# OPTIONS AND DISCONTINUITY: AN ASYMPTOTIC DECOMPOSITION OF HEDGING ERRORS

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# Options and Discontinuity:

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**Abstract**

This paper studies the problem of option hedging in an incomplete market. The market incompleteness comes from the discontinuity of the underlying asset price process, specifically. By adopting an asymptotic approach, letting securities prices converge to continuous processes, we try to improve the classical Black-Scholes hedging strategy.

The first order error term after we hedge an option with the Black-Scholes strategy is decomposed into a part which can be traded away and a part which is purely unreplicable. First, we modify the Black-Scholes hedging strategy by adding the replicable part of the first order error and secondly, we adopt the mean-variance hedging method to take care of the nonreplicable part.

Some results of simulation experiments are also provided. In simulation, we see that the new hedging strategy improves the classical Black-Scholes hedging strategy up to 30% in terms of the mean square of hedging error, when the distribution of log stock price is skewed.

## 1 Introduction

Perfect hedging of an option is impossible in the real world. In a complete financial market, every contingent claim is exactly attainable by investing in the market. But in most real instances, the market is not complete. Market incompleteness may arise for various reasons, including discontinuity of the asset price process, dependency of contingent claims on unavailable stocks for investment, discrete time hedging, or some constraints on the trading strategies. Under the classical Black-Scholes setting in which the stock price process is a geometric Brownian Motion, we can construct a perfect hedging strategy because their setup assures that the market is complete. In other words, every contingent claim is riskless in that setting. However, the stock price process is not a geometric Brownian Motion and even not continuous in reality. Stocks move in fixed increments that is called as a tick size and sometimes there are also big jumps such as market crashes, for example, the big drop of S&P 500 index on October 19, 1987 and more recently, the drop after September 11, 2001. Especially when we look at the intraday trading data, we can see that the more realistic model is a purely discontinuous process rather than a continuous one. Under the discontinuous model for the stock price process, the market is no longer complete. Although Protter [3] showed conditions where the market is complete even with a discontinuous asset price process, the conditions do not hold in most practical markets. Here, we will focus on the case where the stock price process is not continuous, and thus, the market is incomplete.

Our goal is to improve the classical Black-Scholes hedging strategy. We add a correction term to the Black-Scholes hedging strategy by looking closely at the Black-Scholes hedging error. The stock price is allowed to have jumps, and specifically $\log S$ is a compound Poisson process. The detailed model is described in section 2. The market incompleteness originally comes from the discontinuity of the asset price process. However, when we let the price converge to a continuous process, we have an incomplete limiting market where the stock price process follow the geometric Brownian Motion, but there exists a source of randomness that cannot be traded. That is because discontinuity gives a source of another Brownian Motion in the limit. Section 3 deals with the first order remainder term which occurs when we hedge a European style option using classical Black-Scholes hedging strategy. We decompose the remainder term into replicable and non-replicable parts in the limit, and then find the pre-limiting processes which converge weakly to the replicable and non-replicable parts, respectively. These weak convergence results make it possible to perform the transition of the problem to the limiting case. Using the skewness of the log stock price distribution, we deal with the replicable part of the first order remainder term. The mean-variance hedging in section 4 is used to handle the non-replicable part in case where we have the squared loss function. Proofs are in Appendix. Section 5 shows some simulation results.

## 2 The Model

Consider a sequence of discontinuous processes that converges to a geometric Brownian Motion model. Each element of the sequence is a discontinuous process, indexed by $n$. $n$ does not have any practical meaning, but it is used for asymptotic operation. A larger $n$ means that the degree of discontinuity is smaller, *i.e.*, the process is closer to a geometric Brownian Motion model. Although we consider a sequence of processes, we observe only one price process with a certain degree of discontinuity from the market, for a given stock.

Let $(\Omega, \mathcal{F}^{(n)}, P^{(n)})$ be a probability space with the filtration, $\mathcal{F}^{(n)}$, generated by the stock price process $S^{(n)}$ defined below. We suppose that for each $n$, the log stock price process follows a compound Poisson process under $P^{(n)}$ such that

$$\log S_t^{(n)} = \log S_0^{(n)} + \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}, \tag{1}$$

where $N^{(n)}$ is a Poisson process with rate $\lambda_n$, and $Z_i^{(n)}$'s are iid random variables that are independent of $N^{(n)}$. We assume the initial stock price $S_0^{(n)}$ is the same for all $n$. As $n$ goes to $\infty$, $\lambda_n$ goes to $\infty$ and $Z_i^{(n)}$ converges to 0 in distribution. $N_t^{(n)}$ is the number of jumps in the log stock price process up to time $t$, and each $Z_i^{(n)}$ represents the size of the $i$th jump of $\log S^{(n)}$. Since $\lambda_n t$ is the expected number of jumps up to time $t$, we can say that $\lambda_n$ is the jump intensity of the log stock price process. Practically, $\lambda_n$ is related with the level of the trading activity of an individual stock. In other words, a large $\lambda_n$ corresponds to a heavily traded stock and a small $\lambda_n$ corresponds to a rarely traded stock. We define the jump size distribution $Z^{(n)}$ more precisely as follows.

$$Z^{(n)} \stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{\lambda_n}} Q + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2), \tag{2}$$

where $Q$ is a random variable with $EQ = 0$, $EQ^2 = \sigma^2$, $EQ^3 = k_3$, and $EQ^4 = k_4$, under $P^{(n)}$, for all $n$. $\stackrel{\mathcal{D}}{=}$ means that $Z^{(n)}$ has the same distribution as the random variable in the right hand side. For integers $p > 4$, we assume $E(|Q|^p) = o(\lambda_n^{-2+p/2})$. $\mu$ is a constant. It is clear that $Z^{(n)}$ converges to 0 in probability as well as in distribution as $n$ goes to $\infty$, $E(Z^{(n)}) = \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)$, $E(Z^{(n)})^p = O(\lambda_n^{-p/2})$ for $p = 2, 3$ and 4, and $E(Z^{(n)})^p = o(\lambda_n^{-2})$ for $p > 4$. Adding $o(\lambda_n^{-1})$ term to $Z^{(n)}$ will not change anything.

Now, consider the asymptotics as $n$ goes to $\infty$. The conditions above assure that $\log S^{(n)}$ converges to a Brownian Motion with drift as $n$ goes to $\infty$ and we can see how in the following.

**Proposition 2.1** *Assume all the above conditions. Then as n goes to $\infty$, the $\log S^{(n)}$ process converges in distribution to $\log S$ that is*

$$\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \tag{3}$$

*where $B$ is a Brownian Motion under the limiting measure $P$.*

*Proof.* To show that $\log S^{(n)}$ converges in distribution to $\log S$, we need to show that $\log S^{(n)}$ is tight and $\log S$ is the only possible limit.

Consider an auxiliary process $X^{(n)}$, defined by

$$X_t^{(n)} = \log S_t^{(n)} - (\mu - \frac{1}{2}\sigma^2)t.$$

Then $X^{(n)} - X_0^{(n)}$ is a square-integrable martingale under $P^{(n)}$ for each $n$. And the sequence $\{X_0^{(n)}\}$ is tight, because $X_0^{(n)} = \log S_0^{(n)} = \log S_0$ for all $n$ and we can assume $\log S_0$ is a constant for any sample path. Moreover, the quadratic variation sequence $\{< X^{(n)}, X^{(n)} >\}$ is $C$-tight by Theorem VI.4.1 in Jacod and Shiryaev [7] because $< X^{(n)}, X^{(n)} >_t = < \log S^{(n)}, \log S^{(n)} >_t = \lambda_n t E(Z^{(n)})^2 = \sigma^2 t + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)^2 t$. Thus, by Theorem VI.4.13 in Jacod and Shiryaev [7], $X^{(n)} - X_0^{(n)}$ is tight and by Corollary VI.3.33 in Jacod and Shiryaev [7], $\log S^{(n)}$ is also tight because $\{(\mu - \frac{1}{2}\sigma^2)t\}$ is trivially tight. So the tightness is proved.

According to Jacod and Shiryaev [7] Lemma VI.3.19 and Lemma VII.1.3, to show that $\log S$ is the only possible limit is equivalent to show that

$$\log S_t^{(n)} - \log S_u^{(n)} \xrightarrow{\mathcal{D}} \log S_t - \log S_u, \quad \forall\ 0 \le u < t \le T,$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution. Define $Q_{u,i} = \frac{1}{\sigma}Q_{N_u^{(n)}+i}$, for any fixed $0 \le u < t \le T$. Then $Q_{u,i}$'s are iid random variables with zero mean and unit variance. Now,

$$\begin{aligned}
\log S_t^{(n)} - \log S_u^{(n)} &= \sum_{i=N_u^{(n)}+1}^{N_t^{(n)}} Z_i^{(n)} \\
&= \sum_{j=1}^{N_t^{(n)}-N_u^{(n)}} \left( \frac{1}{\sqrt{\lambda_n}} Q_{N_u^{(n)}+j} + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2) \right) \\
&= \sum_{j=1}^{N_t^{(n)}-N_u^{(n)}} \left( \frac{\sigma}{\sqrt{\lambda_n}} Q_{u,j} + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2) \right) \\
&= \frac{\sigma}{\sqrt{\lambda_n}} \sum_{j=1}^{N_t^{(n)}-N_u^{(n)}} Q_{u,j} + \frac{1}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)(N_t^{(n)} - N_u^{(n)}).
\end{aligned}$$

By Doeblin-Anscombe's theorem, (Chow and Teicher [2])

$$\frac{\sigma}{\sqrt{\lambda_n}} \sum_{j=1}^{N_t^{(n)}-N_u^{(n)}} Q_{u,j} \xrightarrow{\mathcal{D}} N(0, \sigma^2(t-u)),$$

where $N(a,b)$ denotes the normal distribution with mean $a$ and variance $b$. Since $\frac{N_t^{(n)}-N_u^{(n)}}{\lambda_n}(\mu - \frac{1}{2}\sigma^2)$ converges to $(\mu - \frac{1}{2}\sigma^2)(t-u)$ in probability by Weak Law of Large Numbers,

$$\begin{aligned}
\log S_t^{(n)} - \log S_u^{(n)} &\xrightarrow{\mathcal{D}} N((\mu - \frac{1}{2}\sigma^2)(t-u), \sigma^2(t-u)) \\
&\stackrel{\mathcal{D}}{=} \log S_t - \log S_u,
\end{aligned}$$

for any $0 \le u < t \le T$. So the proposition is proved.

Alternatively, we can also use the martingale central limit theorem to prove this proposition. □

By defining the jump size distribution as in (2), we can interpret the parameters as follows. $\mu$ and $\sigma$ are the leading terms of the expected rate of return and the volatility, respectively, $\frac{k_3}{\sqrt{\lambda_n}}$ is the leading term of the skewness, and $\frac{k_4}{\lambda_n}$ is the leading term of the kurtosis of the log stock price process.

The compound Poisson model that we are assuming here is a reasonable model mathematically and practically. It has independent increments, and finitely many jumps in a finite time interval. Considering that the Black-Scholes model is used as a reasonable approximation in practice, it is reasonable to have the model that is asymptotically a geometric Brownian Motion. Moreover, this model permits incorporating the skewness and the kurtosis of the log stock price process. Nonzero $k_3$ can be used for a better prediction for the stock price movement. The jump intensity $\lambda_n$ gives another layer of flexibility. According to the level of trading activities of stocks that we are dealing with, we can change the value of $\lambda_n$ so that the model fits with the data.

## 3 Transition to the Limit

Suppose we want to hedge a European style payoff $\eta$ which expires at time $T$. The underlying asset price process follows the model (1). Here, we assume that the volatility $\sigma$ is constant and the interest rate $r$ is 0 for simplicity. As long as the interest rate is deterministic, it is not hard to incorporate nonzero interest rate. Let $X$ be the process of the value of the Black-Scholes hedging portfolio, i.e.,

$$X_t = C(S_0^{(n)}, 0) + \int_0^t C_S(S_{u-}^{(n)}, u) dS_u^{(n)}.$$

Throughout the paper, $C(S_t^{(n)}, t)$ denotes the Black-Scholes option price at time $t$. $C(S_t^{(n)}, t)$ is computed based on the Black-Scholes formula, but it is not observed from the market. On the other hand, $C(S_t, t)$ is also computed by the Black-Scholes formula, but it is the true market price in the limit because the limiting stock price $S$ follows the geometric Brownian Motion model. $C_S$, $C_{SS}$, and $C_{SSS}$ denote the first, second, and third derivative of $C(S_t, t)$ with respect to $S$, respectively. $C_S^{(p)}$ is used for the $p$th derivative of $C(S_t, t)$ with respect to $S$, for $p > 3$.

In a complete market, $X$ would be a perfect hedging for $\eta$, but it is not perfect in this setting. Let us look at the hedging error of the Black-Scholes hedging strategy. Mykland [12] showed that $\sqrt{\lambda_n}(C(S_\cdot^{(n)}, \cdot) - X_\cdot)$ converges jointly with $S^{(n)}$ in distribution to $(R, S)$ where

$$R_t = \frac{1}{2}\int_0^t S_u^2 C_{SS}(S_u, u) d\xi_u + \frac{1}{3!}k_3 \int_0^t \left(3S_u^2 C_{SS}(S_u, u) + S_u^3 C_{SSS}(S_u, u)\right) du, \tag{4}$$

$$\xi_t = c_1 \tilde{W}_t + \frac{k_3}{\sigma} B_t, \quad c_1 = \sqrt{k_4 - \left(\frac{k_3}{\sigma}\right)^2},$$

and $\{\tilde{W}_t\}$ and $\{B_t\}$ are independent Brownian Motions under $P$. As we saw in section 2, the limiting stock price process $S$ follows a geometric Brownian Motion model such as

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{5}$$

Let us look at $R_T$ more closely.

$$\begin{aligned} R_T &= \frac{c_1}{2}\int_0^T S_u^2 C_{SS}(S_u,u)d\tilde{W}_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma} S_u^2 C_{SS}(S_u,u)dB_u + \int_0^T f(S_u,u)du \\ &= \frac{c_1}{2}\int_0^T S_u^2 C_{SS}(S_u,u)d\tilde{W}_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T g(S_u,u)du \end{aligned} \tag{6}$$

where

$$\begin{aligned} f(S_u,u) &= \frac{1}{3!}k_3\left(3S_u^2 C_{SS}(S_u,u) + S_u^3 C_{SSS}(S_u,u)\right), \\ g(S_u,u) &= f(S_u,u) - \frac{k_3\mu}{2\sigma^2} S_u^2 C_{SS}(S_u,u). \end{aligned} \tag{7}$$

Define a measure $P^*$ by

$$\frac{dP^*}{dP}|_{\mathcal{F}_t} = \exp\left(-\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right), \tag{8}$$

where $\{\mathcal{F}_t\}$ is a filtration generated by $(\tilde{W}, B)$. Then, $P^*$ is an equivalent martingale measure for the stock price process in the limit, because $E^*(S_t) < \infty$ for all $t \le T$ and

$$E^*(S_t|\mathcal{F}_s) = \frac{E(S_t\frac{dP^*}{dP}|_{\mathcal{F}_t}|\mathcal{F}_s)}{E(\frac{dP^*}{dP}|_{\mathcal{F}_t}|\mathcal{F}_s)} = S_s,$$

for any $0 \le s \le t \le T$. This measure is, in fact, the same as the minimal martingale measure introduced by Föllmer and Schweizer [6]. We will use this $P^*$ later in order to prove Theorem 3.1. Since $\tilde{W}$ is independent of $\frac{dP^*}{dP}$, $\tilde{W}$ remains a standard Brownian Motion under $P^*$ and it is independent of $B$ under $P^*$, too.

**Theorem 3.1** *Assume the conditions in section 2 and 3. The stock price process is governed by (5) and $R_t$ is defined as in (4). If we let $Y_t$ be $\frac{c_1}{2}S_t^2 C_{SS}(S_t,t)$, then*

$$R_T = \int_0^T Y_u d\tilde{W}_u + \int_0^T \frac{k_3}{2\sigma^2} S_u C_{SS}(S_u,u)dS_u + \int_0^T (T-u)g_S(S_u,u)dS_u. \tag{9}$$

*In particular, when the second derivative of the Black-Scholes price $C(S_t,t)$ exists at the expiration time, then*

$$R_T = Y_T\tilde{W}_T + \int_0^T h(\tilde{W}_u, S_u)dS_u + \int_0^T (T-u)g_S(S_u,u)dS_u,$$

*where* $h(\tilde{W}_u, S_u) = -c_1\tilde{W}_u(S_u C_{SS}(S_u,u) + \frac{1}{2}S_u^2 C_{SSS}(S_u,u)) + \frac{k_3}{2\sigma^2}S_u C_{SS}(S_u,u)$ *and* $g_S(S_u,u)$ *denotes the first derivative of* $g(S_u,u)$ *in (7) with respect to* $S$.

*Proof.* See Appendix 6.1.

From the theorem 3.1, we can see that $R_T$ is divided into two parts: replicable part, $\int_0^T\{\frac{k_3}{2\sigma^2}S_u C_{SS}(S_u,u) + (T-u)g_S(S_u,u)\}dS_u$, and non-replicable part, $\int_0^T Y_u d\tilde{W}_u$. The replicable part, $\int_0^T\{\frac{k_3}{2\sigma^2}S_u C_{SS}(S_u,u) + (T-u)g_S(S_u,u)\}dS_u$, is the stochastic integral with respect to the traded asset $S$, so we can replicate this object exactly by holding $\frac{k_3}{2\sigma^2}S_t C_{SS}(S_t,t) + (T-t)g_S(S_t,t)$ shares of the stock and put everything else in the cash bond at each time $t$. On the other hand, $\int_0^T Y_u d\tilde{W}_u$ is the stochastic integral with respect to a Brownian Motion that is independent of $S$, so we cannot replicate this by trading the underlying asset $S$. Since we have a replicable part in the Black-Scholes hedging error, we try to update the Black-Scholes hedging strategy by including the replicable part. Define $H^{(n)}$ to be the value of the new hedging portfolio as follows.

$$\begin{aligned} H_t^{(n)} &= X_t + \frac{1}{\sqrt{\lambda_n}}\int_0^t \frac{k_3}{2\sigma^2}S_{u-}^{(n)}C_{SS}(S_{u-}^{(n)},u)dS_u^{(n)} \\ &\quad + \frac{1}{\sqrt{\lambda_n}}\int_0^t (T-u)g_S(S_{u-}^{(n)},u)dS_u^{(n)}. \end{aligned} \tag{10}$$

When $C_{SS}$ is bounded and away from 0 for all $t \le T$,

$$H_t^{(n)} = X_t + \frac{1}{\sqrt{\lambda_n}}\int_0^t h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)})dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}}\int_0^t (T-u)g_S(S_{u-}^{(n)},u)dS_u^{(n)},$$

where $R_t^{(n)} = \sqrt{\lambda_n}(C(S_t^{(n)},t) - X_t)$ and

$$\begin{aligned} \tilde{W}_t^{(n)} &= \int_0^t \frac{2}{c_1 {S_{u-}^{(n)}}^2 C_{SS}(S_{u-}^{(n)},u)}dR_u^{(n)} - \int_0^t \frac{k_3}{c_1\sigma^2 S_{u-}^{(n)}}dS_u^{(n)} \\ &\quad - \int_0^t \frac{k_3 S_{u-}^{(n)}C_{SSS}(S_{u-}^{(n)},u)}{3c_1 C_{SS}(S_{u-}^{(n)},u)}du + \frac{k_3(\mu-\sigma^2)t}{c_1\sigma^2}. \end{aligned} \tag{11}$$

Now, after updating Black-Scholes hedging portfolio by $H^{(n)}$, $\sqrt{\lambda_n}(\eta - H_T^{(n)})$ is the only uncontrollable part of the payoff. In fact, $\sqrt{\lambda_n}(\eta - H_T^{(n)})$ is purely non-replicable in the sense that it does not have any replicable component, because it converges to a stochastic integral with respect to untradeable $\tilde{W}$. We can show the following convergence result for the new hedging error $\sqrt{\lambda_n}(\eta - H_T^{(n)})$.

**Theorem 3.2** *Assume the conditions of the theorem 3.1. Then,*

$$\sqrt{\lambda_n}(\eta - H_T^{(n)}) \xrightarrow{\mathcal{D}} \int_0^T Y_u d\tilde{W}_u$$

*where* $H_t^{(n)}$ *is defined as in (10). Moreover, when* $\tilde{W}^{(n)}$ *in (11) is well-defined, subject to the conditions (20) and (21) in Appendix 6.2,*

$$\sqrt{\lambda_n}(\eta - H_T^{(n)}) \xrightarrow{\mathcal{D}} Y_T \tilde{W}_T$$

*where* $H_t^{(n)} = X_t + \frac{1}{\sqrt{\lambda_n}} \int_0^t h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)}) dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)}$.

*Proof.* See Appendix 6.2.

When we use $H^{(n)}$ instead of the Black-Scholes hedging strategy, we improve the Black-Scholes hedging strategy in the sense that the mean square of the limiting hedging error is reduced. Note that $H^{(n)}$ is the same as $X$ when the distribution of the log stock price is symmetric.

**Proposition 3.1** *Assume the conditions of theorem 3.1. Also assume that* $C(S_{\cdot}, \cdot)$ *satisfies* $E(\int_0^T S_t^4 C_{SS}^2(S_t, t_d t) < \infty$. *Note that this additional assumption holds in case of a call option. Under these assumptions,*

$$E(\textit{limiting Black-Scholes hedging error})^2$$
$$\geq E(\textit{limiting hedging error for } H^{(n)})^2.$$

*Proof.* Denote $R'_T$ to be $R_T - \int_0^T Y_u d\tilde{W}_u$. The limiting Black-Scholes hedging error is $R_T$, and the limiting hedging error of the updated hedging, $H^{(n)}$, is $\int_0^T Y_u d\tilde{W}_u$. Then,

$$\begin{aligned} ER_T^2 &= E(R'_T + \int_0^T Y_u d\tilde{W}_u)^2 \\ &= E{R'_T}^2 + E(\int_0^T Y_u d\tilde{W}_u)^2 + 2E(R'_T \int_0^T Y_u d\tilde{W}_u) \\ &= E{R'_T}^2 + E(\int_0^T Y_u d\tilde{W}_u)^2 + 2E(R'_T E(\int_0^T Y_u d\tilde{W}_u | S_u, 0 \leq u \leq T)) \\ &= E{R'_T}^2 + E(\int_0^T Y_u d\tilde{W}_u)^2 \geq E(\int_0^T Y_u d\tilde{W}_u)^2. \quad \square \end{aligned}$$

Since the new hedging error $\sqrt{\lambda_n}(\eta - H_T^{(n)})$ is purely non-replicable, we may want to use $H^{(n)}$ as the final choice for the hedging strategy. But we can hope to do something with $\sqrt{\lambda_n}(\eta - H_T^{(n)})$ because the limit has the integrand $Y$ that is a function of the underlying asset price process. If we specify a certain optimality criterion, we would be able to find the best possible hedging for the new hedging error. Consider a process $K_t^{(n)} = K_0 + \int_0^t \theta_u^{(n)} dS_u^{(n)}$ where $\theta^{(n)}$ is a predictable process with respect to the filtration generated by $S^{(n)}$, satisfying $E(\int_0^T (\theta_u^{(n)} S_{u-}^{(n)})^2 du) < \infty$. We want to find an appropriate $\theta^{(n)}$ in order for us to use $K^{(n)}$ to hedge the new hedging error $\sqrt{\lambda_n}(\eta - H_T^{(n)})$. We can show some weak convergence results for $(H^{(n)}, K^{(n)})$ in the next theorem.

**Theorem 3.3** *Assume the conditions of the theorem 3.1. Define $H^{(n)}$ and $K^{(n)}$ as before. Suppose that there exists a process $\theta$ which is predictable with respect to the limiting filtration generated by $(\tilde{W}, S)$ such that $E(\int_0^T \theta_u^2 S_u^2 du) < \infty$, and $\theta^{(n)} \xrightarrow{\mathcal{D}} \theta$ jointly with $S^{(n)}$ and $R^{(n)}$. Then*

$$(\sqrt{\lambda_n}(\eta - H_T^{(n)}), K^{(n)}) \xrightarrow{\mathcal{D}} (\int_0^T Y_u d\tilde{W}_u, K),$$

*where $H_t^{(n)}$ is defined as in (10). If $\tilde{W}^{(n)}$ in (11) is well-defined, subject to the conditions (20) and (21) in Appendix 6.2,*

$$(\sqrt{\lambda_n}(\eta - H_T^{(n)}), K^{(n)}) \xrightarrow{\mathcal{D}} (Y_T \tilde{W}_T, K),$$

*where $H_t^{(n)} = X_t + \frac{1}{\sqrt{\lambda_n}} \int_0^t h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)}) dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)}$.*

*Proof.* See Appendix 6.2.

Now, the value of our new hedging strategy is

$$L_t^{(n)} = H_t^{(n)} + \frac{K_t^{(n)}}{\sqrt{\lambda_n}}.$$

$L^{(n)}$ converges to the value of the continuous time Black-Scholes hedging portfolio as $n$ goes to $\infty$, but it includes the correction terms for the Black-Scholes hedging error, the first order residual term after we hedge the payoff using the Black-Scholes strategy. We will call this as a compound Poisson hedging strategy in the rest of the paper.

If we denote $M_T$ to be $\int_0^T Y_u d\tilde{W}_u$ or $Y_T\tilde{W}_T$, according to the condition of the second derivative of the Black-Scholes option price described earlier, then $(\sqrt{\lambda_n}(\eta - H_T^{(n)}), K_T^{(n)})$ converges jointly to $(M_T, K_T)$ in distribution as $n$ goes to $\infty$ by Theorem 3.3. Using this, we can make the transition of our problem to the limit. Suppose we impose an optimality criterion of the form $EL(\sqrt{\lambda_n}(\eta - L_T^{(n)}))$, for a given convex function $L : \mathbb{R} \to \mathbb{R}$. $L(x) = x$ gives a rather natural choice for the loss function, and $L(x) = x^2$ gives the expected squared loss function. Since $L$ is convex on $\mathbb{R}$, $L$ is also continuous on $\mathbb{R}$. By continuous mapping theorem, (Billingsley [1]) $L(\sqrt{\lambda_n}(\eta - L_T^{(n)}))$ converges to $L(M_T - K_T)$ in distribution. Subject to the uniform integrability condition, therefore, $EL(\sqrt{\lambda_n}(\eta - L_T^{(n)}))$ also converges to $EL(M_T - K_T)$. So if we find the limiting hedging portfolio $\{K_t\}$ minimizing $EL(M_T - K_T)$, then the corresponding $\{L_t^{(n)}\}$ will minimize $EL(\sqrt{\lambda_n}(\eta - L_T^{(n)}))$ for a large enough $n$.

Now, what we want to do is to find a proper hedging portfolio $\{K_t^{(n)}\}$ and the initial investment. In section 4, we use $L(x) = x^2$ for the optimality criterion, and find a proper hedging portfolio $\{K_t\}$ in the limiting market.

# 4 Mean-Variance Hedging

A substantial body of economics literature has considered the problem of maximizing expected utility. A main contribution is that of Merton, who studied the maximization of expected integrated utility in his papers [10], [11]. Because of its mathematical convenience, the squared loss function has been investigated in many previous papers. In connection with the stochastic calculus formulation of trading strategies in continuous time setting, for example, Duffie and Richardson [5] and Schweizer [14] studied the optimal hedging strategy which gives the minimum value of the expected squared loss for a given target level.

We are also going to work with the expected squared loss to deal with the non-replicable part of the Black-Scholes hedging error, $\sqrt{\lambda_n}(\eta - H_T^{(n)})$. Note that everything in the current section is in the limit, in other words, what we want to hedge is $M_T$, either $\int_0^T Y_u d\tilde{W}_u$ or $Y_T\tilde{W}_T$, and the stock price process $\{S_t\}$ follows a geometric Brownian Motion as in (5).

Consider minimizing $E(M_T - K_T)^2$. If we define $G_t(\theta)$ to be $\int_0^t \theta_u dS_u$ where $\theta \in \Theta =$

$\{\theta$ : predictable with respect to $\mathcal{F}^{\tilde{W},S}, E(\int_0^T \theta_u^2 S_u^2 du) < \infty\}$, then $K_t = K_0 + G_t(\theta)$ and the problem is the same as minimizing $E(M_T - K_0 - G_T(\theta))^2$. $\mathcal{F}^{\tilde{W},S}$ is a filtration generated by $\tilde{W}$ and $S$.

Define $\theta_{K_0}$ to be the $\underset{\theta\in\Theta}{\text{argmin}}\ E(M_T - K_0 - G_T(\theta))^2$ for any given value $K_0$. Let us consider the case where $M_T = \int_0^T Y_u d\tilde{W}_u$, first. By Schweizer [14], $G(\theta_{K_0})$ is a solution, $G^*$, of the SDE (12). Schweizer [14] assumes that $K_0$ is negative, but the same argument works for nonnegative $K_0$'s. Although the explicit form of $G^*$ is not given in Schweizer [14], it can be obtained quite easily. In particular, the explicit solution for our problem is as follows.

**Proposition 4.1** *The optimal hedging portfolio, $\{K_t\}$, that makes the expected squared loss, $E(M_T - K_T)^2$, minimized for a given initial value $K_0$ is*

$$\begin{aligned} K_t = K_0 + \int_0^t \frac{\mu}{\sigma^2 S_u} \left(\frac{S_u}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} \\ \times \left(\int_0^u \left(\frac{S_v}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} Y_v d\tilde{W}_v - K_0\right) dS_u. \end{aligned}$$

*Proof.* By Schweizer [14], $\{G_t(\theta_{K_0})\}$ that minimizes $E(M_T - K_0 - G_T(\theta))^2$ is a solution, $\{G_t^*\}$, of the SDE

$$dG_t^* = \frac{\mu}{\sigma^2 S_t}(\int_0^t Y_u d\tilde{W}_u - K_0 - G_t^*)dS_t \tag{12}$$

with $G_0^* = 0$. Define an auxiliary process $H_t^*$ to be

$$H_t^* = e^{-A_t}(K_0 - \int_0^t e^{A_u} Y_u d\tilde{W}_u) - K_0 + \int_0^t Y_u d\tilde{W}_u,$$

where $e^{A_t} = \left(\frac{S_t}{S_0}\right)^{\frac{\mu}{\sigma^2}} \exp(\frac{\mu^2}{2\sigma^2}t + \frac{\mu}{2}t)$. Then, using Itô's formula,

$$\begin{aligned} dH_t^* &= (K_0 - \int_0^t e^{A_u} Y_u d\tilde{W}_u) de^{-A_t} - Y_t d\tilde{W}_t \\ &\quad - < e^{-A_\cdot}, \int_0^\cdot e^{A_u} Y_u d\tilde{W}_u >_t + Y_t d\tilde{W}_t \\ &= (K_0 - \int_0^t e^{A_u} Y_u d\tilde{W}_u)(-\frac{\mu}{\sigma} e^{-A_t} dB_t - \frac{\mu^2}{\sigma^2} e^{-A_t} dt) \\ &= (-\frac{\mu}{\sigma} dB_t - \frac{\mu^2}{\sigma^2} dt)(H_t^* + K_0 - \int_0^t Y_u d\tilde{W}_u). \end{aligned}$$

Note that the quadratic covariation term is 0 becaues $\tilde{W}$ and $B$ are independent. Since $-\frac{\mu}{\sigma}dB_t - \frac{\mu^2}{\sigma^2}dt = -\frac{\mu}{\sigma^2 S_t}dS_t$,

$$dH_t^* = \frac{\mu}{\sigma^2 S_t}(\int_0^t Y_u d\tilde{W}_u - K_0 - H_t^*)dS_t.$$

$H^*$ satisfies the same SDE with the same initial value as $G^*$, so

$$G_t^* = e^{-A_t}(K_0 - \int_0^t e^{A_u} Y_u d\tilde{W}_u) - K_0 + \int_0^t Y_u d\tilde{W}_u,$$

and by Schweizer [14],

$$\theta_{K_0,t} = \frac{\mu}{\sigma^2 S_t}(\int_0^t Y_u d\tilde{W}_u - K_0 - G_t^*),$$

which is

$$\theta_{K_0,t} = \frac{\mu}{\sigma^2 S_t}\left(\frac{S_t}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)t}\left(\int_0^t \left(\frac{S_u}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} Y_u d\tilde{W}_u - K_0\right).$$

Similar proof can be used for a general setting given in Schweizer [14]. □

In the pre-limiting stage, we adopt the hedging portfolio $K^{(n)}$ for a given $K_0$ such as

$$\begin{aligned} K_t^{(n)} = K_0 + \int_0^t & \frac{\mu}{\sigma^2 S_{u-}^{(n)}}\left(\frac{S_{u-}^{(n)}}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} \\ & \times\left(\int_0^u \left(\frac{S_{v-}^{(n)}}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} dV_v^{(n)} - K_0\right) dS_u^{(n)}, \end{aligned} \tag{13}$$

where $dV_v^{(n)}$ denotes $dR_v^{(n)} - \frac{k_3}{2\sigma^2}S_{v-}^{(n)}C_{SS}(S_{v-}^{(n)}, v)dS_v^{(n)} - (T-v)g_S(S_{v-}^{(n)}, v)dS_v^{(n)}$ and $R_t^{(n)} = \sqrt{\lambda_n}(C(S_t^{(n)}, t) - X_t)$.

So far, what we wanted to hedge, $M_T$, was $\int_0^T Y_u d\tilde{W}_u$. Notice that when we have $M_T = Y_T\tilde{W}_T$, we end up with the same hedging portfolio. In this case, $G(\theta_{K_0})$ minimizing $E(M_T - K_0 - G_T(\theta))^2$ is a solution, $G^*$, of the SDE

$$dG_t^* = (c_1\tilde{W}_t(S_t C_{SS}(S_t,t) + \frac{1}{2}S_t^2 C_{SSS}(S_t,t)) + \frac{\mu}{\sigma^2 S_t}(Y_t\tilde{W}_t - K_0 - G_t^*))dS_t$$

with $G_0^* = 0$. Define $H_t^*$ to be

$$H_t^* = e^{-A_t}(K_0 - \int_0^t e^{A_u} Y_u d\tilde{W}_u) - K_0 + Y_t\tilde{W}_t,$$

where $e^{A_t} = \left(\frac{S_t}{S_0}\right)^{\frac{\mu}{\sigma^2}} \exp(\frac{\mu^2}{2\sigma^2}t + \frac{\mu}{2}t)$. Then, similarly to the proposition 4.1,

$$dH_t^* = \frac{\mu}{\sigma^2 S_t}(Y_t\tilde{W}_t - K_0 - H_t^*)dS_t + c_1\tilde{W}_t(S_tC_{SS}(S_t,t) + \frac{1}{2}S_t^2C_{SSS}(S_t,t))dS_t.$$

Note that $dY_t = c_1(S_tC_{SS}(S_t,t) + \frac{1}{2}S_t^2C_{SSS}(S_t,t))dS_t$. (See Appendix 6.1) Since $H^*$ satisfies the same SDE with the same initial value as $G^*$,

$$G_t^* = e^{-A_t}(K_0 - \int_0^t e^{A_u}Y_u d\tilde{W}_u) - K_0 + Y_t\tilde{W}_t,$$

and by Schweizer [14],

$$\theta_{K_0,t} = c_1\tilde{W}_t(S_tC_{SS}(S_t,t) + \frac{1}{2}S_t^2C_{SSS}(S_t,t)) + \frac{\mu}{\sigma^2 S_t}\left(\frac{S_t}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}t - \frac{\mu}{2}t}$$
$$\times \left(\int_0^t \left(\frac{S_u}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{\mu^2}{2\sigma^2}u + \frac{\mu}{2}u} Y_u d\tilde{W}_u - K_0\right).$$

Therefore, the optimal hedging portfolio for a given initial value $K_0$ is

$$\begin{aligned} K_t = K_0 &+ \int_0^t c_1\tilde{W}_u(S_uC_{SS}(S_u,u) + \frac{1}{2}S_u^2C_{SSS}(S_u,u))dS_u \\ &+ \int_0^t \frac{\mu}{\sigma^2 S_u}e^{-A_u}\left(\int_0^u e^{A_v}Y_v d\tilde{W}_v - K_0\right)dS_u, \end{aligned}$$

where $\exp(A_t) = \left(\frac{S_t}{S_0}\right)^{\frac{\mu}{\sigma^2}} \exp(\frac{\mu^2}{2\sigma^2}t + \frac{\mu}{2}t)$. This is the sum of $\int_0^t c_1\tilde{W}_u(S_uC_{SS}(S_u,u) + \frac{1}{2}S_u^2C_{SSS}(S_u,u))dS_u$ and the optimal hedging for the case of $M_T = \int_0^T Y_u d\tilde{W}_u$, but when we go back to the pre-limiting stage, we get exactly the same hedging strategy. In either case, the value of the resulting pre-limiting hedging strategy is

$$\begin{aligned} L_t^{(n)} &= C(S_0,0) + \int_0^t C_S(S_{u-}^{(n)},u)dS_u^{(n)} \\ &+ \frac{K_0}{\sqrt{\lambda_n}} + \frac{1}{\sqrt{\lambda_n}}\int_0^t \frac{k_3}{2\sigma^2}S_{u-}^{(n)}C_{SS}(S_{u-}^{(n)},u)dS_u^{(n)} \\ &+ \frac{1}{\sqrt{\lambda_n}}\int_0^t (T-u)g_S(S_{u-}^{(n)},u)dS_u^{(n)} \\ &+ \frac{1}{\sqrt{\lambda_n}}\int_0^t \frac{\mu}{\sigma^2 S_{u-}^{(n)}}\left(\frac{S_{u-}^{(n)}}{S_0}\right)^{-\frac{\mu}{\sigma^2}} e^{-\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)u} \\ &\times \left(\int_0^u \left(\frac{S_{v-}^{(n)}}{S_0}\right)^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(\frac{\mu^2}{\sigma^2}+\mu)v} dV_v^{(n)} - K_0\right) dS_u^{(n)}. \end{aligned} \tag{14}$$

Therefore, as far as the squared error loss is concerned, the existence of the second derivative of the Black-Scholes price does not make any difference in the optimal hedging strategy. Notice that the compound Poisson hedging strategy, $L^{(n)}$, that we obtained in this section is determined uniquely for any given value of $K_0$.

The initial investment of the compound Poisson hedging portfolio is the Black-Scholes price plus $K_0/\sqrt{\lambda_n}$. By choosing a reasonable value of $K_0$, we determine the initial investment of our hedging portfolio, and we can use the initial investment as the price of the option. Some reasonable choices for $K_0$ are suggested in Song [15].

# 5 Simulation

We did a simulation experiment with a European call option that is expired in 3 months. The interest rate is assumed to be 0, the expected rate of return of the stock, $\mu$, is set to be 0.15 per annum, and the volatility, $\sigma$, is set to be 0.21 per annum. We tried the strike price $K = \$1000$ and the initial stock price $S_0 = \$950$. We compared the performance of the Black-Scholes and the compound Poisson hedging strategies by calculating the mean squares of hedging errors (abbreviated by MSHE). Hedging error means the option payoff subtracted by the value of the hedging portfolio at the expiration. For compound Poisson model, we used three different jump intensities, $\lambda_n = 1000$, 10000, and 100000. $\lambda_n = 10000$ means that we expect 10,000 jumps per year in the stock price on average. Larger $\lambda_n$ implies that the stock is more heavily traded. The hedging interval is .0001 years that is 52 minutes and 34 seconds. It means that we rebalance the hedging portfolio every 52 minutes and 34 seconds. We choose $K_0 = 0$, which implies that the option price is the same as the Black-Scholes option price.

Any distribution with the moment condition given in (2) can be used as the jump size distribution for the compound Poisson model. For example, $N(0, \sigma^2)$ can be used as a symmetric jump size distribution, and $\sigma - \text{Exp}(\frac{1}{\sigma})$ can be used as a left skewed jump size distribution. We used $\sigma - \text{Exp}(\frac{1}{\sigma})$ as the jump size distribution in the simulation experiment. In this case, $k_3 = -2\sigma^3$, and $k_4 = 9\sigma^4$. The simulation size is 5,000, that is, the number of generated sample paths is 5,000. The Black-Scholes initial price is \$20.59, in this case.

In general, both of the hedging strategies perform much better the bigger $\lambda_n$ gets in terms of the magnitude of MSHE, because the stock price process is getting closer to a geometric Brownian Motion. The compound Poisson hedging has smaller MSHE than the Black-Scholes hedging overall. It means that the value of the compound Poisson hedging portfolio at the expiration time is closer to the payoff in the sense that the mean square of the difference is smaller. We also compare densities of hedging errors in Figure 1. The compound Poisson hedging error (dashed line) has less spread and is more symmetric than the Black-Scholes hedging error (solid line). In other words, the compound Poisson hedging strategy makes the distribution of hedging errors less biased as well as it makes the distribution less variable. The MSHE reduction comes from both of the mean square part and the variance part.

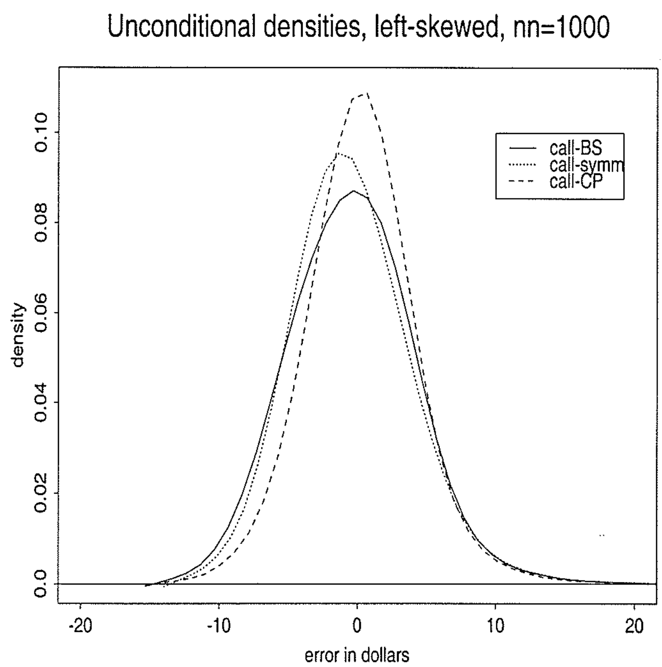


Figure 1: Comparison of densities, left-skewed, $\lambda_n = 1000$

Table 1: Mean squares of hedging errors, left skewed, unit=$^2

| | $\lambda_n = 1,000$ | $\lambda_n = 10,000$ | $\lambda_n = 100,000$ |
|---|---|---|---|
| call−BS | 18.17397 | 2.44599 | 0.67068 |
| call−CP | 12.46278 | 1.75333 | 0.58495 |
| reduction(BS vs. CP) | 31.4% | 28.3% | 12.8% |

# 6 Appendix

## 6.1 Proof of Theorem 3.1

From (6),

$$R_T = \int_0^T Y_u d\tilde{W}_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u, u) dS_u + \int_0^T g(S_u, u) du.$$

We want to show that

$$\int_0^T g(S_u, u) du = \int_0^T (T-u) g_S(S_u, u) dS_u.$$

Since the interest rate $r$ is assumed to be 0, $S$ is a martingale under $P^*$ and we can write

$$dS_u = \sigma S_u dB_u^* \tag{15}$$

where $B^*$ is a martingale under $P^*$. If we let $\{\mathcal{F}_t\}$ be an augmentation of the filtration $\{\mathcal{F}_t^{B^*}\}$ generated by $B^*$, then $\int_0^T g(S_u, u)du$ is $\mathcal{F}_T$-measurable and finite almost surely. By Dudley's theorem, (Karatzas and Shreve [8], p.188 or Duffie [4], p.287) there exists a progressively measurable process $\tilde{Y} = \{\tilde{Y}_t, \mathcal{F}_t; 0 \le t \le T\}$ satisfying $\int_0^T \tilde{Y}_t^2 dt < \infty$ almost surely under $P^*$ such that

$$\int_0^T g(S_u, u) du = \int_0^T \tilde{Y}_u dB_u^* = \int_0^T \tilde{Y}_u \frac{1}{\sigma S_u} dS_u. \tag{16}$$

Note that $g(S_u, u)$ is a $P^*$-martingale because $S_t^p C_S^{(p)}(S_t, t)$ is a $P^*$-martingale for any positive integer $p$. Define $\zeta_t$ to be $E^*(\int_0^T g(S_u, u)du \mid \mathcal{F}_t)$. Then,

$$\begin{aligned}
\zeta_t &= \int_0^t g(S_u, u)du + E^*\left(\int_t^T g(S_u, u)du \mid \mathcal{F}_t\right) \\
&= \int_0^t g(S_u, u)du + \int_t^T E^*(g(S_u, u) \mid \mathcal{F}_t)du \\
&= \int_0^t g(S_u, u)du + (T-t)g(S_t, t).
\end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned}
\zeta_t &= E^*\left(\int_0^T g(S_u, u)du \mid \mathcal{F}_t\right) \\
&= E^*\left(\int_0^T \tilde{Y}_u \frac{1}{\sigma S_u} dS_u \mid \mathcal{F}_t\right) \\
&= \int_0^t \tilde{Y}_u \frac{1}{\sigma S_u} dS_u.
\end{aligned}$$

Thus,

$$\begin{aligned}
<\zeta\ , B^*>_t &= \int_0^t \tilde{Y}_u \frac{1}{\sigma S_u} d <S, B^*>_u \\
&= \int_0^t \tilde{Y}_u \frac{1}{\sigma S_u} \sigma S_u du \\
&= \int_0^t \tilde{Y}_u du,
\end{aligned}$$

and,

$$\tilde{Y}_t = \frac{d}{dt} <\zeta\ , B^*>_t .$$

By the way, from (17),

$$\begin{aligned}
d\zeta_t &= g(S_t, t)dt - g(S_t, t)dt + (T-t)dg(S_t, t) \\
&= (T-t)dg(S_t, t) \\
&= (T-t)(g_S(S_t, t)dS_t + g_t(S_t, t)dt + \frac{1}{2}g_{SS}(S_t, t)\sigma^2 S_t^2 dt).
\end{aligned}$$

Thus,

$$\tilde{Y}_t = \frac{d}{dt} <\zeta\ , B^*>_t = (T-t)g_S(S_t, t)\sigma S_t \tag{18}$$

and this process is progressively measurable and satisfies $\int_0^T \tilde{Y}_t^2 dt < \infty$ almost surely. Combining (16) and (18),

$$R_T = \int_0^T Y_u d\tilde{W}_u + \int_0^T \frac{k_3}{2\sigma^2} S_u C_{SS}(S_u, u) dS_u + \int_0^T (T-u) g_S(S_u, u) dS_u.$$

In case where the second derivative of Black-Scholes price exists at time $T$, by applying Itô's formula to $\int_0^T Y_u d\tilde{W}_u$, we get

$$R_T = Y_T \tilde{W}_T - \int_0^T \tilde{W}_u dY_u + \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u, u) dS_u + \int_0^T g(S_u, u) du.$$

Again, apply the Itô's formula to $Y_t$ to obtain

$$\begin{aligned} dY_t &= d(\frac{c_1}{2} S_t^2 C_{SS}(S_t, t)) \\ &= c_1(S_t C_{SS}(S_t, t) + \frac{1}{2} S_t^2 C_{SSS}(S_t, t)) dS_t \\ &+ \frac{c_1}{2} S_t^2 (C_{SSt}(S_t, t) + \frac{1}{2}\sigma^2(2C_{SS}(S_t, t) + 4S_t C_{SSS}(S_t, t) + S_t^2 C_S^{(4)}(S_t, t))) dt. \end{aligned}$$

On the other hand, we know from the Black-Scholes PDE,

$$-rC + C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} = 0$$

where $C$ is the Black-Scholes price. When $r = 0$,

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} = 0.$$

Thus,

$$\frac{\partial^2}{\partial S^2}(C_t + \frac{1}{2}\sigma^2 S^2 C_{SS}) = C_{SSt} + \frac{1}{2}\sigma^2(2C_{SS} + 4SC_{SSS} + S^2 C_S^{(4)}) = 0.$$

Therefore,

$$dY_t = c_1(S_t C_{SS}(S_t, t) + \frac{1}{2} S_t^2 C_{SSS}(S_t, t)) dSt.$$

Now, $R_T$ becomes

$$\begin{aligned} R_T &= Y_T \tilde{W}_T - c_1 \int_0^T \tilde{W}_u (S_u C_{SS}(S_u, u) + \frac{1}{2} S_u^2 C_{SSS}(S_u, u)) dS_u \\ &+ \frac{1}{2}\int_0^T \frac{k_3}{\sigma^2} S_u C_{SS}(S_u, u) dS_u + \int_0^T g(S_u, u) du \\ &= Y_T \tilde{W}_T + \int_0^T h(\tilde{W}_u, S_u) dS_u + \int_0^T g(S_u, u) du. \end{aligned} \tag{19}$$

Combining (19) and previous arguments,

$$R_T = Y_T \tilde{W}_T + \int_0^T h(\tilde{W}_u, S_u) dS_u + \int_0^T (T-u) g_S(S_u, u) dS_u. \quad \square$$

### 6.2 Proof of Theorems 3.2 and 3.3

Let $M_t^{(n)}$ be $\sqrt{\lambda_n}(C(S_t^{(n)}, t) - H_t^{(n)})$. First, we want to show

$$M_T^{(n)} \xrightarrow{\mathcal{D}} \int_0^T Y_u d\tilde{W}_u,$$

where $H_t^{(n)}$ is defined as in (10). We will show this in the following three steps. For $R_t^{(n)} = \sqrt{\lambda_n}(C(S_t^{(n)}, t) - X_t)$,

(i) $(S^{(n)}, R^{(n)}, f(S^{(n)})) \xrightarrow{\mathcal{D}} (S, R, f(S))$ for any continuous function $f$,

(ii) $(S^{(n)}, R^{(n)}, \int f(S^{(n)})dS^{(n)}) \xrightarrow{\mathcal{D}} (S, R, \int f(S)dS)$, for any continuous function $f$, and

(iii) $M_T^{(n)} \xrightarrow{\mathcal{D}} \int_0^T Y_u d\tilde{W}_u$.

*Proof.*

(i) We know that $(R^{(n)}, S^{(n)})$ converges to $(R, S)$ weakly. By the continuous mapping theorem, for any continuous function $f$,

$$((S^{(n)}, R^{(n)}, f(S^{(n)})) \xrightarrow{\mathcal{D}} (S, R, f(S)).$$

(ii) $S^{(n)}$, $R^{(n)}$, $f(S^{(n)})$ are $\{\mathcal{F}_t^{S^{(n)}}\}$–adapted, cádlág processes, and $S^{(n)}$ is a semimartingale. Since $\log S^{(n)}$ follows (1),

$$\Delta S_t^{(n)} = S_{t-}^{(n)}(\exp(Z_{N_t^{(n)}}^{(n)}) - 1)I_{\{\Delta N_t^{(n)}=1\}}.$$

The first order optional and predictable variations of $S^{(n)}$ are

$$[S^{(n)}]_t = \sum_{i=1}^{N_t^{(n)}} \Delta S_{\tau_i^{(n)}}^{(n)} = \sum_{i=1}^{N_t^{(n)}} S_{\tau_{i-}^{(n)}}^{(n)}(\exp(Z_i^{(n)}) - 1)$$

where $\tau_i^{(n)}$ is the time of the $i$th jump of $\log S^{(n)}$ and

$$< S^{(n)} >_t = E(\exp(Z^{(n)}) - 1) \int_0^t S_{u-}^{(n)} \lambda_n du$$

by the uniqueness of Doob-Meyer decomposition.(Karatzas and Shreve [8], p.24-25) Thus,

$$S^{(n)} = S_0^{(n)} + M^{(n)} + A^{(n)}$$

where $M^{(n)}$ is a martingale which is $[S^{(n)}] - < S^{(n)} >$ and $A^{(n)}$ is a finite variation process which is $< S^{(n)} >$. Since $M^{(n)}$ is a martingale, $\sup_n E(M_n(t)) = 0$, and

$$
\begin{aligned}
T_t(A_n) &= \sup_{n\geq 1} \sum_{k=1}^{2^n} |A_n(\frac{tk}{2^n}) - A_n(\frac{t(k-1)}{2^n})| \\
&= A_n(t) - A_n(0) \\
&= E(\exp(Z^{(n)}) - 1) \int_0^t S_{u-}^{(n)} \lambda_n du,
\end{aligned}
$$

because $A_n(t)$ is monotone. Since $E(\exp(Z^{(n)}) - 1) = \mu\lambda_n^{-1} + O(\lambda_n^{-3/2})$,

$$
E(T_t(A_n)) = (\mu + O(\lambda_n^{-1/2}))S_0^{(n)} \int_0^t \exp(\mu u + O(\lambda_n^{-1/2}))du
$$

and $\sup_n E(T_t(A_n)) < \infty$. Thus, $S^{(n)}$ satisfies the condition of the integrator in Theorem 2.7 in Kurtz and Protter [9] with $\tau_n^\alpha = T \vee (\alpha + 1)$. Therefore, (ii) is proved.

(iii) By (i) and (ii),

$$
\begin{aligned}
M_T^{(n)} &= R_T^{(n)} - \int_0^T (\frac{k_3}{2\sigma^2} S_{u-}^{(n)} C_{SS}(S_{u-}^{(n)}, u) + (T-u) g_S(S_{u-}^{(n)}, u)) dS_u^{(n)} \\
&\xrightarrow{\mathcal{D}} R_T - \int_0^T (\frac{k_3}{2\sigma^2} S_u C_{SS}(S_u, u) + (T-u) g_S(S_u, u)) dS_u \\
&= \int_0^T Y_u d\tilde{W}_u. \qquad \square
\end{aligned}
$$

Secondly, when $C_{SS}$ is bounded and away from 0 and $C_{SSS}$ is defined almost surely, we want to show

$$
M_T^{(n)} \xrightarrow{\mathcal{D}} Y_T \tilde{W}_T,
$$

with $H_t^{(n)} = X_t + \frac{1}{\sqrt{\lambda_n}} \int_0^t h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)}) dS_u^{(n)} + \frac{1}{\sqrt{\lambda_n}} \int_0^t (T-u) g_S(S_{u-}^{(n)}, u) dS_u^{(n)}$. Assume the following conditions.

$$
\begin{aligned}
\sup_n E \int_0^T \{ |C_{SS}(S_{u-}^{(n)}, u)| (S_{u-}^{(n)})^2 + \sum_{v=4}^{\infty} \frac{1}{v!} |C_{\log S}(S_{u-}^{(n)}, u) (Z_{N_u^{(n)}}^{(n)})^v | \lambda_n^{3/2} \\
+ |Z_{N_u^{(n)}}^{(n)}|^3 (l_1 + l_2)(S_u^{(n)}, u) \lambda_n^{3/2} \} du < \infty,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\text{where} \quad l_1(S_u^{(n)}, u) &= \sup_{x \in S_u^{(n)} \pm \Delta S_u^{(n)}} |3x^2 C_{SS}(x, u) + x^3 C_{SSS}(x, u)|, \\
l_2(S_u^{(n)}, u) &= \sup_{x \in S_u^{(n)} \pm \Delta S_u^{(n)}} |C_{\log S}(x, u) - C_{\log S}(S_{u-}^{(n)}, u)|,
\end{aligned}
$$

and

$$E\int_0^T C_{SS}^2(S_{u-}^{(n)},u)(S_{u-}^{(n)})^4du < \infty. \tag{21}$$

Subject to the above conditions, we need to show the following.

(i) $(S^{(n)}, R^{(n)}, \int f(S^{(n)})dR^{(n)}) \xrightarrow{\mathcal{D}} (S, R, \int f(S)dR)$ for any continuous function $f$,

(ii) $(\tilde{W}^{(n)}, R^{(n)}, S^{(n)}) \xrightarrow{\mathcal{D}} (\tilde{W}, R, S)$,

(iii) $(\tilde{W}^{(n)}, R^{(n)}, S^{(n)}, \int f(\tilde{W}^{(n)}, S^{(n)})dS^{(n)}) \xrightarrow{\mathcal{D}} (\tilde{W}, R, S, \int f(\tilde{W}, S)dS)$ for any continuous function $f$, and

(iv) $M_T^{(n)} \xrightarrow{\mathcal{D}} Y_T\tilde{W}_T$.

*Proof.*

(i) We know $(S^{(n)}, R^{(n)}, f(S^{(n)})) \xrightarrow{\mathcal{D}} (S, R, f(S))$ from the previous proof. By Itô's formula and Taylor expansion,

$$\begin{aligned}
R_t^{(n)} &= \sqrt{\lambda_n}(C(S_t^{(n)},t) - X_t)\\
&= \frac{1}{2}\int_0^t C_{SS}(S_{u-}^{(n)},u)(S_{u-}^{(n)})^2 d(\sqrt{\lambda_n}([\log S^{(n)}, \log S^{(n)}]_u - \sigma^2 u))\\
&+ \frac{1}{3!}\sqrt{\lambda_n}\int_0^t \{3(\tilde{Z}_u^{(n)})^2 C_{SS}(\tilde{Z}_u^{(n)},u)\\
&+ (\tilde{Z}_u^{(n)})^3 C_{SSS}(\tilde{Z}_u^{(n)},u)\} d[\log S^{(n)}, \log S^{(n)}, \log S^{(n)}]_u\\
&+ \frac{1}{3!}\sqrt{\lambda_n}\int_0^t (C_{\log S}(\tilde{Z}_u^{(n)},u) - C_{\log S}(S_{u-}^{(n)},u))\\
&\times d[\log S^{(n)}, \log S^{(n)}, \log S^{(n)}]_u\\
&- \sum_{v=4}^{\infty}\frac{1}{v!}\sqrt{\lambda_n}\int_0^t C_{\log S}(S_{u-}^{(n)},u) d[\log S^{(n)}, \cdots, \log S^{(n)}]_u^v,
\end{aligned}$$

where $|\tilde{Z}_t^{(n)} - S_t^{(n)}| \leq |\Delta S_t^{(n)}|$, and $[\log S^{(n)}, \cdots, \log S^{(n)}]^v$ is the $v$th order optional variation of $\log S^{(n)}$. Let us divide $R^{(n)}$ into two parts, the local martingale $M_n$ and the adapted, finite variation process $A_n$. Then,

$$\begin{aligned}
A_n &= \frac{1}{2}\int_0^t C_{SS}(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^2 \frac{1}{\sqrt{\lambda_n}}(\mu - \frac{1}{2}\sigma^2)^2 du \\
&+ \frac{1}{3!}\sqrt{\lambda_n}\int_0^t \{3(\tilde{Z}_u^{(n)})^2 C_{SS}(\tilde{Z}_u^{(n)}, u) \\
&+ (\tilde{Z}_u^{(n)})^3 C_{SSS}(\tilde{Z}_u^{(n)}, u)\} d[\log S^{(n)}, \log S^{(n)}, \log S^{(n)}]_u \\
&+ \frac{1}{3!}\sqrt{\lambda_n}\int_0^t (C_{\log S}(\tilde{Z}_u^{(n)}, u) - C_{\log S}(S_{u-}^{(n)}, u)) \\
&\times d[\log S^{(n)}, \log S^{(n)}, \log S^{(n)}]u \\
&- \sum_{v=4}^{\infty} \frac{1}{v!}\sqrt{\lambda_n}\int_0^t C_{\log S}(S_{u-}^{(n)}, u) d[\log S^{(n)}, \cdots, \log S^{(n)}]_u^v,
\end{aligned}$$

and

$$\begin{aligned}
M_n &= \frac{1}{2}\int_0^t C_{SS}(S_{u-}^{(n)}, u)(S_{u-}^{(n)})^2 \sqrt{\lambda_n} \\
&\times d([\log S^{(n)}, \log S^{(n)}]_u - < \log S^{(n)}, \log S^{(n)} >_u).
\end{aligned}$$

$A_n$ is a finite variation process by Theorem II.17(p.54) and Corollary II.1(p.60) in Protter [13]. By Corollary II.3(p.66) in Protter [13], the local martingale $M_n$ is a martingale because

$$\begin{aligned}
E([M_n, M_n]_t) &= \frac{1}{4} E\int_0^t (C_{SS}(S_{u-}^{(n)}, u))^2 (S_{u-}^{(n)})^4 \lambda_n \\
&\quad \times d[\log S^{(n)}, \log S^{(n)}, \log S^{(n)}, \log S^{(n)}]_u \\
&= \frac{1}{4} E\int_0^t (C_{SS}(S_{u-}^{(n)}, u))^2 (S_{u-}^{(n)})^4 \lambda_n (Z_{N_u^{(n)}}^{(n)})^4 dN_u^{(n)} \\
&= \frac{1}{4} E\int_0^t (C_{SS}(S_{u-}^{(n)}, u))^2 (S_{u-}^{(n)})^4 \lambda_n^2 (Z_{N_u^{(n)}}^{(n)})^4 du \\
&\le \frac{1}{4} E\int_0^T (C_{SS}(S_{u-}^{(n)}, u))^2 (S_{u-}^{(n)})^4 \lambda_n^2 (Z_{N_u^{(n)}}^{(n)})^4 du \\
&= \frac{1}{4}(k_4 + o(1)) \int_0^T E(C_{SS}(S_{u-}^{(n)}, u))^2 (S_{u-}^{(n)})^4) du < \infty,
\end{aligned}$$

by the assumption (21). Therefore,

$$\sup_n E(M_n(t)) = 0. \tag{22}$$

The total variation of $A_n$ on $[0,t]$ is

$$\begin{aligned}
T_t(A_n) &= \sup_{n\geq 1}\sum_{k=1}^{2^n}|A_{n,\frac{tk}{2^n}} - A_{n,\frac{t(k-1)}{2^n}}| \\
&\leq \frac{(\mu-\frac{1}{2}\sigma^2)^2}{2\sqrt{\lambda_n}}\int_0^T |C_{SS}(S_{u-}^{(n)},u)|(S_{u-}^{(n)})^2 du \\
&\quad + \sum_{v=4}^{\infty}\frac{\sqrt{\lambda_n}}{v!}\int_0^T |C_{\log S}(S_{u-}^{(n)},u)(Z_{N_u^{(n)}}^{(n)})^v|dN_u^{(n)} \\
&\quad + \frac{\sqrt{\lambda_n}}{3!}\int_0^T (l_1(S_u^{(n)},u)+l_2(S_u^{(n)},u))|Z_{N_u^{(n)}}^{(n)}|^3 dN_u^{(n)}
\end{aligned}$$

$$\begin{aligned}
\text{where}\quad l_1(S_u^{(n)},u) &= \sup_{x\in S_u^{(n)}\pm\Delta S_u^{(n)}}|3x^2C_{SS}(x,u)+x^3C_{SSS}(x,u)|, \\
l_2(S_u^{(n)},u) &= \sup_{x\in S_u^{(n)}\pm\Delta S_u^{(n)}}|C_{\log S}(x,u)-C_{\log S}(S_{u-}^{(n)},u)|.
\end{aligned}$$

Thus,

$$\begin{aligned}
E(T_t(A_n)) &\leq \frac{(\mu-\frac{1}{2}\sigma^2)^2}{2\sqrt{\lambda_n}}E\int_0^T |C_{SS}(S_{u-}^{(n)},u)|(S_{u-}^{(n)})^2 du \\
&\quad + \sum_{v=4}^{\infty}\frac{\lambda_n^{3/2}}{v!}E\int_0^T |C_{\log S}(S_{u-}^{(n)},u)(Z_{N_u^{(n)}}^{(n)})^v|du \\
&\quad + \frac{\lambda_n^{3/2}}{3!}E\int_0^T (l_1(S_u^{(n)},u)+l_2(S_u^{(n)},u))|Z_{N_u^{(n)}}^{(n)}|^3 du.
\end{aligned}$$

By the assumption (20),

$$\sup_n E(T_t(A_n)) < \infty. \tag{23}$$

Combining (22) and (23), we can see that $R^{(n)}$ satisfies the conditions of the integrator in Theorem 2.7 in Kurtz and Protter [9] with $\tau_n^\alpha = T \vee (\alpha+1)$. Therefore,

$$(S^{(n)}, R^{(n)}, \int f(S^{(n)})dR^{(n)}) \xrightarrow{\mathcal{D}} (S, R, \int f(S)dR).$$

(ii) For continuous functions $f_1$, $f_2$, and $f_3$,

$$\tilde{W}^{(n)} = \int_0^t f_s(S_u^{(n)})dR^{(n)} + \int_0^t f_2(S_u^{(n)})dS_u^{(n)} + \int_0^t f_3(S_u^{(n)})du.$$

We know that $(S^{(n)}, R^{(n)}, f_1(S^{(n)}), f_2(S^{(n)}), \int f_1(S^{(n)})dR^{(n)}, \int f_2(S^{(n)})dS^{(n)})$ converges weakly to $(S, R, f_1(S), f_2(S), \int f_1(S)dR, \int f_2(S)dS)$. Since $f_3(S)$ is continuous, by Proposition VI.1.17 in Jacod and Shiryaev [7] and the continuous mapping theorem,

$$(\tilde{W}^{(n)}, R^{(n)}, S^{(n)}) \xrightarrow{\mathcal{D}} (\tilde{W}, R, S).$$

(iii) By Theorem 2.7 in Kurtz and Protter [9],

$$(\tilde{W}^{(n)}, R^{(n)}, S^{(n)}, \int f(\tilde{W}^{(n)}, S^{(n)})dS^{(n)}) \xrightarrow{\mathcal{D}} (\tilde{W}, R, S, \int f(\tilde{W}, S)dS)$$

for any continuous $f$.

(iv) By (i), (ii), and (iii),

$$\begin{aligned} M_T^{(n)} &= R_T^{(n)} - \int_0^T (h(\tilde{W}_{u-}^{(n)}, S_{u-}^{(n)}) + (T-u)g_S(S_{u-}^{(n)}, u)dS_u^{(n)} \\ &\xrightarrow{\mathcal{D}} R_T - \int_0^T (h(\tilde{W}_u, S_u) + (T-u)g_S(S_u, u))dS_u \\ &= Y_T \tilde{W}_T. \quad \square \end{aligned}$$

Now, for theorem 3.3, we want to show the joint convergence of $(M_T^{(n)}, K^{(n)})$. Since we assume that $\theta^{(n)}$ converges in distribution to $\theta$ jointly with $S^{(n)}$ and $R^{(n)}$, the convergence is trivial from the proof of the theorem 3.2. In case where $M_T = Y_T \tilde{W}_T$, we need the assumptions (20) and (21). □

## References


[1] P. Billingsley, *Probability and Measure*, second ed., Wiley, 1986.

[2] Y. S. Chow and H. Teicher, *Probability Theory: independence, interchangeability, martingales*, third ed., Springer-Verlag, 1997.

[3] M. Dritschel and P. Protter, *"Complete Markets with Discontinuous Security Price"*, Finance and Stochastics **3** (1999), 203–214.

[4] D. Duffie, *Dynamic Asset Pricing Theory*, second ed., Princeton University Press, 1996.

[5] D. Duffie and H. R. Richardson, *"Mean-Variance Hedging in Continuous Time"*, Ann. Appl. Prob. **1** (1991), 1–15.

[6] H. Föllmer and M. Schweizer, *"Hedging of Contingent Claims under Incomplete Information"*, Applied Stochastic Analysis, eds. M. H. A. Davis and R. J. Elliot (1991), 389–414.

[7] J. Jacod and A. N. Shiryeav, *Limit Theorems for Stochastics Processes*, Springer-Verlag, 1987.

[8] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, second ed., Springer-Verlag, 1991.

[9] T. G. Kurtz and P. Protter, *"Weak Limit Theorems for Stochastic Integrals and Stochastic Differential Equations"*, Ann. Prob. **19** (1991), 1035–1070.

[10] R. C. Merton, *"Lifetime Protfolio Selection under Uncertainty:The Continuous Time Case"*, Review of Economics and Statistics **51** (1969), 247–257.

[11] ———, *"Optimum Consumption and Portfolio Rules in a Continuous-Time Model"*, J. Econ. Theory **3** (1971), 373–413.

[12] P. A. Mykland, *"Options Pricing in Incomplete Markets: An Asymptotic Approach"*, Tech. Report 430, Department of Statistics, The University of Chicago, 1996.

[13] P. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag, 1990.

[14] M. Schweizer, *"Mean-Variance Hedging for General Claims"*, Ann. Appl. Prob. **2** (1992), 171–179.

[15] S. Song, *"Options and Discontinuity: An asymptotic decomposition for trading algorithms"*, Ph.D. dissertation, Department of Statistics, University of Chicago (2001).