STRUCTURE FUNCTION FOR ALIASING PATTERNS IN 2^{l-n} DESIGN WITH MULTIPLE GROUPS OF FACTORS

by

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Abstract

A general approach to study fractional factorial designs with multiple groups of factors is proposed. A structure function is generated by the defining contrasts among different groups of factors and the remaining columns. The structure function satisfies a first order partial differential equation. By solving this equation, general results about the structures and properties of the designs are obtained. As an important application, practical rules for the selection of "optimal" single arrays for robust parameter design experiments are derived.

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A general approach to study fractional factorial designs with multiple groups of factors is proposed. A structure function is generated by the defining contrasts among different groups of factors and the remaining columns. The structure function satisfies a first order partial differential equation. By solving this equation, general results about the structures and properties of the designs are obtained. As an important application, practical rules for the selection of "optimal" single arrays for robust parameter design experiments are derived.

1. Introduction. Two-level fractional factorial designs are arguably the most popular experimental plans in practice. Their practical and theoretical importance has long been established [Box, Hunter and Hunter (1978)], and has been further addressed and developed lately [Wu and Hamada (2000)]. Let 2^{l-n} denote a fractional factorial design that involves l factors and has 2^{l-n} runs. Much effort has been dedicated to understanding the structures and properties of fractional factorial designs [Bose (1947)]. Several general criteria, such as maximum resolution [Box and Hunter (1961)] and minimum aberration [Fries and Hunter (1980)], have been proposed to select optimal plans. A 2^{l-n} design is determined by its defining contrast subgroup, denoted by $\mathcal{G}$, which is generated by any n independent defining words. Defining words are factorial effects that are aliased with constant. A simple yet important characteristic of $\mathcal{G}$ is its wordlength pattern, $W = (W_1, W_2, \cdots, W_l)$, where W_i is the number of defining words of length i in $\mathcal{G}$ $(1 \leq i \leq l)$. Wordlength pattern W contains information about the aliasing among factorial effects. Both maximum resolution criterion and minimum aberration criterion are based on wordlength pattern. For fixed run size 2^m $(m = l - n)$, W becomes more complex when the number of factors increases. Tang and Wu (1996) suggested using complementary designs to characterize fractional factorial designs with a large number of factors. This technique has led to many interesting results [Chen and Hedayat (1996)].

Recently fractional factorial designs involving different types of factors have received much attention. Suppose a 2^{l-n} design is employed to investigate l factors. If the l factors do not need to

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be distinguished further, they are said to be symmetric, and the columns of the design matrix are randomly assigned to them. However, this symmetry property does not hold in several interesting designs. For example, blocked fractional factorial designs involve non-blocking factors and blocking factors [Sun et al. (1997) and Sitter et al. (1998)], and split-plot designs involve whole-plot factors and sub-plot factors [Bingham and Sitter (1999)]. Another important case is robust parameter design. Two types of factors, control factors and noise factors, are present in a parameter design experiment. The basic idea of parameter design is to explore the effects of control factors, noise factors and their interactions on certain response of a system, then choose optimal settings of control factors to adjust the mean response on target and "dampen" the variation caused by noise factors. Control factors and noise factors play very different roles in response optimization and variation reduction. They need to be treated separately in any proper experiment planning. Taguchi (1986) proposed the use of cross array (or inner-outer array in his terminology) to run parameter design experiments, which is generated by "crossing" an orthogonal array of control factors with another orthogonal array of noise factors. In order to improve efficiency and run size economy, Welch et al. (1990) and Shoemaker et al. (1991) suggested the use of single arrays. A *single array* is a fractional factorial design with some of its columns assigned to control factors and the rest columns to noise factors. So single arrays are fractional factorial designs with two distinct types of factors. A comprehensive review on parameter design can be found in Wu and Hamada (2000). The selection of optimal single arrays is considered in Wu and Zhu (2001). In general, one can have more than two different groups of factors. We will focus in this paper on the case with only two distinct groups of factors, which are denoted by group I and group II. All the results in this paper can be extended to cover more general cases. And we will only use single arrays for illustration and application.

A fractional factorial design with two different groups of factors is also determined by its defining contrast subgroup $\mathcal{G}$. But W becomes a poor summary of $\mathcal{G}$, because defining words of same length may consist of different numbers of group I factors and group II factors, so that they may have different implications on effect aliasing. For instance, let D^1 and D^2 be two single arrays with $\mathcal{G}_1 = \{I, ABa, Cbc, ABCabc\}$ and $\mathcal{G}_2 = \{I, ABC, abc, ABCabc\}$ respectively, where A, B and C are control factors and a, b and c are noise factors. D^1 and D^2 share the same wordlength pattern $W = (0,0,2,0,0,1)$. But they actually are quite different in the sense of effect aliasing. Assume that effects with order greater than 2 are negligible. All the control-by-noise interactions in D^2

are estimable, while in D^1, only five control-by-noise interactions, Ab, Ac, Bb, Bc and Ca, are estimable. This example shows that it is necessary to distinguish defining words with same length to reflect complex aliasing patterns. Hence a finer summary of $\mathcal{G}$ with the consideration of the difference between the two types of factors is in order.

The purpose of this paper is to develop some theoretical results for fractional factorial designs with distinct types of factors. In Section 2, notation and basic definitions are given. Several new concepts such as wordtype pattern, structure index array N and structure function f are defined. Based on Tang and Wu (1996), a recursive equation for N is derived. In Section 3, a first order partial differential equation of f will be generated. Main theorems about N and a closed form solution to the partial differential equation are obtained. In Section 4, two alternative approaches are introduced and commented. In Section 5, the theoretical results from the previous sections are applied to the selection of "optimal" single arrays.

2. Notation and Definitions. Some concepts and techniques from finite geometry will be used in this section. A brief introduction of them can be found in Bose (1947) and Mukerjee and Wu (1999). Let $\mathbf{F_2}$ be the Galois field $\{0,1\}$, and let $EG(m,2)$ and $PG(m-1,2)$ denote the m-dimensional Euclidean geometry and $(m-1)$-dimensional projective geometry over $\mathbf{F_2}$ respectively. In this paper, we do not distinguish a matrix from the collection of its row vectors. Two matrices with the same collection of row vectors are considered to be identical. Let P be a $m \times (2^m-1)$ matrix whose columns consist of all the distinct points of $PG(m-1,2)$. Sylvester-type Hadamard matrix $H_m(2)$ is defined to be a $2^m \times (2^m-1)$ matrix whose row vectors form the k-dimensional subspace generated by the row vectors of P. Thus there exists an one-to-one correspondence between the columns of $H_m(2)$ and the points in $PG(m-1,2)$. It is well known that the design matrix of a 2^{l-n} design is a collection of l different columns from $H_m(2)$ with rank $m(=l-n)$. Let $2^{(l_1+l_2)-n}$ denote a fractional factorial design with l_1 group I factors, l_2 group II factors and 2^m runs ($m=l_1+l_2-n$). Let $\mathcal{G}$ and D be the associated defining contrast subgroup and the $2^m \times (l_1+l_2)$ design matrix. As discussed in Section 1, wordlength pattern W is not a proper summary of $\mathcal{G}$. Define $A_{i,j}$ to be the number of defining words in $\mathcal{G}$ that consist of i group I factors and j group II factors. Let $A=(A_{i,j})$, i.e., the $(l_1+1)\times(l_2+1)$ matrix with entries $A_{i,j}$. A is called the *wordtype pattern* of the design. The design matrix D has l_1+l_2 columns from $H_m(2)$, among which l_1 columns are assigned to group I factors and the other l_2 columns to group II factors. Let $l_3 = 2^m - l_1 - l_2 - 1$. Marking off the columns used in D from $H_m(2)$, there are l_3 columns left in $H_m(2)$ which can be used to form a design for another l_3 factors. We call these columns the remaining columns, the design the remaining design, and the possible factors the remaining factors. Hence a $2^{(l_1+l_2)-n}$ design induces a 3-way partition of the columns of $H_m(2)$, and it further induces a 3-way partition of $PG(m-1,2)$ because of the correspondence between $H_m(2)$ and $PG(m-1,2)$. Since $D = \{uG : u \in EG(m,2)\}$, where G is an $m \times (l_1 + l_2)$ matrix whose column vectors are different points in $PG(m-1,2)$ with the first l_1 vectors, denoted by $\alpha_1, \alpha_2, \cdots, \alpha_{l_1}$, corresponding to the columns assigned to group I factors, and the other l_2 vectors, denoted by $\beta_1, \beta_2, \cdots, \beta_{l_2}$, corresponding to the columns assigned to group II factors. Denote the remaining points in $PG(m-1,2)$ by $\gamma_1, \gamma_2, \cdots, \gamma_{l_3}$. Let

$$\mathcal{L}_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_{l_1}\}, \mathcal{L}_2 = \{\beta_1, \beta_2, \cdots, \beta_{l_2}\}, \mathcal{L}_3 = \{\gamma_1, \gamma_2, \cdots, \gamma_{l_3}\}.$$

Then $PG(m-1,2) = \mathcal{L}_1 \bigcup \mathcal{L}_2 \bigcup \mathcal{L}_3$. Similar partitions were derived by Chen and Cheng (1998) for studying a general theory of blocked designs and Mukerjee and Wu (2000) for studying mixed-level designs. For any fixed triplet (i,j,k) such that $0 \le i \le l_1$, $0 \le j \le l_2$ and $0 \le k \le l_3$, a collection of i points from $\mathcal{L}_1$, j points from $\mathcal{L}_2$ and k points from $\mathcal{L}_3$ is said to have a $[i,j,k]$-relation, if they sum to be the 0-vector in $EG(m,2)$. Let $N_{i,j,k}$ denote the total number of different $[i,j,k]$-relations and N the $(l_1+1) \times (l_2+1) \times (l_3+1)$ array with entries $N_{i,j,k}$. N is called the *Structure Index Array.* Regarding $H_m(2)$ as a design for group I, group II and remaining factors, $N_{i,j,k}$ represent the number of defining words in the associated defining contrast subgroup which involve i group I factors, j group II factors and k remaining factors. When l_2 equals 0, D becomes a regular fractional factorial design involving only one group of factors, $(N_{i,j,k})$ reduces to be $(N_{i,0,k})$ that is exactly the same as $(N_{i+k}(i))$ defined in Tang and Wu (1996). Clearly wordtype pattern A of D is equivalent to $(N_{i,j,0})$ with $0 \le i \le l_1$ and $0 \le j \le l_2$. Since $\mathcal{L}_1 \bigcap \mathcal{L}_2 = \mathcal{L}_1 \bigcap \mathcal{L}_3 = \mathcal{L}_2 \bigcap \mathcal{L}_3 = \emptyset$, $N_{i,j,k} = 0$ when $1 \le i+j+k \le 2$. For some technical purpose, we define $N_{0,0,0} = 1$.

LEMMA 1. For $i + j + k \ge 2$, $N_{i,j,k}$ satisfy the following iterative equation,

$$(i+1)N_{i+1,j,k} + (j+1)N_{i,j+1,k} + (k+1)N_{i,j,k+1} + N_{i,j,k}$$

$$= \begin{pmatrix} l_1 \\ i \end{pmatrix} \begin{pmatrix} l_2 \\ j \end{pmatrix} \begin{pmatrix} l_3 \\ k \end{pmatrix} - [(l_1 - i + 1)N_{i-1,j,k} + (l_2 - j + 1)N_{i,j-1,k} + (l_3 - k + 1)N_{i,j,k-1}]. \quad (1)$$

PROOF. We know that $PG(m-1,2) = \mathcal{L}_1 \bigcup \mathcal{L}_2 \bigcup \mathcal{L}_3$ with $\mid \mathcal{L}_1 \mid = l_1$, $\mid \mathcal{L}_2 \mid = l_2$, $\mid \mathcal{L}_3 \mid = l_3$ and $l_1 + l_2 + l_3 = 2^m - 1$. There are $\begin{pmatrix} l_1 \\ i \end{pmatrix} \begin{pmatrix} l_2 \\ j \end{pmatrix} \begin{pmatrix} l_3 \\ k \end{pmatrix}$ different ways to select i points, j points and k points from $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ respectively. Suppose one of them is given by $\{\alpha_1, \cdots, \alpha_i\} \subset \mathcal{L}_1$, $\{\beta_1, \cdots, \beta_j\} \subset \mathcal{L}_2$ and $\{\gamma_1, \cdots, \gamma_k\} \subset \mathcal{L}_3$. This combination induces a further partition of $PG(m-1,2)$. Let $A = \{\alpha_1, \cdots, \alpha_i\}$, $B = \mathcal{L}_1 - A$, $C = \{\beta_1, \cdots, \beta_j\}$, $D = \mathcal{L}_2 - C$, $E = \{\gamma_1, \cdots, \gamma_k\}$, $F = \mathcal{L}_3 - E$ and $G = \{0\}$. Now consider $\phi = \alpha_1 + \cdots + \alpha_i + \beta_1 + \cdots + \beta_j + \gamma_1 + \cdots + \gamma_k$. A combination with $\phi \in A$ is said to be of type A, and a combination with $\phi \in B$ is said to be of type B, and so on. We claim that a combination cannot be of two different types simultaneously, and any combination must be of one of the types. We now count the type A combinations. Since $\phi \in A$, there exists an i_0 $(1 \leq i_0 \leq i)$ such that

$$\alpha_1 + \cdots + \alpha_i + \beta_1 + \cdots + \beta_j + \gamma_1 + \cdots + \gamma_k = \alpha_{i_0}.$$

This implies that

$$\alpha_1 + \cdots + \alpha_{i_0-1} + \alpha_{i_0+1} + \cdots + \alpha_i + \beta_1 + \cdots + \beta_j + \gamma_1 + \cdots + \gamma_k = 0,$$

i.e., $\{\alpha_1, \cdots, \alpha_{i_0-1}, \alpha_{i_0+1}, \cdots, \alpha_i, \beta_1, \cdots, \beta_j, \gamma_1, \cdots, \gamma_k\}$ has a $[i-1,j,k]$-relation. So a type A combination corresponds to a $[i-1,j,k]$-relation. In the converse, every $[i-1,j,k]$-relation can generate $(l_1 - i + 1)$ combinations which are of type A. Since different $[i-1,j,k]$-relations must generate different combinations, the number of type A combinations is equal to $(l_1 - i + 1)N_{i-1,j,k}$. Following similar arguments, we have

$$\mid B \mid = (i+1)N_{i+1,j,k}, \mid C \mid = (l_2 - j + 1)N_{i,j-1,k},$$

$$\mid D \mid = (j+1)N_{i,j+1,k}, \mid E \mid = (l_3 - k + 1)N_{i,j,k-1}, \mid F \mid = (k+1)N_{i,j,k+1}.$$

Clearly $\mid G \mid = N_{i,j,k}$. Since

$$\begin{pmatrix} l_1 \\ i \end{pmatrix} \begin{pmatrix} l_2 \\ j \end{pmatrix} \begin{pmatrix} l_3 \\ k \end{pmatrix} = \mid A \mid + \mid B \mid + \mid C \mid + \mid D \mid + \mid E \mid + \mid F \mid + \mid G \mid,$$

equation (1) follows.

The structure index array N of a fractional factorial design with two distinct groups of factors can be used as a good description of the structure and properties of the corresponding design. We define the *structure function* of the associated design by

$$f(x,y,z) = \sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\sum_{k=0}^{l_3} N_{i,j,k}x^i y^j z^k = 1 + \sum_{\substack{i+j+k\geq 3\\ i\geq 0, j\geq 0, k\geq 0}} N_{i,j,k}x^i y^j z^k, \quad (2)$$

where the second equality follows from $N_{i,j,k} = 0$ for $1 \leq i+j+k \leq 2$.

3. Main Results. In this section, we will derive a first order partial differential equation satisfied by f based on (1). The differential equation unveils the intricate relations among the $N_{i,j,k}$. Then an explicit solution of the equation will be obtained. Denote the run size by $r = 2^m$. First, we have the following theorem.

THEOREM 1. The structure function f of a $2^{(l_1+l_2)-n}$ design satisfies the following first order partial differential equation

$$(x^2-1)\frac{\partial f}{\partial x} + (y^2-1)\frac{\partial f}{\partial y} + (z^2-1)\frac{\partial f}{\partial z} - (1+l_1x+l_2y+l_3z)f + (1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3} = 0, \quad (3)$$

where $l_3 = 2^{l_1+l_2-n} - l_1 - l_2 - 1$.

PROOF: Multiplying both sides of (1) by $x^i y^j z^k$, and rearranging the terms, we have

$$\begin{pmatrix} l_1 \\ i \end{pmatrix}\begin{pmatrix} l_2 \\ j \end{pmatrix}\begin{pmatrix} l_3 \\ k \end{pmatrix} x^i y^j z^k = N_{i,j,k}x^i y^j z^k + (l_1-i+1)N_{i-1,j,k}x^i y^j z^k + (i+1)N_{i+1,j,k}x^i y^j z^k$$

$$(l_2-j+1)N_{i,j-1,k} + (j+1)N_{i,j+1,k}x^i y^j z^k + (l_3-k+1)N_{i,j,k-1}x^i y^j z^k + (k+1)N_{i,j,k+1}x^i y^j z^k \quad (4)$$

Summing both sides of (4) over i,j,k with $i+j+k \geq 3$, $i \geq 0$, $j \geq 0$ and $k \geq 0$, we have

$$\sum_{c=3}^{r-1}\sum_{i+j+k=c} N_{i,j,k}x^i y^j z^k = f - 1, \quad (5)$$

$$\sum_{c=3}^{r-1}\sum_{i+j+k=c} (l_1-i+1)N_{i-1,j,k}x^i y^j z^k$$

$$= l_1 x \sum_{c=3}^{r-1} \sum_{\substack{i'+j+k=c \\ i'\geq 0, j\geq 0, k\geq 0}} N_{i',j,k} x^{i'} y^j z^k - \sum_{c=3}^{r-1} \sum_{i'+j+k=c} i' N_{i',j,k} x^{i'+1,j,k}$$

$$= l_1 x (f-1) - x^2 \frac{\partial f}{\partial x}, \tag{6}$$

$$\sum_{c=3}^{r-1} \sum_{i+j+k=c} (i+1) N_{i+1,j,k} x^i y^j z^k = \sum_{c=3}^{+\infty} \sum_{\substack{i'+j+k=c+1 \\ i'\geq 1}} i' N_{i',j,k} x^{i'-1} y^j z^k$$

$$= \frac{\partial f}{\partial x} - \sum_{i+j+k=3} i N_{i,j,k} x^{i-1} y^j z^k. \tag{7}$$

Similarly, we have

$$\sum_{c=3}^{r-1} \sum_{i+j+k=c} [(l_2 - j + 1) N_{i,j-1,k} x^i y^j z^k + (j+1) N_{i,j+1,k} x^i y^j z^k]$$

$$= l_2 y (f-1) - y^2 \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} - \sum_{i+j+k=3} j N_{i,j,k} x^i y^{j-1} z^k \tag{8}$$

and

$$\sum_{c=3}^{r-1} \sum_{i+j+k=c} [(l_3 - k + 1) N_{i,j,k-1} x^i y^j z^k + (k+1) N_{i,j,k+1} x^i y^j z^k]$$

$$= l_3 z (f-1) - z^2 \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z} - \sum_{i+j+k=3} k N_{i,j,k} x^i y^j z^{k-1}. \tag{9}$$

Notice that

$$\sum_{c=3}^{r-1} \sum_{i+j+k=c} \binom{l_1}{i} \binom{l_2}{j} \binom{l_3}{k} x^i y^j z^k = (1+x)^{l_1} (1+y)^{l_2} (1+z)^{l_3}$$

$$-[1 + l_1 x + l_2 y + l_3 z + \frac{l_1(l_1-1)}{2} x^2 + \frac{l_2(l_2-1)}{2} y^2 + \frac{l_3(l_3-1)}{2} z^2 + l_1 l_2 xy + l_1 l_3 xz + l_2 l_3 yz],$$

and

$$\sum_{i+j+k=3} N_{i,j,k} (i x^{i-1} y^j z^k + j x^i y^{j-1} z^k + k x^i y^j z^{k-1})$$

$$= (3N_{3,0,0} + N_{2,1,0} + N_{2,0,1}) x^2 + (3N_{0,3,0} + N_{1,2,0} + N_{0,2,1}) y^2 + (3N_{0,0,3} + N_{1,0,2} + N_{0,1,2}) z^2$$

$+(2N_{2,1,0}+2N_{1,2,0}+N_{1,1,1})xy+(2N_{2,0,1}+2N_{1,0,2}+N_{1,1,1})xz+(2N_{0,2,1}+2N_{0,1,2}+N_{1,1,1})yz.$

Applying (1) again with $i+j+k=2$, we have

$$2N_{2,1,0}+2N_{1,2,0}+N_{1,1,1}=l_1l_2,$$

$$2N_{2,0,1}+2N_{1,0,2}+N_{1,1,1}=l_1l_3,$$

$$2N_{0,2,1}+2N_{0,1,2}+N_{1,1,1}=l_2l_3,$$

$$3N_{3,0,0}+N_{2,1,0}+N_{2,0,1}=\frac{l_1(l_1-1)}{2},$$

$$3N_{0,3,0}+N_{1,2,0}+N_{0,2,1}=\frac{l_2(l_2-1)}{2},$$

$$3N_{0,0,3}+N_{1,0,2}+N_{0,1,2}=\frac{l_3(l_3-1)}{2}.$$

Collecting all the terms, we have (3) and the theorem is proved.

Let $D_{1,2}$, $D_{1,3}$ and $D_{2,3}$ be the designs generated by $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1$ and $\mathcal{L}_3$, and $\mathcal{L}_2$ and $\mathcal{L}_3$ respectively. Then $\{N_{i,j,0}\}$, $\{N_{i,0,k}\}$ and $\{N_{0,j,k}\}$ are the wordtype patterns of the designs correspondingly. Since any of the designs induce the same partition of $PG(m-1,2)$, it determines the other two designs. Intuitively, it is also true that any of the wordtype patterns determines the other two wordtype patterns, and further determines all the structure indices $N_{i,j,k}$, which are only dependent on the partition. For instance, if we know $\{N_{0,j,k}\}$, all the other $N_{i,j,k}$ can be uniquely determined. This provides an opportunity to study $D_{1,2}$ in terms of $D_{1,3}$ or $D_{2,3}$ whichever is simpler. Employing the derived partial differential equation of f, not only can we show the above result rigorously, we can also derive the relations explicitly. In the following, we will first state a theorem on the existence and uniqueness of the solution to (3) given $\{N_{0,j,k}\}$.

THEOREM 2. For given $\{N_{0,j,k}\}$, there exists a unique solution f to the first order partial differential equation in (3).

PROOF. Since $\{N_{0,j,k}\}$ are given, $f(0,y,z)=1+\sum_{j=0}^{l_2}\sum_{k=0}^{l_3}N_{0,j,k}y^jz^k$ is determined. Introduce u, v and t as the new parameters for the desired surface $w=f(x,y,z)$. Let $h(u,v)=f(0,u,v)$. The existence and uniqueness problem of (3) given $\{N_{0,j,k}\}$ is equivalent to the existence and uniqueness problem of the following system of ordinary differential equations:

$$\frac{dx}{dt} = x^2 - 1, \tag{10}$$

$$\frac{dy}{dt} = y^2 - 1, \tag{11}$$

$$\frac{dz}{dt} = z^2 - 1, \tag{12}$$

$$\frac{dw}{dt} = (1 + l_1 x + l_2 y + l_3 z)w - (1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3}, \tag{13}$$

with the initial conditions

$$x(u, v, 0) = 0, \tag{14}$$

$$y(u, v, 0) = u, \tag{15}$$

$$z(u, v, 0) = v, \tag{16}$$

$$w(u, v, 0) = \sum_{j=0}^{l_1} \sum_{k=0}^{l_2} N_{0,j,k} = h(u, v). \tag{17}$$

Applying the standard existence and uniqueness theorem for a system of linear ordinary differential equations [John (1971)], the theorem is proved.

With the help of the associated system of ordinary differential equations, we can get an analytic solution of (3) directly and the analytical solution shows explicitly how $N_{i,j,k}$ are related to $\{N_{0,j,k}\}$. First we solve the initial problem for the ordinary differential equation system given by (14), (15), (16) and (17), and x, y, z and w can be expressed as functions of the parameters u, v and t. Then we solve the system of functional equations involving $x = x(u, v, t)$, $y = y(u, v, t)$ and $z = z(u, v, t)$ to represent u, v and t in terms of x, y and z. Finally, replace the variables of w with $u = u(x, y, z)$, $v = v(x, y, z)$ and $t = t(x, y, z)$, and we get an explicit expression of $f = w(u(x, y, z), v(x, y, z), t(x, y, z))$. From (10),

$$\frac{dx}{x^2 - 1} = dt \text{ implies } (\frac{1}{x+1} - \frac{1}{x-1})dx = dt.$$

So a general solution for (10) is

$$\frac{1}{2}(\log\frac{1+x}{1-x}) = t + c.$$

Because $x(u,v,0)=0$, we have

$$x=\frac{-1+e^{-2t}}{1+e^{-2t}}. \tag{18}$$

Similarly, based on (11), (15), (12) and (16), we have

$$y(u,v,t)=\frac{-1+ce^{-2t}}{1+ce^{-2t}}, \quad \text{where} \quad c=\frac{1+u}{1-u}, \tag{19}$$

$$z(u,v,t)=\frac{-1+de^{-2t}}{1+ce^{-2t}}, \quad \text{where} \quad d=\frac{1+v}{1-v}. \tag{20}$$

For (13) and (17), the solution given x, y and z is

$$w=(-\int_0^t(1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3}\exp(-\int_0^t(1+l_1x+l_2y+l_3z)dt)dt+h(u,v))$$
$$\exp(\int_0^t(1+l_1x+l_2y+l_3z)dt) \tag{21}$$

Replacing x, y and z with (18), (19) and (20), we have

$$w(u,v,t)=2^{l_1}(1+c)^{l_2}(1+d)^{l_3}(1+e^{2t})^{-l_1}(c+e^{2t})^{-l_2}(d+e^{2t})^{-l_3}\exp(rt)h(u,v)$$
$$-2^{l_1+l_2+l_3}c^{l_2}d^{l_3}(1+e^{2t})^{-l_1}(c+e^{2t})^{-l_2}(d+e^{2t})^{-l_3}\frac{\exp(rt)-1}{r}. \tag{22}$$

Because

$$e^{2t}=\frac{1-x}{1+x}, \tag{23}$$

$$c=\frac{1+y}{1-y}\frac{1-x}{1+x}, \tag{24}$$

$$d=\frac{1+z}{1-z}\frac{1-x}{1+x}, \tag{25}$$

$$u=\frac{y-x}{1-yx}, \quad v=\frac{z-x}{1-zx}, \tag{26}$$

$w(u,v,t)$ can be re-expressed in terms of x, y and z. After some routine but cumbersome calculations, we have

$$f(x,y,z)=w(u(x,y,z),v(x,y,z),t(x,y,z))$$
$$=(1+x)^{l_1-r/2}(1-x)^{r/2-l_2-l_3}\sum_{j,k}N_{0,j,k}(y-x)^j(1-yx)^{l_2-j}(z-x)^k(1-zx)^{l_3-k}$$
$$-\frac{1}{r}(1+x)^{l_1-r/2}(1-x)^{r/2}(1+y)^{l_2}(1+z)^{l_3}+\frac{1}{r}(1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3}. \tag{27}$$

Similarly, by following the same argument, f can also be expressed in terms of $N_{i,0,k}$ or in terms of $N_{i,j,0}$ as follows,

$$f(x,y,z) = (1+y)^{l_2-r/2}(1-y)^{r/2-l_1-l_3}\sum_{i,k} N_{i,0,k}(x-y)^i(1-yx)^{l_1-i}(z-y)^k(1-yz)^{l_3-k}$$

$$-\frac{1}{r}(1+y)^{l_2-r/2}(1-y)^{r/2}(1+x)^{l_1}(1+z)^{l_3} + \frac{1}{r}(1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3}, \quad (28)$$

$$f(x,y,z) = (1+z)^{l_3-r/2}(1-x)^{r/2-l_1-l_2}\sum_{i,j} N_{i,j,0}(x-z)^i(1-xz)^{l_1-i}(y-z)^j(1-yz)^{l_3-j}$$

$$-\frac{1}{r}(1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3-r/2}(1-z)^{r/2} + \frac{1}{r}(1+x)^{l_1}(1+y)^{l_2}(1+z)^{l_3}. \quad (29)$$

In the following, we will obtain an exact relation between general $N_{i,j,k}$ and $\{N_{0,j,k}\}$ by expanding f. First we define

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{cases} 0 & \text{if } k < 0 \text{ or } k \text{ is not an integer} \\ 1 & \text{if } k = 0 \\ \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1} & \text{otherwise} \end{cases} \quad (30)$$

Now consider the following identity

$$(x-y)^k(1-xy)^{n-k} = \sum_{i=0}^{k}\sum_{j=0}^{n-k}(-1)^{i+j}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} n-k \\ j \end{pmatrix} x^{k-i+j}y^{i+j}. \quad (31)$$

Applying the transformation $T : (i,j) \to (s,t) : k-i+j = s, i+j = t$, (31) becomes

$$(x-y)^k(1-xy)^{n-k} = \sum_{(s,t)\in T([0,k]\times[0,n-k])} (-1)^t \begin{pmatrix} k \\ (t-s+k)/2 \end{pmatrix}\begin{pmatrix} n-k \\ (t+s-k)/2 \end{pmatrix} x^s y^t. \quad (32)$$

Because of the definition in (30), for $(s,t) \in [0,+\infty)\times[0,+\infty) - T([0,k]\times[0,n-k])$,

$$\begin{pmatrix} k \\ (t-s+k)/2 \end{pmatrix}\begin{pmatrix} n-k \\ (s+t-k)/2 \end{pmatrix} = 0.$$

We have

$$(x-y)^k(1-xy)^{n-k} = \sum_{(s,t)\in[0,+\infty]\times[0,+\infty]} (-1)^t \begin{pmatrix} k \\ (t-s+k)/2 \end{pmatrix}\begin{pmatrix} n-k \\ (t+s-k)/2 \end{pmatrix} x^s y^t.$$

Let

$$Q_{n,k}(s,t) = (-1)^t \binom{k}{(t-s+k)/2} \binom{n-k}{(s+t-k)/2}. \tag{33}$$

It is clear that $Q_{n,k}(s,t) = 0$ for $\max(s,t) > n$. Hence (31) can be rewritten as

$$(x-y)^k (1-xy)^{n-k} = \sum_{s=0}^{n} \sum_{t=0}^{n} Q_{n,k}(s,t) x^s y^t. \tag{34}$$

Now consider another expression $(1+x)^{l_1 - r/2}(1-x)^{l_1 - r/2 + 1} = (1-x^2)^{l_1 - r/2}(1-x)$. Because

$$(1-x)(1-x^2)^{l_1 - r/2} = (1-x)[1 + \sum_{k=1}^{+\infty} (-1)^k \binom{l_1 - r/2}{k} x^{2k}]$$

$$= 1 - x + \sum_{k=1}^{+\infty} (-1)^k \binom{l_1 - k/2}{k} x^{2k} + \sum_{k=1}^{+\infty} (-1)^{k+1} \binom{l_1 - r/2}{k} x^{2k+1},$$

we have

$$(1+x)^{l_1 - r/2}(1-x)^{l_1 - r/2 + 1} = \sum_{n=0}^{+\infty} (-1)^{[\frac{n}{2}] + I(n)} \binom{l_1 - r/2}{[\frac{n}{2}]} x^n, \tag{35}$$

where $I(n) = 0$ for even n is an even integer and $I(n) = 1$ for odd n. With the help of (34) and (35), the first term of (27) can be expanded as follows,

$$(1+x)^{l_1 - r/2}(1-x)^{r/2 - l_2 - l_3} \sum_{j,k} N_{0,j,k} (y-x)^j (1-yx)^{l_2 - j} (z-x)^k (1-zx)^{l_1 - k}$$

$$= (1+x)^{l_1 - r/2}(1-x)^{l_1 - r/2 + 1} \sum_{j,k} N_{0,j,k} (-1)^j (-1)^k \sum_{s_2, t_2 = 0}^{l_2} Q_{l_2, j}(s_2, t_2) y^{s_2} x^{t_2} \sum_{s_3, t_3}^{l_3} Q_{l_3, k}(s_3, t_3) x^{s_3} z^{t_3}$$

$$= \sum_{i=0}^{l_1} \sum_{j=0}^{l_2} \sum_{k=0}^{l_3} c_{i,j,k} x^i y^j z^k,$$

where

$$c_{i,j,k} = \sum_{t_1 + t_2 = i} \sum_{s_2 + s_3 = t_1} \sum_{u,v} (-1)^{[t_2/2] + I(t_2)} \binom{l_1 - r/2}{[t_2/2]} (-1)^{u+v} N_{0,u,v} Q_{l_2,u}(s_2, j) Q_{l_3,v}(s_3, k).$$

It is easy to expand the other two terms. Collecting all the terms from the expansion of equation (27) and comparing coefficients with the definition of f, we have

$$N_{i,j,k} = \frac{1}{r} \binom{l_1}{i} \binom{l_2}{j} \binom{l_3}{k} - \frac{1}{r} \sum_{i_1 + i_2 = i} (-1)^{i_2} \binom{l_1 - r/2}{i_1} \binom{r/2}{i_2} \binom{l_2}{j} \binom{l_3}{k}$$

$$+\sum_{t_1+t_2=i}\sum_{s_2+s_3=t_1}\sum_{u,v}(-1)^{[t_2/2]+I(t_2)}\binom{l_1-r/2}{[t_2/2]}(-1)^{u+v}N_{0,u,v}Q_{l_2,u}(s_2,j)Q_{l_3,v}(s_3,k). \quad (36)$$

In particular, we have

$$N_{i,j,0}=\frac{1}{r}\binom{l_1}{i}\binom{l_2}{j}-\frac{1}{r}\sum_{i_1+i_2=i}(-1)^{i_2}\binom{l_1-r/2}{i_1}\binom{r/2}{i_2}\binom{l_2}{j}$$

$$+\sum_{t_1+t_2=i}\sum_{s_2+s_3=t_1}\sum_{u}(-1)^{[t_2/2]+I(t_2)}\binom{l_1-r/2}{[t_2/2]}(-1)^{u+s_3}N_{0,u,s_3}Q_{l_2,u}(s_2,j), \quad (37)$$

and

$$N_{i,0,k}=\frac{1}{r}\binom{l_1}{i}\binom{l_3}{k}-\frac{1}{r}\sum_{i_1+i_2=i}(-1)^{i_2}\binom{l_1-r/2}{i_1}\binom{r/2}{i_2}\binom{l_3}{k}$$

$$+\sum_{t_1+t_2=i}\sum_{s_2+s_3=t_1}\sum_{v}(-1)^{[t_2/2]+I(t_2)}\binom{l_1-r/2}{[t_2/2]}(-1)^{s_2+v}N_{0,s_2,v}Q_{l_3,v}(s_3,k). \quad (38)$$

4. Alternative Approaches. It is well known that fractional factorial designs are equivalent to linear codes [Bose (1961)]. Most concepts in fractional factorial designs, such as wordlength pattern, resolution and defining relation ect., have their counterparts in the context of linear codes [Sue et al. (1998) and Chen and Cheng (1998)]. General theory of linear codes can be found in MacWilliams and Sloane (1978). Fractional factorial designs with two groups of factors are equivalent to 2-way split linear codes. Theoretical results for 2-way split linear codes can also be used to study the structure indices $N_{i,j,k}$ defined in Section 3. A brief account of 2-way split linear codes is given as follows. Let C and $\overline{C}$ be a 2-way split linear $[l, l-n]$ code and its dual code respectively. Each code word of C is divided into two subwords, $c=(c_1,c_2)$, where c_1 consists of l_1 components and c_2 consists of l_2 components ($l=l_1+l_2$). The split weight of c is defined to be $w(c)=(w(c_1),w(c_2))$, where $w(c_i)$ is the number of 1's in c_i for $i=1,2$. Define the split weight enumerator of C as follows,

$$W_C(x,y,s,t)=\sum_{c\in C}x^{w(c_1)}y^{l_1-w(c_1)}s^{w(c_2)}t^{l_2-w(c_2)}=\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}B_{i,j}x^iy^{l_1-i}s^jt^{l_2-j}; \quad (39)$$

similarly for $\overline{C}$ with the same split, its split weight enumerator is defined as

$$W_{\overline{C}}(x,y,s,t) = \sum_{\bar{c}\in\overline{C}} x^{w(\bar{c}_1)}y^{l_1-w(\bar{c}_1)}s^{w(\bar{c}_2)}t^{l_2-w(\bar{c}_2)} = \sum_{i=0}^{l_1}\sum_{j=0}^{l_2} \overline{B}_{i,j}x^iy^{l_1-i}s^jt^{l_2-j}, \tag{40}$$

where $(B_{i,j})_{0\le i\le l_1, 0\le j\le l_2}$ and $(\overline{B}_{i,j})_{0\le i\le l_1, 0\le j\le l_2}$ are called the split weight distributions for C and $\overline{C}$ respectively. The following MacWilliams identity holds for W_C and $W_{\overline{C}}$.

LEMMA 2. For any 2-way split linear code C and its dual code $\overline{C}$ with the 2-way split weight enumerators as in (39) and (40), the following identities hold,

$$W_{\overline{C}}(x,y,s,t) = \frac{1}{\mid C\mid} W_C(y-x, y+x, t-s, t+s) \tag{41}$$

$$\overline{B}_{i,j} = \frac{1}{\mid C\mid}\sum_{k_1=0}^{l_1}\sum_{k_2=0}^{l_2} B_{k_1,k_2}P_{l_1-i}(k_1,l_1)P_{l_2-j}(k_2,l_2), \tag{42}$$

where $\mid C\mid$ denotes the number of code words in C and

$$P_k(m_1,m_2) = \sum_{k_3=0}^{k}(-1)^{k_3}\binom{m_1}{k_3}\binom{m_2-m_1}{k-k_3}$$

are the Krawtchouk polynomials.

Recall that $D_{1,2}$ is a fractional factorial design with l_1 group I factors, l_2 group II factors and 2^m runs. So $D_{1,2}$ is a 2-way split linear code. Its dual code, denoted by $\overline{D}_{1,2}$, is also a 2-way split code. $\overline{D}_{1,2}$ is equivalent to the defining contrast subgroup and the split weight distribution of $\overline{D}_{1,2}$ is equivalent to the wordtype pattern of $D_{1,2}$. As discussed previously, $D_{1,2}$ induces a 3-way partition of $H_m(2)$ and generates another two associated designs, which are $D_{1,3}$ and $D_{2,3}$. Clearly $D_{2,3}$ and $D_{1,3}$ are two-way split linear codes and $H_m(2)$ is a 3-way split code. Define $\{n_{i,j,k}\}$, $\{n^{12}_{i,j}\}$, $\{n^{13}_{i,k}\}$ and $\{n^{23}_{j,k}\}$ as the split weight distributions of $H_k(2)$, $D_{1,2}$, $D_{1,3}$ and $D_{2,3}$. Let $\overline{H}_m(2)$, $\overline{D}_{1,2}$, $\overline{D}_{1,3}$ and $\overline{D}_{2,3}$ be their dual codes respectively. It is not difficult to see that $\{N_{i,j,k}\}$, $\{N_{i,j,0}\}$, $\{N_{i,0.k}\}$ and $\{N_{0,j,k}\}$ are the corresponding split weight distributions of $\overline{H}_m(2)$, $\overline{D}_{1,2}$, $\overline{D}_{1,3}$ and $\overline{D}_{2,3}$. Applying MacWilliams identity to $\{N_{i,j,k}\}$ and $\{n_{i,j,k}\}$, we have

$$N_{i,j,k} = \frac{1}{2^m}\sum_{s=0}^{l_1}\sum_{t=0}^{l_2}\sum_{u=0}^{l_3} n_{s,t,u}P_{l_1-i}(s,l_1)P_{l_2-j}(t,l_2)P_{l_3-k}(u,l_3). \tag{43}$$

Applying MacWilliams identity to $\{N_{0,j,k}\}$ and $\{n^{23}_{j,k}\}$, we have

$$N_{0,j,k} = \frac{1}{2^m} \sum_{s=0}^{l_2} \sum_{t=0}^{l_3} n^{23}_{s,t} P_{l_2-j}(s, l_2) P_{l_3-k}(t, l_3), \tag{44}$$

where $P_u(m_1, m_2)$ are the Krawtchouk polynomials defined in LEMMA 2. Note that $n_{i,j,k} = 0$ when $i + j + k \neq 2^{k-1} - 1$ and $n_{i,j,k} = n^{23}_{j,k}$ when $i + j + k = 2^{k-1}$. Using the relations between $n_{i,j,k}$ and $n^{23}_{j,k}$, we can get equations that are analogous to (36), (37) and (38), if not more complicated.

The results in this paper provide an approach to look at the interconnections among the three portions of $H_k(2)$ induced by any design involving two different groups of factors in a more intuitive and explicit way. It also shows that the approach of Tang and Wu (1996) and the approach of Suen, Chen and Wu (1997) are indeed equivalent. Both can be applied according to different situations and needs.

Another important approach to studying factorial designs is to use finite Abelian group theory. A general framework developed by Bailey and her associates can accommodate symmetric and asymmetric factorial designs with flexible factor levels [Bailey (1982, 1985, 1989)]. The case of multiple groups of factors can be easily treated in this framework. A full factorial design for l factors is identified with an Abelian group D of order 2^l, where each element of D represents a factorial run. D can be represented as $D = \langle g_1 \rangle \otimes \langle g_2 \rangle \cdots \otimes \langle g_l \rangle$, where g_1, g_2, ..., g_l are the generators with order 2. Naturally the generators correspond to the factors. Suppose the factors, or the generators correspondingly, are divided into two groups, e.g., l_1 group I factors and l_2 group II factors, D becomes

$$D = \langle g'_1 \rangle \otimes \cdots \otimes \langle g'_{l_1} \rangle \otimes \langle g''_1 \rangle \otimes \cdots \otimes \langle g''_{l_2} \rangle,$$

where g'_i belongs to Group I for $1 \leq i \leq l_1$, and g''_j belongs to Group II for $1 \leq j \leq l_2$. The dual group D^* of D is composed of the irreducible characters of D, i.e., the homomorphisms $\chi : D \to \{1, -1\}$. D and D^* are in fact isomorphic. For $1 \leq i \leq l_1$ and $1 \leq j \leq l_2$, define χ_i and η_j as follows: for any $g \in \{g'_1, \cdots, g'_{l_1}, g''_1, \cdots, g''_{l_2}\}$, $\chi_i(g) = -1$ if $g = g'_i$, =1 otherwise; $\eta_j(g) = -1$, if $g = g''_j$, =1 otherwise. Then $\{\chi_1, \cdots, \chi_{l_1}, \eta_1, \cdots, \eta_{l_2}\}$ becomes a set of generators for D^*. For any given $\theta \in D^*$, it can be uniquely represented as a product of some of the generators. The split weight of θ is defined by $w(\theta) = (w_1(\theta), w_2(\theta))$, where $w_1(\theta)$ is the number of χ_i in θ and $w_2(\theta)$ is the number of η_j in θ. The generators are identified with the main effects of the group

I factors and of the group II factors. In general, $\theta \in D^*$ with $w(\theta) = (i,j)$ represents a factorial effect involving i group I factors and j group II factors. Now any 2^{l-n} fractional factorial design with two groups of factors, denoted by $D_{1,2}$ as before, is a subgroup of D with order 2^{l-n}. Let $D_{1,2}^{\circ} = \{\tau \in D^* : \tau(\alpha) = 1, \text{for any } \alpha \in D_{1,2}\}$. It is clear that $D_{1,2}^{\circ}$ is a subgroup of D^*, and it is the defining contrast subgroup $\mathcal{G}$ of $D_{1,2}$. Define $A_{i,j}$ to be the number of $\tau \in D_{l,p}^{\circ}$ such that $w(\tau) = (i,j)$, where $0 \le i \le l_1$ and $0 \le j \le l_2$. Then $(A_{i,j})$ is the wordtype pattern of the $2^{(l_1+l_2)-n}$ design defined previously. Therefore, all the results regarding wordlength pattern or wordtype pattern can be developed and applied in the framework based on the finite Abelian group approach. Though a complete development of the results is interesting, it is not straitforward and beyond the scope of the current paper.

5. Application and Example. In this section, the theoretical results derived in the previous sections will be applied to the selection of "optimal" single arrays for parameter design experiments. As defined in Section 1, single arrays are typical examples of fractional factorial designs with two groups of factors, which are control factors and noise factors. A single array of l_1 control factors, l_2 noise factors and $2^{l_1+l_2-n}$ runs induces a partition of $PG(m-1,2)$ ($m = l_1 + l_2 - n$), that is, $PG(m-1,2) = \mathcal{L}_1 \bigcup \mathcal{L}_2 \bigcup \mathcal{L}_3$, where $\mathcal{L}_1$ includes the points corresponding to the control factors, $\mathcal{L}_2$ includes the points corresponding to the noise factors and $\mathcal{L}_3$ the points to the remaining columns. Wu and Zhu (2001) proposed an index vector $J = (J_1, J_2, J_3, J_4, J_5, J_6)$ to describe the aliasing severity of a single array, where $J_1 = 4(N_{2,1,0} + N_{1,2,0} + N_{2,2,0})$, $J_2 = 3N_{3,0,0} + 3N_{3,1,0} + N_{2,1,0}$, $J_3 = N_{1,2,0} + 3N_{1,3,0} + 3N_{0,3,0}$, $J_4 = 3N_{3,0,0} + 3N_{3,1,0} + N_{2,1,0}$, $J_5 = 6N_{4,0,0}$ and $J_6 = N_{2,2,0}$. And they use the following minimum J-aberration criterion to select optimal single arrays.

DEFINITION. For any two single arrays D^1 and D^2, if there exists i_0 such that $J_i^1 = J_i^2$ for $i \le i_0 - 1$ and $J_{i_0}^1 < J_{i_0}^2$, D^1 is said to have less J-aberration than D^2. If there is no other designs with less J-aberration than D^1, D^1 is said to have minimum J- aberration.

When l_1 and l_2 are large, $\{N_{i,j,0}\}$ become very complicated. Since all $N_{i,j,k}$ are intricately related as indicated by the results in Section 3, it is easier to consider $D_{1,3}$ and $D_{2,3}$ generated by $\mathcal{L}_1$ and $\mathcal{L}_3$ and by $\mathcal{L}_2$ and $\mathcal{L}_3$, whichever is simpler. Applying (37), we have the following corollary.

COROLLARY 1.

$$N_{3,0,0} = \text{Constant} - \sum_{j+k=3} N_{0,j,k}, \tag{45}$$

$$N_{2,1,0} = \text{Constant} + \sum_{j+k=3} jN_{0,j,k}, \tag{46}$$

$$N_{1,2,0} = \text{Constant} - (N_{0,2,1} + 3N_{0,3,0}), \tag{47}$$

$$N_{1,3,0} = \text{Constant} - N_{0,3,0} - (N_{0,3,1} + 4N_{0,4,0}), \tag{48}$$

$$N_{2,2,0} = \text{Constant} + (N_{0,2,1} + 3N_{0,3,0}) + (N_{0,2,2} + 3N_{0,3,1} + 6N_{0,4,0}), \tag{49}$$

$$N_{3,1,0} = \text{Constant} - \sum_{j+k=3} jN_{0,j,k} - \sum_{j+k=4} jN_{0,j,k}, \tag{50}$$

$$N_{4,0,0} = \text{Constant} + \sum_{j+k=3} N_{0,j,k} + \sum_{j+k=4} N_{0,j,k}. \tag{51}$$

Based on Corollary 1, The expression of J in terms of $\{N_{0,j,k}\}$ can be derived as follows, $J_1 = \text{Constant} + \sum_{j+k=3} 4jN_{0,j,k} + (4N_{0,2,2} + 12N_{0,3,1} + 24N_{0,4,0})$, $J_2 = \text{Constant} - \sum_{j+k=3}(3 + 2j)N_{0,j,k} - \sum_{j+k=4} 3jN_{0,j,k}$, $J_3 = \text{Constant} - (N_{0,2,1} + 3N_{0,3,0}) - (3N_{0,3,1} + 12N_{0,4,0})$, $J_4 = \text{Constant} + 6\sum_{j+k=3} N_{0,j,k} + 6\sum_{j+k=4} N_{0,j,k}$, $J_5 = \text{Constant} + (N_{0,2,1} + 3N_{0,3,0}) + (N_{0,2,2} + 3N_{0,3,1} + 6N_{0,4,0})$, $J_6 = \text{Constant} + 6N_{0,4,0}$. Similar to the approaches in Tang and Wu (1996) and Chen and Cheng (1998), based on the equations above, we can establish some general rules to identify minimum J-aberration single arrays.

Rule 1. A single array $D^\star_{1,2}$ has minimum J-aberration if
(i) $\sum_{j+k=3} 4jN_{0,j,k} + (4N_{0,2,2} + 12N_{0,3,1} + 24N_{0,4,0})$ of $D^\star_{2,3}$ is the minimum among all possible $D_{2,3}$,
(ii) $D^\star_{1,2}$ is the unique single array satisfying (i).

Rule 2. A single array $D^\star_{1,2}$ has minimum J-aberration if
(i) $\sum_{j+k=3} 4jN_{0,j,k} + (4N_{0,2,2} + 12N_{0,3,1} + 24N_{0,4,0})$ of $D^\star_{2,3}$ is the minimum among all possible $D_{2,3}$,
(ii) $\sum_{j+k=3}(3 + 2j)N_{0,j,k} + \sum_{j+k=4} 3jN_{0,j,k}$ of $D^\star_{2,3}$ is the maximum among all possible $D_{2,3}$ with J_1 the same as of $D^\star_{2,3}$

Design	Defining relation	Nonzero $N_{i,j,k}$
$D^1_{2,3}$	$I = abr_1 = acr_2 = bcr_1r_2$	$N_{0,0,0} = N_{0,2,2} = 1,\ N_{0,2,1} = 2$
$D^2_{2,3}$	$I = abc = ar_1r_2 = bcr_1r_2$	$N_{0,0,0} = N_{0,1,2} = N_{0,2,2} = N_{0,3,0} = 1$
$D^3_{2,3}$	$I = abr_1 = cr_1r_2 = abcr_2$	$N_{0,0,0} = N_{0,1,2} = N_{0,2,1} = N_{0,3,1} = 1$
$D^4_{2,3}$	$I = abc$	$N_{0,0,0} = N_{0,3,0} = 1$
$D^5_{2,3}$	$I = abr_1$	$N_{0,0,0} = N_{0,2,1} = 1$
$D^6_{2,3}$	$I = ar_1r_2$	$N_{0,0,0} = N_{0,1,2} = 1$
$D^7_{2,3}$	$I = abcr_1$	$N_{0,0,0} = N_{0,3,1} = 1$
$D^8_{2,3}$	$I = abr_1r_2$	$N_{0,0,0} = N_{0,2,2} = 1$
$D^9_{2,3}$	$I = abcr_1r_2$	$N_{0,0,0} = N_{0,3,2} = 1$

Table 1: All possible $D_{2,3}$'s with l_1=10, l_2=3 and r=16

(iii) $D^*_{1,2}$ is the unique single array satisfying (i) and (ii).

Rule 1 only involves J_1 and Rule 2 only involves J_1 and J_2. Similarly we can develop Rule i ($3 \leq i \leq 6$) that involves the first i J indices based on the idea of sequentially minimizing J_1, J_2, J_3, J_4, J_5 and J_6.

EXAMPLE 1. Suppose we want to obtain a 16-run single arrays with minimum J-aberration for 10 control factors and 3 noise factors. So $l_1 = 10$, $l_2 = 3$ and $l_3 = 2$. It is clear that $N_{0,1,3} = N_{0,4,0} = N_{0,0,4} = 0$. Sequentially minimizing J_1, J_2, J_3, J_4, J_5 and J_6 is equivalent to sequentially minimizing $\sum_{j+k=3} jN_{0,j,k} + N_{0,2,2}$, maximizing $\sum_{j+k=3}(3+2j)N_{0,j,k} + 6N_{0,2,2} + 9N_{0,3,1}$, maximizing $N_{0,2,1} + 3N_{0,3,0} + 3N_{0,3,1}$, minimizing $\sum_{j+k=3} N_{0,j,k} + N_{0,2,2} + N_{0,3,1}$ and $J_5 = (N_{0,2,1} + 3N_{0,3,0})) + (N_{0,2,2} + 3N_{0,3,1})$. Notice that $J_6 = 0$. Now, we only need to consider the wordtype patterns of the complementary designs $D_{2,3}$ with 3 noise factors, 2 remaining factors and 16 runs. Note that the complementary designs could either be two folds of a 2^{5-2} design or a 2^{5-1} design. Denote the three noise factors by a, b and c, and the two remaining factors by r_1 and r_2. There are 9 non-equivalent designs as shown in Table 1. $\sum_{j+k=3} jN_{0,j,k} + N_{0,2,2}$ is minimized to be zero by $D^7_{2,3}$ and $D^9_{2,3}$. Since $D^7_{2,3}$ has a bigger value of $\sum_{j+k=3}(3+2i)N_{0,j,k} + 6N_{0,2,2} + 9N_{0,3,1}$, applying Rule 2, we conclude that the corresponding design $D^7_{1,2}$ is the only single array with minimum J-aberration. Based on $D^7_{2,3}$, $D^7_{1,2}$ can be constructed in the following way: let $H_4(2)$ be the Hadamard matrix consisting of 15 columns with the first four columns independent and the remaining columns being all possible linear combinations (modulus 2) of the first four columns. Select any other four independent columns, such as 12, 23, 34 and 234, assign 12, 23 and 34 to a, b and c respectively, delete 234 and 14 and assign the left columns to the 10 control factors randomly. Thus we have derived the design matrix of $D^7_{1,2}$. It is easy to write down the corresponding defining contrast subgroup.

Generally, any properties of a design that are determined by $(N_{i,j,0})_{i\geq 0, j\geq 0}$ can be studied by its complementary designs. The indices $N_{i,j,k}$ with $i > 0$, $j > 0$ and $k > 0$, which can be accommodated easily in our approach, can provide further insights about the design and its structure. In some applications such as split-plot design and blocked design, the induced partitions of the Hadamard matrix or $PG(m-1, 2)$ are not arbitrary. And $(N_{i,j,0})$ needs to satisfy certain constraints. How to consider these constraints in the complementary design approach and how they can be used to develop efficient search algorithms for optimal designs are two interesting questions that need further investigation.

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