{N_t^{(n)}} (1 - c^* E(e^{Z_i^{(n)}} - 1)) | N_t^{(n)})) \\ &= \exp(c^* \lambda_n t E(e^{Z^{(n)}} - 1)) \sum_{k=0}^{\infty} E(\prod_{i=1}^{N_t^{(n)}} (1 - c^* E(e^{Z_i^{(n)}} - 1)) | N_t^{(n)} = k) P(N_t^{(n)} = k) \\ &= 1. \end{aligned} \tag{5}$$

We follow analogous steps of Theorem 2 of Lee [10] to show the remaining part. ■

We recall some useful results of Föllmer and Schweizer [4]. They showed these results when the price process has continuous sample paths, but we can easilty check that these still hold when the price process is a special semimartingale with the canonical decomposition of the local martingale part and the predictable part. (See Lee [9].) Suppose that $H(S_T^{(n)})$ is our contingent claim and $M^{(n)}$ denotes the martingale part of $S^{(n)}$ under $\mathbf{P}^{(n)}$. We also assume that $V_t^{(n)} = E_{\mathbf{Q}^{(n)}}[H(S_T^{(n)}) | \mathcal{F}_t^{(n)}]$ has a decomposition $V_t^{(n)} = V_0 + \int_0^t \xi_s^H dS_s^{(n)} + L_t^{(n)}$, where $L^{(n)}$ is a square integrable $\mathbf{P}^{(n)}$-martingale that is

[1] Notice that $c^*(\exp(Z^{(n)}) - 1)$ is $O_p(\lambda_n^{-1/2})$. Thus, $P(c^*(\exp(Z^{(n)}) - 1) \geq 1)$ converges to 0 as n goes to ∞.

orthogonal to $M^{(n)}$ under $\mathbf{P}^{(n)}$. Then the local risk minimization strategy ξ^H exists and is obtained by

$$\xi_t^H = \frac{d\langle V^{(n)}, S^{(n)}\rangle_t}{d\langle S^{(n)}, S^{(n)}\rangle_t}, \tag{6}$$

where the quadratic variations are calculated under $\mathbf{P}^{(n)}$.

Therefore, with the decomposition of $V^{(n)}$, it remains to calculate $d\langle V^{(n)}, S^{(n)}\rangle_t$ and $d\langle S^{(n)}, S^{(n)}\rangle_t$. The latter is relatively simple, but the former, in general, does not give an explicit form for computation. Instead of finding its exact form, we try to approximate $d\langle V^{(n)}, S^{(n)}\rangle_t$ as the jump rate gets larger. It will be discussed in Section 3.2. We close this subsection with the existence theorem of the decomposition of $V^{(n)}$. The proof of the following theorem is analogous to Theorem 6 of Lee [10] and omitted here.

Theorem 2 *Let $M^{(n)}$ be the martingale part of $S^{(n)}$ and ξ^H be as above. Then, $V_t^{(n)} = E_{\mathbf{Q}^{(n)}}[H(S_T^{(n)})|\mathcal{F}_t^{(n)}]$ has a decomposition*

$$V_t^{(n)} = V_0 + \int_0^t \xi_s^H dS_s^{(n)} + L_t^{(n)},$$

where $L^{(n)}$ is a square integrable $\mathbf{P}^{(n)}$ martingale such that $\langle L^{(n)}, M^{(n)}\rangle_t = 0$ under $\mathbf{P}^{(n)}$. In other words, there exists the local risk minimization strategy.

3.2 An Asymptotic Local Risk Minimization

Let $H(S_T^{(n)})$ be a European style contingent claim that expires at time T. We assume that the interest rate r is 0, for simplicity. We denote $C(x,t)$ the solution of the Black-Scholes PDE at time $t < T$ with the terminal condition $C(x,T) = H(x)$. For each n, we calculate $C(S_t^{(n)}, t)$ with the corresponding stock price process $S^{(n)}$. In other words, $C(S_t^{(n)}, t)$ is calculated by the Black-Scholes PDE but is not the market price of the contingent claim. With further approximation, we can assume that $C(x,t)$ is infinitely differentiable with respect to the state variable. C_S, C_{SS}, and C_{SSS} denote the first, second, and third derivatives of $C(x,t)$ with respect to x, respectively. $C_S^{(p)}$ is used for the pth derivative of $C(x,t)$ with respect to x, for $p > 3$.

Theorem 3 *Suppose that for every positive integer v,*

$$\int_0^T E_{\mathbf{Q}^{(n)}}(C_S^{(v)}(S_{t-}^{(n)}, t)(S_{t-}^{(n)})^v)dt < \infty,$$

and

$$\int_t^T E_{\mathbf{Q}^{(n)}}(K_S^{(v)}(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^v|\mathcal{F}_t^{(n)})du < \infty,$$

where $K(x,t) = C_{SS}(x,t)$. The local risk minimization strategy of a European contingent claim, $C(S_T^{(n)}, T)$, is written as

$$\xi_t^H = C_S(S_{t-}^{(n)}, t) + \frac{1}{\sqrt{\lambda_n}} \frac{k_3}{2\sigma^2} C_{SS}(S_{t-}^{(n)}, t) S_{t-}^{(n)} + o(\lambda_n^{-1/2}).$$

Proof: To find the local risk minimizing strategy, it is enough to find $\langle V^{(n)}, S^{(n)} \rangle$ and $\langle S^{(n)}, S^{(n)} \rangle$, by (6). One can easily see

$$\langle S^{(n)}, S^{(n)} \rangle_t = E(\exp(Z^{(n)}) - 1)^2 \int_0^t (S_{u-}^{(n)})^2 \lambda_n du, \tag{7}$$

by the uniqueness of Doob-Meyer decomposition, (Karatzas and Shreve [8], p.24-25) because the second order optional variation of $S^{(n)}$ is

$$[S^{(n)}, S^{(n)}]_t = \sum_{i=1}^{N_t^{(n)}} (\Delta S_{\tau_i^{(n)}}^{(n)})^2 = \sum_{i=1}^{N_t^{(n)}} (S_{\tau_{i-}^{(n)}}^{(n)})^2 (\exp(Z_i^{(n)}) - 1)^2,$$

where $\tau_i^{(n)}$ is the time of the ith jump of $S^{(n)}$, and

$$\sum_{i=1}^{N_t^{(n)}} (S_{\tau_{i-}^{(n)}}^{(n)})^2 (\exp(Z_i^{(n)}) - 1)^2 - E(\exp(Z^{(n)}) - 1)^2 \int_0^t (S_{u-}^{(n)})^2 \lambda_n du$$

defines a martingale.

We can show that

$$V_0^{(n)} = E_{\mathbf{Q}^{(n)}} C(S_T^{(n)}, T) = C(S_0^{(n)}, 0) + \frac{k_3 T}{2\sqrt{\lambda_n}} (1 - \frac{\mu}{\sigma^2}) C_{SS}(S_0, 0) S_0^2 + \frac{k_3 T}{6\sqrt{\lambda_n}} C_{SSS}(S_0, 0) S_0^3 + o(\lambda_n^{-1/2}).$$

(See Song and Mykland [18] for details.) Similarly, for any $0 < t < T$,

$$\begin{aligned} V_t^{(n)} = E_{\mathbf{Q}^{(n)}}(C(S_T^{(n)}, T) | \mathcal{F}_t^{(n)}) = C(S_t^{(n)}, t) &+ \frac{k_3(T-t)}{2\sqrt{\lambda_n}} (1 - \frac{\mu}{\sigma^2}) C_{SS}(S_{t-}^{(n)}, t)(S_{t-}^{(n)})^2 \\ &+ \frac{k_3(T-t)}{6\sqrt{\lambda_n}} C_{SSS}(S_{t-}^{(n)}, t)(S_{t-}^{(n)})^3 + o(\lambda_n^{-1/2}), \end{aligned} \tag{8}$$

where $K(x,t) = C_{SS}(x,t)$. We use the notation $V_t^{(n)}$ for $E_{\mathbf{Q}^{(n)}}(C(S_T^{(n)}, T) | \mathcal{F}_t^{(n)})$ to be consistent with notations that we used in Section 3.1. From (8), it is clear that

$$\langle V_t^{(n)}, S_t^{(n)} \rangle = \langle C(S_t^{(n)}, t), S_t^{(n)} \rangle,$$

for all $t < T$.

On the other hand, by Itô's formula,

$$\begin{aligned}C(S_t^{(n)},t) &= C(S_0,0) + \int_0^t C_S(S_{u-}^{(n)},u)dS_u^{(n)} + \int_0^t C_t(S_{u-}^{(n)},u)du \\ &+ \frac{1}{2}\int_0^t C_{SS}(S_{u-}^{(n)},u)d[S^{(n)},S^{(n)}]_u^c \\ &+ \sum_{u\le t}(C(S_u^{(n)},u) - C(S_{u-}^{(n)},u) - C_S(S_{u-}^{(n)},u)\Delta S_u^{(n)}).\end{aligned}$$

Since $S^{(n)}$ is a pure jump process, it becomes

$$\begin{aligned}dC(S_t^{(n)},t) &= C_S(S_{t-}^{(n)},t)dS_t^{(n)} + C_t(S_{t-}^{(n)},t)dt + \Delta C(S_t^{(n)},t) - C_S(S_{t-}^{(n)},t)\Delta S_t^{(n)} \\ &= C_t(S_{t-}^{(n)},t)dt + \Delta C(S_t^{(n)},t).\end{aligned}$$

We apply the Taylor expansion to $C(S_t^{(n)},t)$ to get

$$\begin{aligned}dC(S_t^{(n)},t) &= C_t(S_{t-}^{(n)},t)dt + \sum_{v=1}^{\infty} C_S^{(v)}(S_{t-}^{(n)},t)(\Delta S_t^{(n)})^v\frac{1}{v!} \\ &= C_t(S_{t-}^{(n)},t)dt + \sum_{v=1}^{\infty} C_S^{(v)}(S_{t-}^{(n)},t)\frac{1}{v!}d[S^{(n)},\cdots,S^{(n)}]_t^v,\end{aligned}$$

where $[S^{(n)},\cdots,S^{(n)}]^v$ is the vth order optional variation of $S^{(n)}$. Again, since $S^{(n)}$ is a pure jump process,

$$[S^{(n)},\cdots,S^{(n)}]_t^v = \sum_{i=1}^{N_t^{(n)}}(\Delta S_{\tau_i^{(n)}}^{(n)})^v = \sum_{i=1}^{N_t^{(n)}}(S_{\tau_{i-}^{(n)}}^{(n)})^v(\exp(Z_i^{(n)})-1)^v,$$

where $\tau_i^{(n)}$ is the time of the ith jump of $S^{(n)}$. As in (7), by the uniqueness of Doob-Meyer decomposition,

$$\langle S^{(n)},\cdots,S^{(n)}\rangle_t^v = E(\exp(Z^{(n)})-1)^v\int_0^t (S_{u-}^{(n)})^v\lambda_n du.$$

Thus,

$$\begin{aligned}d\langle C(S_t^{(n)},t),S_t^{(n)}\rangle &= \sum_{v=1}^{\infty} C_S^{(v)}(S_{t-}^{(n)},t)\frac{1}{v!}d\langle S^{(n)},\cdots,S^{(n)}\rangle_t^{v+1} \\ &= \sum_{v=1}^{\infty} C_S^{(v)}(S_{t-}^{(n)},t)(S_{t-}^{(n)})^{v+1}\frac{\lambda_n}{v!}E(e^{Z^{(n)}}-1)^{v+1}dt.\end{aligned} \tag{9}$$

Note that

$$E(e^{Z(n)}-1)^2 = \frac{\sigma^2}{\lambda_n} + \frac{k_3}{\lambda_n^{3/2}} + o(\lambda_n^{-3/2}),$$
$$E(e^{Z(n)}-1)^3 = \frac{k_3}{\lambda_n^{3/2}} + o(\lambda_n^{-3/2}),$$
$$E(e^{Z(n)}-1)^p = o(\lambda_n^{-3/2}),$$

where p is an integer greater than 3. Combining this with (7) and (9), we get the locally risk minimizing strategy

$$\xi_t^H = C_S(S_{t-}^{(n)}, t) + \frac{1}{\sqrt{\lambda_n}}\frac{k_3}{2\sigma^2}C_{SS}(S_{t-}^{(n)}, t)S_{t-}^{(n)} + o(\lambda_n^{-1/2}).$$

This ends the proof. ■

We will call

$$\tilde{\xi}_t = C_S(S_{t-}^{(n)}, t) + \frac{1}{\sqrt{\lambda_n}}\frac{k_3}{2\sigma^2}C_{SS}(S_{t-}^{(n)}, t)S_{t-}^{(n)}$$

as the *asymptotic local risk minimization strategy of* $C(S_T^{(n)}, T)$.

The asymptotic local risk minimization strategy has the Black-Scholes delta hedging strategy as its leading term, which is sensible because the underlying stock price process converges in distribution to the geometric Brownian motion. As we noted earlier, the asymptotic local risk minimization strategy allows us to incorporate the nonzero skewness. The first order term $\frac{1}{\sqrt{\lambda_n}}\frac{k_3}{2\sigma^2}C_{SS}(S_{t-}^{(n)}, t)S_{t-}^{(n)}$ corrects the Black-Scholes strategy so that we can deal with asymmetric return distribution.[2] On the other hand, the expected rate of return is still not involved in the asymptotic local risk minimization. Considering that it is hard for us to estimate the expected rate of return from the data, it is an advantage that we can avoid this estimation problem.

4 Numerical Results

This section presents numerical results on the asymptotic local risk minimization strategy. Consider a European call option that expires in 3 months. The interest rate is assumed to be 0, μ is set to be 0.15 per annum, and σ is set to be 0.2 per annum. We try the strike price $K = \$65$ and the initial stock price $S_0 = \$60$. We use three different jump intensities, $\lambda_n = 1,000$, $10,000$, and $100,000$. $\lambda_n = 10,000$ means that we expect 10,000 jumps per year in the stock price on average. Larger λ_n implies that the stock is more heavily traded. The hedging interval is .0001 years which means that we rebalance the hedging portfolio once in approximately one hour.

[2]When $k_3 = 0$, the asymptotic local risk minimization strategy does not make any difference from the Black-Scholes hedging strategy. We follow the Black-Scholes delta hedging strategy when the distribution of the stock return is symmetric up to the order of $\sqrt{\lambda_n}$.

Table 1: Mean squares of hedging errors, unit=$\$^2$

	$\lambda_n = 1,000$	$\lambda_n = 10,000$	$\lambda_n = 100,000$
Black Scholes	0.059271	0.007787	0.002269
Asymptotic LRM	0.044443	0.006064	0.002098
Compound Poisson	0.037228	0.005408	0.001964

Any distribution with the moment conditions given in Section 2 can be used as the jump size distribution for the compound Poisson model. For example, Unif$(-\sqrt{3}\sigma, \sqrt{3}\sigma)$ can be used for the distribution of Q in (2) as a symmetric jump size case and $\sigma - \text{Exp}(\frac{1}{\sigma})$ can be used as a left skewed jump size case. We use $\sigma - \text{Exp}(\frac{1}{\sigma})$ as the distribution of Q in the simulation experiment. In this case, $k_3 = -2\sigma^3$. Note that it is easy to check that this jump size distribution satisfies the condition (4). The simulation size is 5,000, that is, the number of generated sample paths is 5,000 for each jump intensity. For reference, the Black-Scholes initial price is \$0.75.

We compare the performances of Black-Scholes and the asymptotic local risk minimization strategy obtained in Section 3.2 by calculating the mean squares of hedging errors (abbreviated by MSHE). For convenience of the comparison, we assume that the initial investments of both hedging strategies are the same, which is the Black-Scholes price. By hedging error, we mean the option payoff subtracted by the value of the hedging portfolio at the expiration. For example, MSHE of the asymptotic local risk minimization strategy is

$$E(H(S_T^{(n)}) - C(S_0, 0) - \int_0^T \tilde{\xi}_t dS_t^{(n)})^2,$$

and MSHE of the Black-Scholes strategy is

$$E(H(S_T^{(n)}) - C(S_0, 0) - \int_0^T C_S(S_{t-}^{(n)}, t) dS_t^{(n)})^2.$$

In general, both of the hedging strategies perform better as λ_n gets larger in terms of the magnitude of MSHE, because the stock price process is getting closer to a geometric Brownian Motion. The asymptotic local risk minimization strategy provides smaller MSHE than the Black-Scholes hedging strategy overall. It means that the value of the asymptotic risk minimization hedging portfolio at the expiration is closer to the payoff in the sense that the mean square of the difference is smaller. In the presence of asymmetry, it makes more sense to use the asymptotic local risk minimization than the Black-Scholes hedging strategy.

We also compare the asymptotic risk minimization hedging strategy with the compound Poisson hedging strategy proposed by Song and Mykland [17]. Let us call it as

η. Then

$$\eta_t = \tilde{\xi}_t + \frac{1}{\sqrt{\lambda_n}}(T-t)g_S(S_{t-}^{(n)}, t)$$
$$+ \frac{1}{\sqrt{\lambda_n}} \frac{\mu}{\sigma^2 S_{t-}^{(n)}} \left(\frac{S_{t-}^{(n)}}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)t} \left(\int_0^t \left(\frac{S_{v-}^{(n)}}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} dV_v^{(n)}\right), \quad (10)$$

where

$$g_S(x,t) = \frac{k_3}{6}\frac{\partial}{\partial x}\left(3x^2 C_{SS}(x,t) + x^3 C_{SSS}(x,t) - \frac{\mu}{2\sigma^2}x^2 C_{SS}(x,t)\right),$$
$$dV_t^{(n)} = dR_t^{(n)} - \frac{k_3}{2\sigma^2} S_{t-}^{(n)} C_{SS}(S_{t-}^{(n)}, t) dS_t^{(n)} - (T-t) g_S(S_{t-}^{(n)}, v) dS_t^{(n)},$$

and

$$R_t^{(n)} = \sqrt{\lambda_n}(C(S_t^{(n)}, t) - C(S_0, 0) - \int_0^t C_S(S_{u-}^{(n)}, u) dS_u^{(n)}).$$

The idea of the compound Poisson hedging strategy is that we keep the Black-Scholes hedging strategy as the leading term and find the correction term. They first found the law limit of the Black-Scholes hedging error and decomposed it to a part that can be replicated by trading the underlying stock and a part that is purely nonreplicable. They included the replicable part as a part of the correction term to the Black-Scholes hedging strategy, and then applied the mean-variance hedging method to the nonreplicable part to get the final correction term. They showed that the compound Poisson hedging strategy minimizes the mean square of the hedging error asymptotically. See Song and Mykland [17] for details. Because it minimizes the mean square of the hedging error in the limit, we expect that the asymptotic local risk minimization strategy would have larger MSHE than the compound Poisson strategy, which we can see in Table 1.

The absolute terms of the mean square of hedging errors in Table 1 are small, but the difference in MSHE between strategies is not negligible in terms of percentage. For example, the percentage gain in MSHE by using the asymptotic local risk minimization over the Black-Scholes when λ_n is 1000 is 25%. Moreover, if we have different values of parameters, then we may also obtain more reduction in absolute terms.

As mentioned earlier, the hedging error of the asymptotic local risk minimization strategy is slightly bigger than that of compound Poisson hedging strategy, which has a little more complicated form. However, asymptotics that we used to obtain two strategies are a bit different. We may say that we use a weaker type of asymptotics for the local risk minimization strategy. Compound Poisson hedging strategy is obtained by looking at the limit of the Black-Scholes hedging error. Thus, what to hedge in the first place is the limit of the hedging error. On the other hand, the asymptotic local risk minimization strategy is obtained by applying asymptotics to $E_{\mathbf{Q}^{(n)}}(H(S_T^{(n)}|\mathcal{F}_t^{(n)})$. What we try to hedge is the option payoff itself in this case. It turns out that the compound Poisson hedging strategy has the form of the asymptotic local risk minimization strategy plus some other terms, as in (10).

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