OPTIONS AND DISCONTINUITY: AN ASYMPTOTIC APPROACH FOR PRICING OPTIONS

by

Seongjoo Song and Per A. Mykland
Purdue University   University of Chicago

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Department of Statistics
Purdue University
West Lafayette, IN USA

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Seongjoo Song
Department of Statistics
Purdue University

Per A. Mykland
Department of Statistics
University of Chicago

Abstract

This paper studies the problem of option pricing in an incomplete market. The market incompleteness comes from the discontinuity of the underlying asset price process, specifically. Since the price of any contingent claim cannot be defined uniquely under the market incompleteness, we try to find some reasonable prices by adopting an asymptotic approach, letting securities prices converge to continuous processes.

Assuming that we use the compound Poisson hedging strategy proposed in Song and Mykland [10], three different choices for an option price are suggested in this paper as the form of the initial investment of the compound Poisson hedging strategy. The classical Black-Scholes price is a simple choice, and we study the case where we want to invest as little as possible with the expected squared loss bounded by a certain level. Thirdly, we study to find an equivalent martingale measure that converges to the minimal martingale measure. The expected value under this equivalent martingale measure is another possible amount to invest at the beginning. The choice between these possibilities is left for practitioners.

1 Introduction

Unlike other derivative securities such as futures or forwards, options give the owner the right to buy or sell the underlying securities, not the obligation. Therefore, options are traded at a certain price and the price of an option is an important issue in a financial market. Even for other derivative securities, how much we should invest to hedge the payoff completely is always an important and interesting issue. In a complete financial market, the price of any contingent claim is uniquely determined. It is the same as the initial investment of the replicating portfolio, and also the same as the expectation of the discounted payoff under the unique equivalent martingale measure. When it comes to an incomplete market, there are, in general, no replicating portfolios for a contingent claim, and there may be many different equivalent martingale measures. Therefore, the price of a contingent claim may not be determined uniquely.

Since we do not have the unique price for a contingent claim in an incomplete market, there are attempts to find the price range for the actual market price of a given contingent claim such as El Karoui and Quenez [3] and Eberlein and Jacod [2]. El Karoui and Quenez [3] determined the price range and showed that the maximum price is the smallest price that allows the seller to hedge completely by a controlled portfolio of the basic securities. They developed the optional version of the Doob-Meyer decomposition which holds simultaneously for all equivalent martingale measures. Kramkov [7] and Föllmer and Kramkov [4] studied more about the optional decomposition and its usage for hedging contingent claims under incompleteness. Eberlein and Jacod [2] found a price range for a general payoff function under the incompleteness due to discontinuity of the asset price process. They showed that the prices for such hedging strategies are impracticably high. Mykland [8] studied conservative hedging price under the existence of unavailable assets for hedging. In a case study of convex European options hedged in a stock and a zero coupon bond, he gave the explicit expression for the conservative hedging price using upper limit of the cumulative volatility and upper and lower limit of the cumulative interest rate. The upper bounds of a contingent claim appeared in literature guarantee that we hedge the payoff completely, but they provide very high starting prices for hedging strategies.

On the other hand, since there can be many different equivalent martingale measures in any incomplete market, there are attempts to choose a particular probability measure among them. He and Pearson [6] introduced the notion of minimax local martingale measure. Under this martingale measure, the investors do not want to hedge the unhedgeable uncertainty. Since the investor can increase his/her utility by hedging the unhedgeable risks, they choose the martingale measure to minimize the maximum attainable utility. Föllmer and Schweizer [5] introduced a minimal martingale measure, which is the martingale measure that preserves the structure of the real measure as far as possible. They constructed the unique optimal strategy that minimizes the intrinsic risk of a general claim using the minimal martingale measure.

In this paper, we study the pricing problem of a contingent claim in an incomplete market. The market incompleteness specifically comes from the discontinuity of the underlying asset price process. Section 2 describes the model of the underlying asset price process, and sections 3, 4, and 5 propose several possible prices for European style derivatives. The choice between suggested prices depends on the practitioner who needs to use the price. Section 3 suggests the classical Black-Scholes price, and section 4 studies the case where we want to invest as small as possible with a certain constraint. Section 5 studies a problem of choosing an adequate equivalent martingale measure under the discontinuity of the asset price process. Once we select an equivalent martingale measure, we can use the expected value of the discounted payoff of a contingent claim under this measure as the price of the contingent claim.

2 The Model

Consider a sequence of discontinuous processes that converges to a geometric Brownian Motion model. Each element of the sequence is a discontinuous process, indexed by n. n does not have any practical meaning, but it is used for asymptotic operation. A larger n means that the degree of discontinuity is smaller, *i.e.*, the process is closer to a geometric Brownian Motion model. Although we consider a sequence of processes, we observe only one price process with a certain degree of discontinuity from the market, for a given stock.

Let $(\Omega, \mathcal{F}^{(n)}, P^{(n)})$ be a probability space with $\{\mathcal{F}^{(n)}\}$ generated by the stock price process $S^{(n)}$ defined below. We suppose that for each n, the log stock price process follows a compound Poisson process under $P^{(n)}$ such that

$$\log S_t^{(n)} = \log S_0^{(n)} + \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}, \tag{1}$$

where $N^{(n)}$ is a Poisson process with rate λ_n, and $Z_i^{(n)}$'s are iid random variables that are independent of $N^{(n)}$. We assume the initial stock price $S_0^{(n)}$ is the same for all n. As n goes to ∞, λ_n goes to ∞ and $Z_i^{(n)}$ converges to 0 in distribution. $N_t^{(n)}$ is the number of jumps in the log stock price process up to time t, and each $Z_i^{(n)}$ represents the size of the ith jump of $\log S^{(n)}$. Since $\lambda_n t$ is the expected number of jumps up to time t, we can say that λ_n is the jump intensity of the log stock price process. Practically, λ_n is related with the level of the trading activity of an individual stock. A large λ_n corresponds to a heavily traded stock and a small λ_n corresponds to a rarely traded stock. We define the jump size distribution $Z^{(n)}$ more precisely as follows.

$$Z^{(n)} \stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{\lambda_n}}Q + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2), \tag{2}$$

where Q is a random variable with $EQ = 0$, $EQ^2 = \sigma^2$, $EQ^3 = k_3$, and $EQ^4 = k_4$, under $P^{(n)}$, for all n. For integers $p > 4$, we assume $E(|Q|^p) = o(\lambda_n^{-2+p/2})$. μ is a constant. It is clear that $Z^{(n)}$ converges to 0 in probability as well as in distribution as n goes to ∞, $E(Z^{(n)}) = \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)$, $E(Z^{(n)})^p = O(\lambda_n^{-p/2})$ for $p = 2, 3$ and 4, and $E(Z^{(n)})^p = o(\lambda_n^{-2})$ for $p > 4$. We can add $o(\lambda_n^{-1})$ term to $Z^{(n)}$ if we want, and it will not change anything. Consider the asymptotics as n goes to ∞. The conditions above assure that as n goes to ∞, $\log S^{(n)}$ converges in distribution to $\log S$ that is

$$\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \tag{3}$$

where B is a Brownian Motion under the limiting measure P. (See Song and Mykland [10].)

Suppose we want to hedge a European style payoff η that expires at time T. We assume that the interest rate r is 0 for simplicity. Let $C(S_t^{(n)}, t)$ and $C(S_t, t)$ denote the Black-Scholes option price at time t before the limit and in the limit, respectively. If X is the process of the value of the Black-Scholes hedging portfolio, then

$$X_t = C(S_0^{(n)}, 0) + \int_0^t C_S(S_{u-}^{(n)}, u) dS_u^{(n)}.$$

By Song and Mykland [10], the scaled Black-Scholes hedging error, $\sqrt{\lambda_n}(C(S_\cdot^{(n)}, \cdot) - X_\cdot)$ converges to a continuous process and its limit at time T is decomposed into two parts, so-called a replicable part and a non-replicable part. The replicable part is a stochastic integral with respect to the limiting stock price process and the non-replicable part is a stochastic integral with respect to a Brownian Motion that is independent of the stock price process under P. We denote the non-replicable part by $\int_0^T Y_u d\tilde{W}_u$ where the integrand Y is a function of S, and $\tilde{W}$ is independent of S under P. It can also be shown that $\tilde{W}$ is independent of S under P^*, which is the minimal martingale measure proposed by Föllmer and Scheweizer [5]. For more details, see Song and Mykland [10].

What can we do with the limit of the Black-Scholes hedging error for the hedging purpose? First, using the replicable part of the Black-Scholes hedging error, we update the Black-Scholes hedging strategy as

$$H_t^{(n)} = X_t + \frac{1}{\sqrt{\lambda_n}} \int_0^t \frac{k_3}{2\sigma^2} S_{u-}^{(n)} C_{SS}(S_{u-}^{(n)}, u) dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)}. \tag{4}$$

To deal with the non-replicable part of the Black-Scholes hedging error, we find the hedging portfolio, K, that makes the expected squared loss in the limiting market, $E(\int_0^T Y_u d\tilde{W}_u - K_T)^2$, minimized for a given initial value K_0. Taking the non-replicable part into account, we end up with the compound Poisson hedging strategy as follows. (See Song and Mykland [10])

$$\begin{aligned} L_t^{(n)} &= C(S_0, 0) + \frac{K_0}{\sqrt{\lambda_n}} + \int_0^t C_S(S_{u-}^{(n)}, u) dS_u^{(n)} \\ &+ \frac{1}{\sqrt{\lambda_n}} \int_0^t \frac{k_3}{2\sigma^2} S_{u-}^{(n)} C_{SS}(S_{u-}^{(n)}, u) dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)} \\ &+ \frac{1}{\sqrt{\lambda_n}} \int_0^t \frac{\mu}{\sigma^2 S_{u-}^{(n)}} \left(\frac{S_{u-}^{(n)}}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} \left(\int_0^u \left(\frac{S_{v-}^{(n)}}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} dV_v^{(n)} - K_0 \right) dS_u^{(n)}, \end{aligned} \tag{5}$$

where C_S, C_{SS}, and C_{SSS} denote the first, second, and third derivative of $C(S_t, t)$ with respect to S, respectively, $g(S_u, u) = \frac{1}{3!} k_3 \left(3S_u^2 C_{SS}(S_u, u) + S_u^3 C_{SSS}(S_u, u)\right) - \frac{k_3 \mu}{2\sigma^2} S_u^2 C_{SS}(S_u, u)$, g_S is the first derivative of g with respect to S, $dV_v^{(n)}$ denotes $dR_v^{(n)} - \frac{k_3}{2\sigma^2} S_{v-}^{(n)} C_{SS}(S_{v-}^{(n)}, v) dS_v^{(n)} - (T-v) g_S(S_{v-}^{(n)}, v) dS_v^{(n)}$ and $R_t^{(n)} = \sqrt{\lambda_n}(C(S_t^{(n)}, t) - X_t)$. The compound Poisson hedging strategy is determined uniquely for any given value of K_0, and its initial investment is the Black-Scholes price plus $K_0/\sqrt{\lambda_n}$. By choosing a reasonable value of K_0, we determine the initial investment of the compound Poisson hedging portfolio. At the same time, we can use it as the price of the European option. In the following sections, we try to find a reasonable value for K_0.

3 Black-Scholes price

To invest only the Black-Scholes price at the beginning is a good choice in several ways. First of all, it means that we choose $K_0 = 0$ in the compound Poisson hedging in (5) and it makes the hedging strategy simpler. Moreover, the investment of the Black-Scholes price minimizes the mean square of the limiting hedging error over the values of K_0, as follows.

As we saw in section 2, the limiting stock price process S follows a geometric Brownian Motion model such as $dS_t = \mu S_t dt + \sigma S_t dB_t$. And the minimal martingale measure, P^*, (See Föllmer and Schweizer [5]) is defined as

$$\frac{dP^*}{dP}|_{\mathcal{F}_t} = \exp\left(-\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right), \tag{6}$$

where $\{\mathcal{F}_t\}$ is a filtration generated by $(\tilde{W}, B)$. In the next proposition, we find the initial investment that makes the expected squared loss minimized.

Proposition 3.1 *Π is an $\mathcal{F}_T$-measurable random variable satisfying $\Pi \in \mathcal{L}^p(P)$ for some $p > 2$. Define $G(\theta)$ to be $\{\int_0^t \theta_u dS_u : 0 \leq t \leq T\}$ where θ is predictable with respect to $\mathcal{F}$ and $E(\int_0^T \theta_u^2 S_u^2 du) < \infty$. Then, the initial investment K_0 that minimizes $E(\Pi - K_0 - G_T(\theta))^2$ is $E^*(\Pi)$, where E^* is the expectation under the minimal martingale measure P^*.*

Proof. Define Θ to be a set $\{\theta :$ predictable with respect to $\mathcal{F},\ E(\int_0^T \theta_u^2 S_u^2 du) < \infty\}$, and define G^* to be $\{\int_0^t \theta_{K_0} dS_u\}$ where θ_{K_0} is the optimal hedging strategy minimizing $E(\Pi - K_0 - \int_0^T \theta_u dS_u)^2$ over Θ. To find K_0 that minimizes $\min_{\theta\in\Theta} E(\Pi - K_0 - G_T(\theta))^2$, differentiate $\min_{\theta\in\Theta} E(\Pi - K_0 - G_T(\theta))^2$ with respect to K_0 as follows.

$$\begin{aligned}\frac{\partial}{\partial K_0}\min_{\theta\in\Theta} E(\Pi - K_0 - G_T(\theta))^2 &= \frac{\partial}{\partial K_0} E(\Pi - K_0 - G_T^*)^2 \\ &= E(2(\Pi - K_0 - G_T^*)(-1 - \frac{\partial}{\partial K_0} G_T^*)).\end{aligned}$$

The differentiation under the expectation can be easily shown to be legitimate. By Schweizer [9], G^* is the solution of the stochastic differential equation

$$dG_t^* = (\hat{\theta}_t + \frac{\mu}{\sigma^2 S_t}(\hat{V}_t - K_0 - G_t^*))dS_t$$

with $G_0^* = 0$. $\hat{V}_t$ is defined as $E^*(\Pi|\mathcal{F}_t)$ and $\hat{V}_t$ can be written as

$$\hat{V}_t = \hat{\Pi}_0 + \int_0^t \hat{\theta}_u dS_u + \int_0^t \nu_u d\tilde{W}_u.$$

Schweizer only provides the above SDE, but it is easy to show that

$$G_t^* = e^{-A_t} \int_0^t e^{A_u} (\hat{\theta}_u + \frac{\mu}{\sigma^2 S_u}(\hat{V}_u - K_0))(dS_u + \mu S_u du)$$

where $e^{A_t} = (S_t/S_0)^{\mu/\sigma^2} \exp(\frac{\mu^2}{2\sigma^2}t + \frac{\mu}{2}t)$. Then, we can show that

$$\begin{aligned}\frac{\partial}{\partial K_0} G_T^* &= -e^{-A_t} \int_0^t e^{A_u} (\frac{\mu}{\sigma^2 S_u} dS_u + \frac{\mu^2}{\sigma^2} du) \\ &= -e^{-A_t}(e^{A_t} - e^{A_0}) = e^{-A_t} - 1.\end{aligned}$$

Thus,

$$\begin{aligned}E(2(\Pi - K_0 - G_T^*)(-1 - \frac{\partial}{\partial K_0} G_T^*)) &= E(2(\Pi - K_0 - G_T^*)(-e^{A_T})) \\ &= E(2(K_0 + G_T^* - \Pi) e^{-\frac{\mu^2}{\sigma^2}T} \frac{dP^*}{dP} | \mathcal{F}_T) \\ &= E^*(2e^{-\frac{\mu^2}{\sigma^2}T}(K_0 + G_T^* - \Pi)) \\ &= 2e^{-\frac{\mu^2}{\sigma^2}T}(K_0 - E^*(\Pi)).\end{aligned}$$

$E^*(G_T^*)$ is 0 since G^* is a martingale under the minimal martingale measure. $\min_{\theta \in \Theta} E(\Pi - K_0 - G_T(\theta))^2$ is convex in K_0, so $E^*(\Pi)$ is the value of K_0 that minimizes $\min_{\theta \in \Theta} E(\Pi - K_0 - G_T(\theta))^2$.
□

Similar arguments will work for more general setting that is given in Schweizer's [9]. In our setting, we have $\Pi = \int_0^T Y_u d\tilde{W}_u$. Since $\tilde{W}$ is a Brownian Motion under the minimal martingale measure P^*, $K_0 = 0$ makes the expected squared loss the smallest, assuming that $C(S_\cdot, \cdot)$ satisfies $E^*(\int_0^T S_t^4 C_{SS}^2(S_t, t)dt) < \infty$. Therefore, if we want $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta))^2$ as small as possible, the Black-Scholes initial value $C(S_0, 0)$ is the right amount to invest in the pre-limiting stage, as long as we use the hedging strategy proposed in (5).

4 Bounded Expected Loss

Suppose we have a certain upper bound U of the expected squared loss in mind and want to invest the smallest amount possible while the expected squared loss bounded by U. In other words, we want to minimize the starting value K_0 subject to $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta))^2 \leq U$ for a given level U. Note that if any pair (K_0, θ) solves the problem of minimizing K_0 subject to $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta))^2 \leq U$, then (K_0, θ_{K_0}) also solves the same problem, where $\theta_{K_0} = \underset{\theta \in \Theta}{\operatorname{argmin}} E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta))^2$. Thus, we can restrict our attention to a smaller set $\Theta' = \{\theta \in \Theta : \theta \text{ minimizes } E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta))^2 \text{ for some value of } K_0\}$ for the proper set of hedging strategies.

$C(S_t, t)$ is the Black-Scholes option price at time t when the stock price S follows a geometric Brownian Motion. Let us assume that $C(S_\cdot, \cdot)$ satisfies

$$E(\int_0^T S_t^4 C_{SS}^2(S_t, t)dt) < \infty. \tag{7}$$

It can be easily shown that this assumption is satisfied in case of a call option. The assumption (7) assures that $\{\int_0^t Y_u d\tilde{W}_u : 0 \leq t \leq T\}$ is a square-integrable martingale with zero means and finite variances $E(\int_0^t Y_u^2 du)$. By Song and Mykland [10],

$$G_t(\theta_{K_0}) = e^{-A_t}(K_0 - \int_0^t e^{A_u} Y_u d\tilde{W}_u) - K_0 + \int_0^t Y_u d\tilde{W}_u,$$

where $e^{A_t} = (S_t/S_0)^{\mu/\sigma^2} \exp(\frac{\mu^2}{2\sigma^2}t + \frac{\mu}{2}t)$. Under the assumption (7),

$$\begin{aligned}
E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta_{K_0}))^2 &= E(e^{-A_T}\int_0^T e^{A_u} Y_u d\tilde{W}_u - e^{-A_T}K_0)^2 \\
&= K_0^2 E(e^{-2A_T}) - 2K_0 E(e^{-2A_T}\int_0^T e^{A_u} Y_u d\tilde{W}_u) \\
&\quad + E(e^{-2A_T}(\int_0^T e^{A_u} Y_u d\tilde{W}_u)^2) \\
&= K_0^2 e^{-\frac{\mu^2}{\sigma^2}T} - 2K_0 E(E(e^{-2A_T}\int_0^T e^{A_u} Y_u d\tilde{W}_u | (S_u), 0 \leq u \leq T)) \\
&\quad + E(E(e^{-2A_T}(\int_0^T e^{A_u} Y_u d\tilde{W}_u)^2 | (S_u), 0 \leq u \leq T)) \qquad (8) \\
&= K_0^2 e^{-\frac{\mu^2}{\sigma^2}T} + E(e^{-2A_T}\int_0^T e^{2A_u} Y_u^2 du) \\
&= K_0^2 e^{-\frac{\mu^2}{\sigma^2}T} + \int_0^T E((\frac{S_T}{S_u})^{-\frac{2\mu}{\sigma^2}} e^{-(\frac{\mu^2}{\sigma^2}+\mu)(T-u)} Y_u^2) du \\
&= K_0^2 e^{-\frac{\mu^2}{\sigma^2}T} + \int_0^T E(S_u^{\frac{2\mu}{\sigma^2}} Y_u^2 e^{-(\frac{\mu^2}{\sigma^2}+\mu)(T-u)} E(S_T^{-\frac{2\mu}{\sigma^2}} | S_u)) du \\
&= K_0^2 e^{-\frac{\mu^2}{\sigma^2}T} + \int_0^T E(e^{-\frac{\mu^2}{\sigma^2}(T-u)} Y_u^2) du.
\end{aligned}$$

Since our constraint is $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta_{K_0}))^2 \leq U$,

$$K_0^2 \leq e^{\frac{\mu^2}{\sigma^2}T}\left(U - \int_0^T E(e^{-\frac{\mu^2}{\sigma^2}(T-u)}Y_u^2)du\right).$$

Therefore,

Proposition 4.1 *The minimum investment subject to* $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta))^2 \leq U$ *is*

$$\tilde{K}_0 = -e^{\frac{\mu^2}{2\sigma^2}T}\sqrt{U - \int_0^T E(e^{-\frac{\mu^2}{\sigma^2}(T-u)}Y_u^2)du}.$$

Thus, the minimum investment with the bounded loss constraint is $C(S_0, 0) + \frac{\tilde{K}_0}{\sqrt{\lambda_n}}$.

In case where the contingent claim is a call option, we can get a more explicit expression. If the option is a European call with the strike price K and expiration time T, then

$$E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta_{K_0}))^2 = K_0^2 e^{-\frac{\mu^2}{\sigma^2}T} + \frac{c_1^2}{8\pi\sigma^2}\int_0^T \frac{1}{\sqrt{T^2-u^2}}\exp(\frac{B(T)+C(T)u}{\sigma^2(T+u)})du$$

where $c_1 = \sqrt{k_4 - (\frac{k_3}{\sigma})^2}$, $B(T) = -(\log(\frac{S_0}{K}))^2 + \sigma^2 T\log(S_0K) - \frac{\sigma^4}{4}T^2 - \mu^2T^2$ and $C(T) = \sigma^2\mu T + 2(\mu+\sigma^2)\log K - 2\mu\log S_0$. And

$$\begin{aligned} K_0^2 &\leq e^{\frac{\mu^2}{\sigma^2}T}\left(U - \int_0^T E(e^{-\frac{\mu^2}{\sigma^2}(T-u)}Y_u^2)du\right) \\ &= e^{\frac{\mu^2}{\sigma^2}T}\left(U - \frac{c_1^2}{8\pi\sigma^2}\int_0^T \frac{1}{\sqrt{T^2-u^2}}\exp(\frac{B(T)+C(T)u}{\sigma^2(T+u)})du\right). \end{aligned}$$

Thus,

$$\tilde{K}_0 = -e^{\frac{\mu^2}{2\sigma^2}T}\sqrt{\left(U - \frac{c_1^2}{8\pi\sigma^2}\int_0^T \frac{1}{\sqrt{T^2-u^2}}\exp(\frac{B(T)+C(T)u}{\sigma^2(T+u)})du\right)}.$$

Note that $\tilde{K}_0$ is always negative. It means that we can reduce the initial investment from the Black-Scholes price. It is reasonable in the sense that $K_0 = 0$ minimizes $E(\int_0^T Y_u d\tilde{W}_u - K_0 - G_T(\theta_{K_0}))^2$ over the values of K_0. Here, we are willing to tolerate a higher level of error at the end, so we should be able to lower the initial investment at the beginning.

Remark 4.1 Since we have two interesting values for the initial investment, which are $C(S_0, 0)$ and $C(S_0, 0) + \frac{\tilde{K}_0}{\sqrt{\lambda_n}}$, we want to compare two different cases in order to see which one is more appealing. Let us see the limiting case, where we compare $K_0 = 0$ and $K_0 = \tilde{K}_0$. Suppose we invest $\tilde{K}_0$ at the beginning. Then we gain $|\tilde{K}_0|$ in the initial investment over the other case. How much do we lose in terms of the mean square of the hedging error at the end? Note that

$$E(\tilde{K}_0 + G_T(\theta_{\tilde{K}_0}) - \int_0^T Y_u d\tilde{W}_u) - E(G_T(\theta_0) - \int_0^T Y_u d\tilde{W}_u) = E(e^{-A_T}\tilde{K}_0) = \tilde{K}_0 e^{-\frac{\mu^2}{\sigma^2}T},$$

$$\mathrm{Var}(\tilde{K}_0 + G_T(\theta_{\tilde{K}_0}) - \int_0^T Y_u d\tilde{W}_u) - \mathrm{Var}(G_T(\theta_0) - \int_0^T Y_u d\tilde{W}_u) = \tilde{K}_0^2(e^{-\frac{\mu^2}{\sigma^2}T} - e^{-\frac{2\mu^2}{\sigma^2}T}).$$

Since $\tilde{K}_0$ is negative, we lose in both of expectation and variance over $K_0 = 0$ case. But the loss at the expiration is less than the gain at the beginning in the sense that when we invest $K_0 = \tilde{K}_0$ at $t = 0$, our squared gain over $K_0 = 0$ is $\tilde{K}_0^2$ and the expected squared loss over $K_0 = 0$ at $t = T$ is less than $\tilde{K}_0^2$, because

$$E(\int_0^T Y_u d\tilde{W}_u - \tilde{K}_0 - G_T(\theta_{\tilde{K}_0}))^2 = \tilde{K}_0^2 e^{-\frac{\mu^2}{\sigma^2}T} + E(\int_0^T Y_u d\tilde{W}_u - G_T(\theta_0))^2.$$

Remark 4.2 Let us compare numerically the gain and loss we will have in cases of investing $K_0 = 0$ and $K_0 = \tilde{K}_0$. In both cases, we use the compound Poisson hedging strategy obtained in (5). The contingent claim is a European call and $Q \sim N(0, \sigma^2)$. The parameter values are $T = 0.25$, $\sigma = 0.21$, $\mu = 0.15$, $S_0 = 950$, and $K = 1000$.

Suppose $\lambda_n = 10000$ and $U = 15000$. If we invest $K_0 = \tilde{K}_0$, then we gain 0.93373 dollars at the beginning. But at the end, we lose 0.76744 square dollars in the mean square of the hedging error, $E(\int_0^T Y_u d\tilde{W}_u - \tilde{K}_0 - G^*_{T,\tilde{K}_0})^2$. We lose 0.82191 dollars in terms of expectation of hedging error and lose 0.09190 square dollars in terms of variance of hedging error. The most of the loss at the end is coming from the bias part, so $\tilde{K}_0$ is not much more attractive than 0 in this case. However, when the time to expiration is longer and the volatility is small relative to the level of the expected rate of return, $\tilde{K}_0$ becomes more attractive.

Suppose $\mu = 0.5$, $\sigma = 0.21$ and $T = 0.5$. Then when we invest $\tilde{K}_0$ at $t = 0$, we gain 4.63739 dollars at the beginning and we lose 1.2634 square dollars in the mean square hedging error at the end. The loss in terms of the expectation is 0.2724 dollars and the loss in terms of the variance is 1.1892 square dollars.

5 Equivalent martingale measure in the pre-limiting stage

As we saw in section 2, the limiting log stock price process is a Brownian Motion with drift as (3) under P. Under the minimal martingale measure P^* defined as (6), the limiting log stock price is

$$\log S_t = \log S_0 - \frac{1}{2}\sigma^2 t + \sigma B_t^*,$$

where B^* is a standard Brownian Motion under P^*. We still assume that the interest rate r is 0.

Since the fair price of a contingent claim in a complete market is the expected value of the payoff under the unique equivalent martingale measure, we can try to find an appropriate equivalent martingale measure in the pre-limiting stage for the pricing purpose. One reasonable choice will be an equivalent martingale measure, $P^{*(n)}$, which converges weakly to the minimal martingale measure P^* for the stock price process in the limit. Let us start with the following assumptions.

Assumption 1 *(Distribution of the stock price process under $P^{*(n)}$)*

(i) $\{N_t^{(n)}\}$ *is a Poisson process with the compensator* $\lambda_n^* t$ *under* $P^{*(n)}$.

(ii) Q *has the density* f_n^* *under* $P^{*(n)}$ *and* f *under* $P^{(n)}$, *for all* n.

(iii) $N^{(n)}$ *and* Q *are independent under* $P^{*(n)}$, *for all* n.

Define $\frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}}$ as

$$\frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}} = \left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}} e^{(\lambda_n - \lambda_n^*)t} \prod_{i=1}^{N_t^{(n)}} \frac{f_n^*(Q_i)}{f(Q_i)}, \tag{9}$$

where $\{\mathcal{F}_t^{(n)}\}$ is a filtration generated by $S^{(n)}$. Under the assumption 1, we can easily check $\{\frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}}\}$ is a martingale under $P^{(n)}$. For $P^{*(n)}$ to be a martingale measure, we need $\{S_t^{(n)}\}$

to be a martingale under $P^{*(n)}$, because the interest rate r is 0. For any $0 \leq u \leq t \leq T$,

$$
\begin{aligned}
E^*(S_t^{(n)}|\mathcal{F}_u^{(n)}) &= S_u^{(n)} E^*(\prod_{i=N_u^{(n)}+1}^{N_t^{(n)}} e^{Z_i^{(n)}}|\mathcal{F}_u^{(n)}) \\
&= S_u^{(n)} E^*((E^*(e^{Z^{(n)}}))^{N_t^{(n)}-N_u^{(n)}}|\mathcal{F}_u^{(n)}) \\
&= S_u^{(n)} E^*(E^*(e^{Z^{(n)}}))^{N_t^{(n)}-N_u^{(n)}}) \\
&= S_u^{(n)} E^*(\exp((\log E^*(e^{Z^{(n)}}))(N_t^{(n)} - N_u^{(n)}))) \\
&= S_u^{(n)} \exp(\lambda_n^*(t-u)(E^*(e^{Z^{(n)}}) - 1)).
\end{aligned}
$$

Therefore, in order for $E^*(S_t^{(n)}|\mathcal{F}_u^{(n)}) = S_u^{(n)}$ for any $0 \leq u \leq t \leq T$, we need

$$
E^*(e^{Z^{(n)}}) = 1. \tag{10}
$$

We also want $\log \frac{dP^{*(n)}}{dP^{(n)}} = O_p(1)$ because we want $P^{*(n)}$ to converge to the minimal martingale measure. It is also necessary for the asymptotic equivalence between two measures.

Define

$$
\tilde{X}_i = \frac{\log \frac{f_n^*(Q_i)}{f(Q_i)} - E \log \frac{f_n^*(Q)}{f(Q)}}{\sqrt{\operatorname{Var} \log \frac{f_n^*(Q)}{f(Q)}}}
$$

and look at $\log \frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}}$ more closely.

$$
\begin{aligned}
\log \frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}} &= N_t^{(n)}(\log \lambda_n^* - \log \lambda_n) + (\lambda_n - \lambda_n^*)t + \sum_{i=1}^{N_t^{(n)}} \log \frac{f_n^*(Q_i)}{f(Q_i)} \\
&= \sum_{i=1}^{N_t^{(n)}} (\log \frac{f_n^*(Q_i)}{f(Q_i)} - E \log \frac{f_n^*(Q)}{f(Q)}) \\
&\quad + N_t^{(n)}(\log \lambda_n^* - \log \lambda_n + E \log \frac{f_n^*(Q)}{f(Q)}) - t(\lambda_n^* - \lambda_n).
\end{aligned}
$$

Using the variable $\tilde{X}_i$,

$$\begin{aligned}
\log \frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}} &= \sqrt{\lambda_n \mathrm{Var} \log \frac{f_n^*(Q)}{f(Q)}} \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i \\
&+ \frac{N_t^{(n)} - \lambda_n t}{\sqrt{\lambda_n}} \sqrt{\lambda_n} (\log \lambda_n^* - \log \lambda_n + E \log \frac{f_n^*(Q)}{f(Q)}) \\
&+ \lambda_n t (\log \lambda_n^* - \log \lambda_n + E \log \frac{f_n^*(Q)}{f(Q)}) - t(\lambda_n^* - \lambda_n) \\
&= \sqrt{\lambda_n \mathrm{Var} \log \frac{f_n^*(Q)}{f(Q)}} \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i \\
&+ \frac{N_t^{(n)} - \lambda_n t}{\sqrt{\lambda_n}} \sqrt{\lambda_n} (\log \lambda_n^* - \log \lambda_n + E \log \frac{f_n^*(Q)}{f(Q)}) \\
&+ \lambda_n t (E \log \frac{f_n^*(Q)}{f(Q)} - \frac{1}{2} (\log \lambda_n^* - \log \lambda_n)^2 - \frac{1}{6} (\log \lambda_n^* - \log \lambda_n)^3 + \cdots).
\end{aligned} \tag{11}$$

Based on (11) and the lemma 6.1 in Appendix 6.1, we can find a set of sufficient conditions for $\log \frac{dP^{*(n)}}{dP^{(n)}} = O_p(1)$ as

$$\begin{aligned}
\lambda_n \mathrm{Var}(\log \frac{f_n^*(Q)}{f(Q)}) &= O(1), \\
\log \lambda_n^* - \log \lambda_n &= O(\lambda_n^{-1/2}), \\
E \log \frac{f_n^*(Q)}{f(Q)} &= O(\lambda_n^{-1}).
\end{aligned}$$

Now, suppose

$$\begin{aligned}
f_n^*(Q) &= f(Q) e^{g_{\lambda_n}(Q)}, \\
\sqrt{\lambda_n} (\log \lambda_n^* - \log \lambda_n) &\longrightarrow \eta, \\
\sqrt{\lambda_n \mathrm{Var}(g_{\lambda_n}(Q))} &\longrightarrow |\alpha|, \\
\lambda_n E(g_{\lambda_n}(Q)) &\longrightarrow \beta.
\end{aligned} \tag{12}$$

Theorem 5.1 *Subject to the assumption 1, (10), and (12),* $(\log \frac{dP^{*(n)}}{dP^{(n)}}, \log S^{(n)})$ *converges jointly in distribution to* $(\xi, \log S)$ *where* $\xi_t = -\frac{\mu}{\sigma} B_t + \eta W_t^1 - \frac{1}{2}(\alpha^2 + \eta^2)t$, *where* W^1 *is a Browninan Motion under* P *and is independent of* B.

Proof. See Appendix 6.1.

Let us check if the measure defined by the limit of $\log \frac{dP^{*(n)}}{dP^{(n)}}$ is a martingale under the limiting measure P. For any $0 \leq u \leq t \leq T$, we want

$$E(\exp(-\frac{\mu}{\sigma}B_t + \eta W_t^1 - \frac{1}{2}(\alpha^2+\eta^2)t|\mathcal{F}_u) = \exp(-\frac{\mu}{\sigma}B_u + \eta W_u^1 - \frac{1}{2}(\alpha^2+\eta^2)u),$$

and hence, we need $\alpha^2 = \frac{\mu^2}{\sigma^2}$ or $|\alpha| = \frac{|\mu|}{\sigma}$, because

$$\begin{aligned}
&E(\exp(-\frac{\mu}{\sigma}B_t + \eta W_t^1 - \frac{1}{2}(\alpha^2+\eta^2)t|\mathcal{F}_u) \\
&= \exp(-\frac{\mu}{\sigma}B_u + \eta W_u^1 - \frac{1}{2}(\alpha^2+\eta^2)u)E(\exp(-\frac{\mu}{\sigma}(B_t - B_u) + \eta(W_t^1 - W_u^1) - \frac{1}{2}(\alpha^2+\eta^2)(t-u)) \\
&= \exp(-\frac{\mu}{\sigma}B_u + \eta W_u^1 - \frac{1}{2}(\alpha^2+\eta^2)u)\exp(\frac{\mu^2}{2\sigma^2}(t-u) + \frac{1}{2}\eta^2(t-u) - \frac{1}{2}(\alpha^2+\eta^2)(t-u)) \\
&= \exp(-\frac{\mu}{\sigma}B_u + \eta W_u^1 - \frac{1}{2}(\alpha^2+\eta^2)u)\exp(\frac{1}{2}(\frac{\mu^2}{\sigma^2} - \alpha^2)(t-u)).
\end{aligned}$$

It can be easily shown that the limiting stock price process S is a martingale under the measure P^* defined by the limit of $\log \frac{dP^{*(n)}}{dP^{(n)}}$. Using $\alpha^2 = \frac{\mu^2}{\sigma^2}$, for all $t \leq T$,

$$\log \frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}} \xrightarrow{\mathcal{D}} -\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t + \eta W_t^1 - \frac{1}{2}\eta^2 t.$$

If $\eta = 0$, in other words, λ_n and λ_n^* are very close in the sense that $\lambda_n^* = \lambda_n(1 + o(\lambda_n^{-1/2}))$, then

$$\log \frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}} \xrightarrow{\mathcal{D}} -\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t,$$

i.e., $P^{*(n)}$ converges to the minimal martingale measure in (6). In particular, if there is no change in the Poisson process under the measure change, then $\prod_{i=1}^{N_t^{(n)}} \frac{f_n^*(Q_i)}{f(Q_i)}$ converges weakly to the minimal martingale measure assuming $\lambda_n \mathrm{Var}(\log \frac{f_n^*(Q)}{f(Q)}) = O(1)$ and $E \log \frac{f_n^*(Q)}{f(Q)} = O(\lambda_n^{-1})$.

Now, we have another candidate for the price of the option.

Proposition 5.1 *Suppose we choose an equivalent martingale measure $P^{*(n)}$ that converges to the minimal martingale measure for the pricing purpose. Then the price of a European contingent claim $C(S_T^{(n)}, T)$ is*

$$\begin{aligned}
E^*C(S_T^{(n)}, T) = C(S_0, 0) &+ \frac{k_3 T}{\sqrt{\lambda_n}}\{\frac{1}{2}C_{SS}(S_0,0)S_0^2 + \frac{1}{6}C_{SSS}(S_0,0)S_0^3\} \\
&+ \frac{T}{2\sqrt{\lambda_n}}(\sqrt{\lambda_n}E(Q^2 g_{\lambda_n}(Q)))C_{SS}(S_0,0)S_0^2 + o(\lambda_n^{-1/2}).
\end{aligned}$$

Proof. See Appendix 6.2.

In the above proposition, we need the limit of $E(Q^2 g_{\lambda_n}(Q))$ for general distribution of Q. Let us take the distribution of Q such that $E(Q^2 g_{\lambda_n}(Q))$ is $-\frac{\sigma^2\eta}{\sqrt{\lambda_n}} + o(\lambda_n^{-1/2})$. The reasoning behind this assumption is the following.

In the limit, the quadratic variation of $\log S$ at t is $\sigma^2 t$ under P and P^*. But before the limit, the predictable quadratic variation, $< \log S^{(n)}, \log S^{(n)} >_t$, is not the same under $P^{(n)}$ and under $P^{*(n)}$. It is easy to see that

$$< \log S^{(n)}, \log S^{(n)} >_t = (\sigma^2 + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)^2)t,$$
$$< \log S^{(n)}, \log S^{(n)} >_t^* = (\sigma^2 + \frac{1}{\sqrt{\lambda_n}}(\sigma^2\eta + \sqrt{\lambda_n}E(Q^2 g_{\lambda_n}(Q))) + o(\lambda_n^{-1/2}))t.$$

It might be good to choose the distribution of Q that makes $\lambda_n^{-1/2}$ term to be 0.

With $\eta = 0$, the choice of K_0 will be $k_3 T\{\frac{1}{2}C_{SS}(S_0,0)S_0^2 + \frac{1}{6}C_{SSS}(S_0,0)S_0^3\}$, and the price of a European contingent claim is

$$C(S_0,0) + \frac{k_3 T}{\sqrt{\lambda_n}}\{\frac{1}{2}C_{SS}(S_0,0)S_0^2 + \frac{1}{6}C_{SSS}(S_0,0)S_0^3\}.$$

Remark 5.1 Subject to the assumption 1, (10), and (12), it is easy to show that

$$\left(\frac{\lambda_n^*}{\lambda_n}\right)^{N_t^{(n)}} e^{(\lambda_n - \lambda_n^*)t} \xrightarrow{\mathcal{D}} \exp(\eta W_t^1 - \frac{1}{2}\eta^2 t)$$
$$\prod_{i=1}^{N_t^{(n)}} \frac{f_n^*(Q_i)}{f(Q_i)} \xrightarrow{\mathcal{D}} -\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t.$$

As we defined $\frac{dP^{*(n)}}{dP^{(n)}}$ in (9), it is the product of the Radon-Nikodym derivative induced from the jump intensity process, $N^{(n)}$, and the Radon-Nikodym derivative induced from the jump.size distribution, Q. We can see that the N-component of $\frac{dP^{*(n)}}{dP^{(n)}}$ converges weakly to $\exp(\eta W_t^1 - \frac{1}{2}\eta^2 t)$ and Q-component of $\frac{dP^{*(n)}}{dP^{(n)}}$ converges weakly to $-\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t$.

6 Appendix

6.1 Proof of Theorem 5.1

Lemma 6.1 $\left(\frac{N_t^{(n)} - \lambda_n t}{\sqrt{\lambda_n}}, \frac{1}{\sqrt{\lambda_n}}\sum_{i=1}^{N_t^{(n)}} \tilde{X}_i\right)$ *converges in distribution to a two-dimensional Brownian Motion under P.*

Proof. Define $M_t^{(n)}$ and $\tilde{M}_t^{(n)}$ to be $\frac{N_t^{(n)}-\lambda_n t}{\sqrt{\lambda_n}}$ and $\frac{1}{\sqrt{\lambda_n}}\sum_{i=1}^{N_t^{(n)}}\tilde{X}_i$, respectively. Then, they are square-integrable martingales satisfying $< M^{(n)}, M^{(n)} >_t = t$, $< \tilde{M}^{(n)}, \tilde{M}^{(n)} >_t = t$, and $< M^{(n)}, \tilde{M}^{(n)} >_t = 0$. Consider local square integrable martingales $M_\epsilon^{(n)}$ and $\tilde{M}_\epsilon^{(n)}$ which include all the jumps of the original martingales in absolute value than ϵ. Since the jump size of $\frac{N_t^{(n)}-\lambda_n t}{\sqrt{\lambda_n}}$ at any time point is either 0 or $\frac{1}{\sqrt{\lambda_n}}$, we can define $M_\epsilon^{(n)} = 0$. In the case of $\tilde{M}^{(n)}$, define

$$\tilde{M}_{\epsilon,t}^{(n)} = \frac{1}{\sqrt{\lambda_n}}\sum_{i=1}^{N_t^{(n)}}\tilde{X}_i I_{\{|\tilde{X}_i|>\epsilon\sqrt{\lambda_n}\}} - \sqrt{\lambda_n}tE(\tilde{X}I_{\{|\tilde{X}|>\epsilon\sqrt{\lambda_n}\}}).$$

Then,

$$|\Delta\tilde{M}_t^{(n)} - \Delta\tilde{M}_{\epsilon,t}^{(n)}| = \frac{1}{\sqrt{\lambda_n}}|\tilde{X}_{N_t^{(n)}}|I_{\{|\tilde{X}_{N_t^{(n)}}|\le\epsilon\sqrt{\lambda_n}\}} \le \epsilon,$$

$$[\tilde{M}_\epsilon^{(n)}, \tilde{M}_\epsilon^{(n)}]_t = \frac{1}{\lambda_n}\sum_{i=1}^{N_t^{(n)}}\tilde{X}_i^2 I_{\{|\tilde{X}_i|>\epsilon\sqrt{\lambda_n}\}},$$

and

$$< \tilde{M}_\epsilon^{(n)}, \tilde{M}_\epsilon^{(n)} >_t = tE(\tilde{X}^2 I_{\{|\tilde{X}|>\epsilon\sqrt{\lambda_n}\}}).$$

$< \tilde{M}_\epsilon^{(n)}, \tilde{M}_\epsilon^{(n)} >_t$ converges to 0 as n goes to ∞ because $\tilde{X}^2$ is integrable and $P(|\tilde{X}| > \epsilon\sqrt{\lambda_n})$ goes to 0. Now, Rebolledo's theorem(Andersen, Borgen, Gill and Keiding [1], p. 83) completes the proof. □

Suppose (12). It implies that $g_{\lambda_n}(Q) = O_p(\lambda_n^{-1/2})$ because if we take $K_\epsilon > \frac{|\alpha|+1}{\epsilon}$, then

$$\begin{aligned} P(|\sqrt{\lambda_n}g_{\lambda_n}(Q)| \ge K_\epsilon) &\le \frac{\sqrt{\lambda_n E(g_{\lambda_n}^2(Q))}}{K_\epsilon} \\ &= \frac{\sqrt{\lambda_n \mathrm{Var}(g_{\lambda_n}(Q)) + \lambda_n (E(g_{\lambda_n}(Q)))^2}}{K_\epsilon} \\ &\le \frac{|\alpha|+1}{K_\epsilon} < \epsilon, \end{aligned}$$

for a large enough n. By (11), (12), and lemma 6.1,

$$\log\frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}} \xrightarrow{\mathcal{D}} |\alpha|W_t^2 + \eta W_t^1 + (\beta - \frac{1}{2}\eta^2)t$$

where W^1 and W^2 are independent Brownian Motions under P. For the asymptotic equivalence,

$$E(\exp(|\alpha|W_t^2 + \eta W_t^1 + (\beta - \frac{1}{2}\eta^2)t)) = \exp((\beta - \frac{1}{2}\eta^2)t + \frac{1}{2}(\alpha^2 + \eta^2)t)$$
$$\stackrel{\text{set}}{=} 1.$$

So we get

$$\beta = -\frac{1}{2}\alpha^2.$$

On the other hand, recall we made the assumption $E^*(e^{Z^{(n)}}) = 1$ to make $P^{*(n)}$ to be a martingale measure. It is equivalent to say

$$E(e^{\frac{1}{\sqrt{\lambda_n}}Q + g_{\lambda_n}(Q)}) = e^{-\frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)}.$$

Since

$$e^{-\frac{1}{\lambda_n}(\mu-\frac{1}{2}\sigma^2)} = 1 - \frac{\mu}{\lambda_n} + \frac{\sigma^2}{2\lambda_n} + o(\lambda_n^{-1}),$$
$$E(e^{\frac{1}{\sqrt{\lambda_n}}Q+g_{\lambda_n}(Q)}) = E(1 + \frac{1}{\sqrt{\lambda_n}}Q + g_{\lambda_n}(Q) + \frac{1}{2}(\frac{1}{\sqrt{\lambda_n}}Q + g_{\lambda_n}(Q))^2 + o_p(\lambda_n^{-1}))$$
$$= 1 + E(g_{\lambda_n}(Q)) + \frac{\sigma^2}{2\lambda_n} + \frac{1}{\sqrt{\lambda_n}}E(Qg_{\lambda_n}(Q)) + \frac{1}{2}\mathrm{Var}(g_{\lambda_n}(Q)) + o(\lambda_n^{-1}),$$

we have

$$E(g_{\lambda_n}(Q)) + \frac{1}{\sqrt{\lambda_n}}E(Qg_{\lambda_n}(Q)) + \frac{1}{2}\mathrm{Var}(g_{\lambda_n}(Q)) = -\frac{\mu}{\lambda_n} + o(\lambda_n^{-1}).$$

Thus,

$$\sqrt{\lambda_n}E(Qg_{\lambda_n}(Q)) \longrightarrow -\mu \tag{13}$$

because $\lambda_n E(g_{\lambda_n}(Q)) \longrightarrow \beta = -\frac{1}{2}\alpha^2$ and $\lambda_n \mathrm{Var}(g_{\lambda_n}(Q)) \longrightarrow \alpha^2$.

We are ready to consider the joint convergence of $\frac{N_t^{(n)}-\lambda_n t}{\sqrt{\lambda_n}}$, $\frac{1}{\sqrt{\lambda_n}}\sum_{i=1}^{N_t^{(n)}}\tilde{X}_i$, and $\log S_t^{(n)} - (\mu - \frac{1}{2}\sigma^2)t$. Define $M_t^{(n)}$, $\tilde{M}_t^{(n)}$, and $\hat{M}_t^{(n)}$ to be $\frac{N_t^{(n)}-\lambda_n t}{\sqrt{\lambda_n}}$, $\frac{1}{\sqrt{\lambda_n}}\sum_{i=1}^{N_t^{(n)}}\tilde{X}_i$, and $\log S_t^{(n)} - (\mu - \frac{1}{2}\sigma^2)t$, respectively. They are square-integrable martingales and we know from the lemma 6.1,

$$(M^{(n)}, \tilde{M}^{(n)}) \xrightarrow{\mathcal{D}} (W^1, W^2)$$

where W^1 and W^2 are independent Brownian Motions under P. Moreover,

$$[\hat{M}^{(n)}, \hat{M}^{(n)}]_t = \sum_{i=1}^{N_t^{(n)}} (Z_i^{(n)})^2,$$

$$[M^{(n)}, \hat{M}^{(n)}]_t = \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)},$$

$$[\tilde{M}^{(n)}, \hat{M}^{(n)}]_t = \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i Z_i^{(n)}.$$

Thus,[1]

$$< \hat{M}^{(n)}, \hat{M}^{(n)} >_t = \sigma^2 t + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)^2 t \underset{n\to\infty}{\longrightarrow} \sigma^2 t,$$

$$< M^{(n)}, \hat{M}^{(n)} >_t = \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2) t \underset{n\to\infty}{\longrightarrow} 0,$$

$$< \tilde{M}^{(n)}, \hat{M}^{(n)} >_t = \frac{E(Q g_{\lambda_n}(Q))t}{\sqrt{\mathrm{Var}(g_{\lambda_n}(Q))}} \underset{n\to\infty}{\longrightarrow} -\frac{\mu}{|\alpha|} t.$$

If we define

$$\hat{M}_{\epsilon,t}^{(n)} = \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)} I_{\{|Z_i^{(n)}|>\epsilon\}} - \lambda_n t E(Z^{(n)} I_{\{|Z^{(n)}|>\epsilon\}}),$$

$\hat{M}_\epsilon^{(n)}$ is a martingale containing all jumps of $\hat{M}^{(n)}$ whose size is bigger than ϵ in absolute value. Its quadratic variations are

$$[\hat{M}_\epsilon^{(n)}, \hat{M}_\epsilon^{(n)}]_t = \sum_{i=1}^{N_t^{(n)}} {Z_i^{(n)}}^2 I_{\{|Z_i^{(n)}|>\epsilon\}},$$

$$< \hat{M}_\epsilon^{(n)}, \hat{M}_\epsilon^{(n)} >_t = \lambda_n t E({Z^{(n)}}^2 I_{\{|Z^{(n)}|>\epsilon\}},$$

and $< \hat{M}_\epsilon^{(n)}, \hat{M}_\epsilon^{(n)} >_t$ goes to 0 because $\lambda_n {Z^{(n)}}^2$ is integrable and $P(|\sqrt{\lambda_n} Z| > \epsilon\sqrt{\lambda_n})$ goes to 0 as n goes to ∞.

By Rebolledo's theorem,

$$\left(\frac{N_t^{(n)} - \lambda_n t}{\sqrt{\lambda_n}}, \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{N_t^{(n)}} \tilde{X}_i, \log S_t^{(n)} - (\mu - \frac{1}{2}\sigma^2)t \right) \xrightarrow{\mathcal{D}} (W_t^1, W_t^2, \sigma B_t),$$

[1] I used (12) and (13) for the limit of $< \tilde{M}^{(n)}, \hat{M}^{(n)} >$.

where W^1 is independent of W^2 and B, $< W^2, B >_t = -\frac{\mu}{\sigma|\alpha|}t$, and they are all Brownian Motions under P. Therefore, $W^2 = -\frac{\mu}{\sigma|\alpha|}B$ and

$$\left(\log \frac{dP^{*(n)}}{dP^{(n)}}|_{\mathcal{F}_t^{(n)}}, \log S_t^{(n)}\right) \xrightarrow{\mathcal{D}} (-\frac{\mu}{\sigma}B_t + \eta W_t^1 + -\frac{1}{2}(\alpha^2+\eta^2)t, \log S_t).$$

6.2 Proof of Proposition 5.1

By Itô's formula and Taylor expansion,

$$\begin{aligned} dC(S_t^{(n)},t) &= \dot{C}_t(S_{t-}^{(n)},t)dt + \dot{C}_S(S_{t-}^{(n)},t)dS_t^{(n)} + \Delta C(S_t^{(n)},t) - \dot{C}_S(S_{t-}^{(n)},t)\Delta S_t^{(n)} \\ &= \Delta C(S_t^{(n)},t) + \dot{C}_t(S_{t-}^{(n)},t)dt \\ &= \sum_{v=1}^{\infty} C_S^{(v)}(S_{t-}^{(n)},t)(\Delta S_t^{(n)})^v \frac{1}{v!} + \dot{C}_t(S_{t-}^{(n)},t)dt. \end{aligned}$$

Thus,

$$\begin{aligned} E^*C(S_T^{(n)},T) &= C(S_0,0) + \int_0^T E^*\dot{C}_t(S_{t-}^{(n)},t)dt \\ &\quad + \int_0^T \sum_{v=1}^{\infty} \frac{1}{v!}E^*(C_S^{(v)}(S_{t-}^{(n)},t)d < S^{(n)},\cdots,S^{(n)} >_t^*) \\ &= C(S_0,0) + \int_0^T E^*\dot{C}_t(S_{t-}^{(n)},t)dt \\ &\quad + \int_0^T \sum_{v=1}^{\infty} \frac{1}{v!}E^*(e^{Z^{(n)}}-1)^v\lambda_n E^*(C_S^{(v)}(S_{t-}^{(n)},t)S_{t-}^{(n)^v})dt. \end{aligned}$$

By Black-Scholes PDE,

$$\begin{aligned} E^*C(S_T^{(n)},T) &= C(S_0,0) + \int_0^T E^*(e^{Z^{(n)}}-1)\lambda_n E^*(\dot{C}_S(S_{t-}^{(n)},t)S_{t-}^{(n)})dt \\ &\quad + \int_0^T \frac{1}{2}(E^*(e^{Z^{(n)}}-1)^2\lambda_n - \sigma^2)E^*(\ddot{C}_{SS}(S_{t-}^{(n)},t)S_{t-}^{(n)^2})dt \\ &\quad + \int_0^T \frac{1}{6}E^*(e^{Z^{(n)}}-1)^3\lambda_n E^*(C_S^{(3)}(S_{t-}^{(n)},t)S_{t-}^{(n)^3})dt \\ &\quad + \int_0^T \sum_{v=4}^{\infty} \frac{1}{v!}E^*(e^{Z^{(n)}}-1)^v\lambda_n E^*(C_S^{(v)}(S_{t-}^{(n)},t)S_{t-}^{(n)^v})dt \\ &= C(S_0,0) + \frac{1}{2}\left(\frac{k_3}{\sqrt{\lambda_n}} + \frac{\sqrt{\lambda_n}E(Q^2 g_{\lambda_n}(Q))}{\sqrt{\lambda_n}} + O(\lambda_n^{-1})\right)\int_0^T E^*(\ddot{C}_{SS}(S_{t-}^{(n)},t)S_{t-}^{(n)^2})dt \\ &\quad + (\frac{k_3}{6\sqrt{\lambda_n}} + O(\lambda_n^{-1}))\int_0^T E^*(C_S^{(3)}(S_{t-}^{(n)},t)S_{t-}^{(n)^3})dt \\ &\quad + \int_0^T \sum_{v=4}^{\infty} \frac{1}{v!}O(\lambda_n^{-v/2+1})E^*(C_S^{(v)}(S_{t-}^{(n)},t)S_{t-}^{(n)^v})dt. \end{aligned}$$

Since $\ddot{C}_{SS}(S_{t-}^{(n)},t){S_{t-}^{(n)}}^2$ and $C_S^{(3)}(S_{t-}^{(n)},t){S_{t-}^{(n)}}^3$ satisfy the Black-Scholes PDE, we can show that

$$E^*(\ddot{C}_{SS}(S_{t-}^{(n)},t){S_{t-}^{(n)}}^2) = \ddot{C}_{SS}(S_0,0)S_0^2 + O(\lambda_n^{-1/2})$$

and

$$E^*(C_S^{(3)}(S_{t-}^{(n)},t){S_{t-}^{(n)}}^3) = C_S^{(3)}(S_0,0)S_0^3 + O(\lambda_n^{-1/2}).$$

Therefore,

$$\begin{aligned} E^*C(S_T^{(n)},T) = C(S_0,0) &+ \frac{k_3 T}{\sqrt{\lambda_n}}\{\frac{1}{2}\ddot{C}_{SS}(S_0,0)S_0^2 + \frac{1}{6}C_S^{(3)}(S_0,0)S_0^3\} \\ &+ \frac{T}{2\sqrt{\lambda_n}}(\sqrt{\lambda_n}E(Q^2 g_{\lambda_n}(Q)))\ddot{C}_{SS}(S_0,0)S_0^2 + o(\lambda_n^{-1/2}). \end{aligned}$$

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