ON THE DETERMINACY OF POWERS, PRODUCTS AND CONVOLUTIONS

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On the Determinacy of Powers, Products and Convolutions

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Abstract

It has been known for sometime that the cube of a normal random variable is not determined by its moments in the sense of Hamburger. Similarly, convolutions of determined random variables need not be determined. In this article, we give some general results on the determinacy and indeterminacy of powers, products, and convolutions of random variables. The results apply to some common distributions as corollaries, including the normal, logistic, and the double exponential.

The central theme is that slight perturbations of a determined random variable stay determined, but adequately large powers are typically undetermined. For convolutions, the one with the dominating tail will typically determine the determinacy of the convolution; some results of Simeon Berman on tails of convolutions are used to show this. It is also shown that the product of two iid normals is determined while the product of three is not. This latter result needs some work as the density of the product of three normals is highly complex. On the other hand, we show that the product of just two iid double exponentials is already undetermined.

The methods include computation of Krein's entropy integrals and use of the familiar Carleman conditions. Most of the results are illustrated by further examples.

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1 Introduction

Theory of moments has flourished since the publication of the first examples of undetermined measures by Stieltjes about a century ago (Stieltjes(1894)). Körner (1990) says: "I always regarded this as a curiosity until I saw Yves Myer applying it to wavelets." For a lovely all around discussion of how moment theory has been greatly useful in probability and statistics, see Diaconis (1987). One should also see Kemperman (1968, 1972, 1987), Karlin and Studden (1966), and Stoyanov (1987) for careful developments of some of the techniques, the associated geometry, and interesting examples. Stoyanov (1987) in particular is largely responsible for making some of the most applicable results of Krein accessible to a large audience (one can see the classic reference Akhiezer (1965) for much of Krein's work on moment theory). It will be useful and nice to have a source with illustrative examples describing the fundamental techniques of moment theory and how they apply to statistics and probability; this appears to be still lacking.

It is well known that the $N(0,1)$ distribution is determined; frequently one simply states this as Z is determined if $Z \sim N(0,1)$. Indeed, with the solitary exception of the lognormal distribution, the common statistical distributions are generally determined. In two interesting articles, Berg (1985, 1988), it is shown that convolutions and powers of determined random variables may be undetermined. In particular, Z^3 is undetermined if $Z \sim N(0,1)$. Likewise, $|Z|^\alpha$ is undetermined as a Stieltjes moment problem if $\alpha > 4$ (i.e., there exist other nonnegative random variables with the same moment sequence if $\alpha > 4$; if the restriction of nonnegativity is removed, $\alpha > 2$ suffices. That is, $|Z|^\alpha$ is undetermined as a Hamburger moment problem for $\alpha > 2$). The set of all measures with the same moment sequence is convex and weakly compact and generally admits a choquet representation. Explicit description of all the extreme points of this set is generally impossible, but some special extremal measures and their Stieltjes transforms can be described (in principle). These are the Nevanlinna extremal measures, all discrete. See Akhiezer (1965), Berg (1995), and Shobat and Tamarkin (1950).

This article considers the question of determinacy of powers, convolutions and products in a bit more generality and attempts in particular to present a bit of coherent structure to the interesting fact that Z is determined but Z^3 is not. We show, by theorems and examples, that generally "slight perturbations" of a determined random variable are determined, but large powers are frequently undetermined. We also consider the corresponding natural questions for products; i.e., what can we say about the determinacy of $\prod_{i=1}^{n} x_i$ or $\prod_{i=1}^{n} |x_i|^{\alpha_i}$ if $x_1, x_2, \ldots, x_n$ are iid copies of a determined random variable?

In Section 2, we first give two illustrative results: one on powers and one on products. The basic underlying density for these two results is $ce^{-a|x|^\alpha}$, for $c, a, \alpha > 0$; for given α and a, c is fixed but its expression is unimportant for the results. It is shown that if x_i are iid with the density as above, then $\prod_{i=1}^{n} |x_i|^{\alpha_i}$ is Stieltjes-determined if $\sum_{i=1}^{n} \alpha_i \leq 2\alpha$; on the other hand a single power $|x|^\delta$ is Stieltjes-undetermined if $\delta > 2\alpha$. The two cases $\alpha = 1, 2$ are special; for these two values of α, Krein's entropy integral for $|x|^\delta$ is computed and plotted to aid in the understanding of indeterminacy for $\delta > 2\alpha$.

In Section 3, we give three general results. One says that if a random variable x is determined, then a "nearly linear" monotone function $h(x)$ is also determined and an example shows that the near linearity cannot be relaxed; one needs a minor condition on x. We give an application to Bayesian Statistics: for estimating the unknown mean of a normal distribution, the sampling distribution of a Bayes estimate is often determined by its moments. A second result shows that the convolution of a $N(0, 1)$ with another random variable will often be determined. A third result says that (in addition), under quite mild conditions on the density $q(\cdot)$ of $\varepsilon, Z+ \in$ and $\in$ will be simultaneously determined or undetermined. This result is derived by making use of the order of tails of a convolution; see Berman (1992).

In Section 4, the central message is that high powers, on the other hand, are generally undetermined. A result is proved that the cube of a convolution, $(Z+ \in)^3$, is generally undermined when Z is $N(0, 1)$. Then we specialize to the densities $ce^{-a|x|^\alpha}$ of Section 2, and prove that if $x_1, \ldots, x_n$ are iid from such a density, and $T_n = \max(x_1, \ldots, x_n)$, then irrespective of the value of n, T_n^k is undetermined if k is odd and $k > \alpha$, and it is Stieltjes-undetermined if k is even and $k > 2\alpha$. Thus, the threshold power between determinacy and indeterminacy depends only on α, not on n. Thus for $\alpha \geq 1$, when the density $ce^{-a|x|^\alpha}$ is log concave, a third power is undetermined.

The article closes in Section 5 with the result that if $z_1, \ldots, z_n$ are iid $N(0, 1)$, then $\prod_{i=1}^{n} Z_i$ is determined if $n = 2$ but undetermined if $n = 3$. This result is perhaps expected as the cube of a standard normal is undetermined; however, since the density of the product of 3 is a complicated Meijer function, the result is not very easy to prove and depends on a saddle point approximation. To demonstrate the subtlety of the topic, this is preceded by the result that it only takes two iid double exponentials for the product to be undetermined!!

To summarize, the main messages of our results are the following:

(a) If a random variable x is determined, then slight perturbations of x are often determined and the determinacy of a convolution is often decided by the dominating tail;

(b) If a higher odd power x^{2k+1} is determined, then x itself is also of course determined;

(c) Large powers are often undetermined; in particular, for many log concave densities, a cube is already undetermined. This includes, from our theorem, the normal, double exponential and the logistic . In fact, the cube of the maximum (and the minimum) of any number of iid random variables of these types is undetermined;

(d) In some cases, products of a few (2 or 3) iid random variables are also undetermined. In this sense, products of many and powers of one act similarly.

In addition, the results are illustrated by examples and applications, including one application to Bayesian Statistics.

2 Two Illustrative Results

In this section, we give two illustrative results on products and powers for the basic underlying density

$$f(x) = ce^{-a|x|^{\alpha}}, c, a, \alpha > 0, -\infty < x < \infty. \qquad (1)$$

The exact expression for $c = c(a, \alpha)$ is unimportant.

Proposition 1 *Let $x_i \overset{iid}{\sim} f(x)$ and $\alpha_i > 0$ given constants, $1 \leq i \leq n$. Then $\prod_{i=1}^{n} |x_i|^{\alpha_i}$ is Stieltjes-determined if $\sum_{i=1}^{n} \alpha_i \leq 2\alpha$.*

Proof In the following, the notation $a_k \sim b_k$ will be used to mean that the sequence $a_k = O(b_k)$ and it will be helpful to let us ignore explicit calculation of constants.

For given n, denote $x = \prod_{i=1}^{n} |x_i|^{\alpha_i}$ and denote the k^{th} moment of x by μ_k. Thus,

$$\mu_k = E\left(\prod_{i=1}^{n} |X_i|^{\alpha_i}\right)^k = \prod_{i=1}^{n} E|X_i|^{k\alpha_i} = \prod_{i=1}^{n} E|X_1|^{k\alpha_i} . \tag{2}$$

Now, for any $\mathcal{S} > 0$,

$$E|X_i|^{\mathcal{S}} = 2e\int_0^{\infty} t^{\mathcal{S}} e^{-at^{\alpha}} dt = \text{constant } \frac{\Gamma\left(\frac{\mathcal{S}+1}{\alpha}\right)}{a^{\frac{\mathcal{S}+1}{\alpha}}}, \tag{3}$$

where the constant depends on α, but not on $\mathcal{S}$. From (2) and (3),

$$\mu_k \sim \prod_{i=1}^{n} \frac{\Gamma\frac{(k\alpha_i+1)}{\alpha}}{a^{\frac{k\alpha_i+1}{\alpha}}} \tag{4}$$

Applying Stirling's approximation to the Gamma function, from(4) one concludes on calculations

$$\mu_k^{\frac{1}{2k}} \sim k^{\frac{\sum \alpha_i}{2\alpha}}, \tag{5}$$

and so $\sum_{k=1}^{\infty} \frac{1}{\mu_k^{\frac{1}{2k}}} = \infty$ if $\sum_{i=1}^{n} \alpha_i \leq 2\alpha$. This will imply by the familiar result of Carleman (see Shiryayev (1980)) that $x = \prod_{i=1}^{n} |X_i|^{\alpha_i}$ is Stieltjes-determined if $\sum_{i=1}^{n} \alpha_i \leq 2\alpha$.

Corollary 1 *Let $X_1 \sim f(x)$ as in (1). Then $|X_1|^{\delta}$ is Stieltjes-determined if $\delta \leq 2\alpha$.*

The next result says that $|X_1|^{\delta}$ is Stieltjes-undetermined if $\delta > 2\alpha$. In this result and the subsequent sections, two results due to Krein are extensively used. We state them as lemmas; see Stoyanov(1987) and Berg (1995).

Definition. Consider $X \sim f(x)$, a general given density f. If X is nonnegative, the entropy integral of X (or of f) is defined as

$$E_1(f) = \int_0^\infty \frac{\log f(x)}{\sqrt{x}(1+x)}\, dx; \tag{6}$$

otherwise, the entropy integral is defined as

$$E_2(f) = \int_{-\infty}^{\infty} \frac{\log f(x)}{1+x^2}\, dx. \tag{7}$$

Lemma 1 *Let X be a real valued random variable with density $f(x)$. X is determined in the sense of Hamburger if and only if $E_2(f) = -\infty$.*

Lemma 2 *Let X be a nonnegative random variable with density $f(x)$. Then X is Stieltjes-undetermined (i.e., the moments of X can be duplicated by another nonnegative random variable) if $E_1(f) > -\infty$.*

Proposition 2 *Let $x_1 \sim f(x)$ given in (1). Then $|X_1|^\delta$ is Stieltjes-undetermined if $\delta > 2\alpha$.*

Proof: By a straightforward calculation, the density of $Y = |X_1|^\delta$ is

$$\begin{aligned} g(y) &= \text{constant } y^{\frac{1}{\delta}-1}\, e^{-ay^{\frac{\alpha}{\delta}}} \\ \Rightarrow \log g(y) &= \text{constant } + \left(\frac{1}{\delta}-1\right)\log y - ay^{\frac{\alpha}{\delta}} \end{aligned} \tag{8}$$

Since $\int_0^\infty \frac{1}{\sqrt{y}(1+y)}dy$, $\int_0^\infty \frac{\log y}{\sqrt{y}(1+y)}dy$ are both finite real numbers and $\int_0^\infty \frac{y^{\frac{\alpha}{\delta}}}{\sqrt{y}(1+y)}\, dy$ is also finite if $\delta > 2\alpha$, it follows from Lemma 2 that $Y = |X_1|^\delta$ is Stieltjes-undetermined.

Example 1. By Lemma 2, if the entropy integral $E_1(f) > -\infty$, then the corresponding random variable is Stieltjes-undetermined. For the densities $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\frac{1}{2}e^{-|x|}$, $E_1(f) > -\infty$ for the random variable $Y = |X_1|^\delta$ when $\delta > 4(2)$ respectively. It could be interesting to see a plot of $E_1(f)$ for these two cases to see the convergence to $-\infty$ as δ approaches the threshold value. This is described in Figure 1.

3 Determinacy of Slight Perturbations

In this section, we give two results that convey the message that if a random variable X is determined, then a slight perturbation is also often determined. The perturbation could be a nearly linear function $h(x)$ or a convolution $X+\in$. Of course, the latter result is not always true.

Theorem 1 *Let a real valued random variable X with a bounded density $f(x)$ be determined. Suppose $h \in C_1(\mathcal{R})$ and there exist $0 < \delta \leq K < \infty$ such that $\delta \leq h'(x) \leq K$ for all x. Then $Y = h(X)$ is determined.*

Proof: Note that $h(x)$ is monotone increasing due to the hypothesis. Hence the density of Y is

$$\begin{aligned} g(y) &= \frac{f\left(h^{-1}(y)\right)}{h'\left(h^{-1}(y)\right)} \\ \Rightarrow \log g(y) &= \log f(\left(h^{-1}(y)\right) - \log h'\left(h^{-1}(y)\right). \end{aligned} \tag{9}$$

Also, by hypothesis, f is bounded and we may assume without loss of generality in the following proof that $f \leq 1$; thus $-\log f \geq 0$. Also note that we may assume $h(0) = 0$ and that by hypothesis and an application of the fundamental theorem of calculus, $h^2(x) \leq k^2x^2$. Hence,

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{-\log f(h^{-1}(y))}{1+y^2}\, dy \\ = & \int_{-\infty}^{\infty} \frac{-\log f(x)}{1+h^2(x)}\, h'(x)\, dx \\ \geq & \ \delta \int_{-\infty}^{\infty} \frac{-\log f(x)}{1+h^2(x)}\, dx \\ \geq & \ \delta \int_{-\infty}^{\infty} \frac{-\log f(x)}{1+x^2} \frac{1+x^2}{1+k^2x^2}\, dx \\ \geq & \ \min(1, \frac{1}{k^2}) \cdot \delta \cdot \int_{-\infty}^{\infty} \frac{-\log f(x)}{1+x^2}\, dx \end{aligned} \tag{10}$$

On the other hand, $\int_{-\infty}^{\infty} \frac{\log h'(h^{-1}(y))}{1+y^2}\, dy > \log \delta \int_{-\infty}^{\infty} \frac{1}{1+y^2}\, dy > -\infty$. Hence, from (9), (10) and Lemma 1, $\int_{-\infty}^{\infty} \frac{-\log g(y)}{1+y^2}\, dy = \infty$ as f is assumed to be determined. Now Lemma 1 applies again and $Y = h(X)$ is seen to be determined.

Corollary 2 *Suppose $X \sim N(\theta, 1)$ and $\delta(X)$ is the posterior mean of θ with respect to some prior CDF G. Then the sampling distribution of $\delta(X)$ is determined by its moments if $\inf_x \left(\frac{d^2}{dx^2} \log m(x)\right) > -1$ and $\sup_x \left(\frac{d^2}{dx^2} \log m(x)\right) < \infty$, where $m(x)$ is the marginal density $\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} dG(\theta)$.*

Proof: It is well known (see Brown (1986)) that $\delta(x) = x + \frac{d}{dx} \log m(x)$. So under the hypothesis of Corollary 2, $\inf \delta'(x) > 0$ and $\sup \delta'(x) < \infty$. δ certainly belongs to $C_1(R)$ (in fact it is in $C_\infty(R)$; see Brown (1986) again). So Theorem 1 implies the assertion of Corollary 2.

Remark. For many common choices of the prior G, the conditions on $\log m(x)$ in Corollary 2 are verifiable. These include all normal and double exponential prior distributions. We could not verify if these conditions hold for t prior distributions as well.

Example 2. Consider $Z \sim N(0, 1)$ and the function $h(z) = e^{\lambda z^{\frac{1}{2K+1}}}$, where $\lambda > 0$ and K is any nonnegative integer. $h'(z)$ diverges near $z = 0$ as well as at the two tails. Calculation shows that $h(z)$ is not determined. Thus a very nonlinear function of a determined random variable need not be determined (of course, z^3 is another example of this).

Instead of a functional perturbation, one may look at random perturbations, like convolutions. One result is given in the following proposition. The result can be stated in a more general form.

Proposition 3 *Let $x, \in$ be random variables with moment sequences $\{\mu_k\}, \{\nu_k\}$. If $\{\mu_k\}$ satisfies $\lim \sup_{n\to\infty} \frac{1}{2k} \mu_{2k}^{\frac{1}{2k}} < \infty$ and so does $\{\nu_k\}$, then $x+ \in$ is determined.*

Remark. If $Z \sim N(0,1)$, then $\mu_{2k} = E(Z^{2k}) = \frac{2^k\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}$ and so, by Stirling's approximation the condition $\limsup_{k\to\infty} \frac{1}{2k}\mu_{2k}^{\frac{1}{2k}} < \infty$ is satisfied. So if Z is convolved with any $\in$ whose moment sequence $\{\nu_k\}$ satisfies the above condition of Proposition 3, then $Z+\in$ will be determined.

Proof of Proposition 3: Follows from the elementary inequality

$$\begin{aligned}
\gamma_{2k} &= E|X+\in|^{2k} \leq 2^{2k-1}\left(\mu_{2k}+\nu_{2k}\right) \\
\Rightarrow \gamma_{2k}^{\frac{1}{2k}} &\leq 2\left(\mu_{2k}^{\frac{1}{2k}}+\nu_{2k}^{\frac{1}{2k}}\right) \\
&\Rightarrow \limsup_{k\to\infty} \frac{1}{2k}\gamma_{2k}^{\frac{1}{2k}} < \infty. \qquad (11)
\end{aligned}$$

Now one applies the Carlman sufficient condition for determinacy (see Shiryayev (1980)).

The final result of this section says that if $Z \sim N(0,1)$, then a convolution $Z+\in$ will often be simultaneously determined or undetermined with $\in$. The restriction to $Z \sim N(0,1)$ can be removed by saying that the convolution has often the same determinacy character as the one with the dominating tail. We also take $\in$ to be symmetric only for convenience .

Theorem 2 *Let $\in$ have the density function $q(|\in|)$ which satisfies:*

(i) For every $\delta > 0, q(x)e^{\delta x^2}$ is increasing for all sufficiently large x;

(ii) $\frac{q'(x)}{q(x)}$ is of regular variation;

(iii) $q(x)$ is continuous and ultimately monotone;

(iv) $\lim_{x\to\infty} \frac{q'(x)}{q(x)} > 0$.

Let $Z \sim N(0,1)$ and let $f(x) = (\phi * q)(x)$ denote the density of the convolution $x = Z+\in$. Then the entropy integrals of X and $\in$ converge or diverge simultaneously.

Remark. Many logconcave densities (which necessarily have all moments) with tail heavier than that of a normal satisfy conditions i-iv.

Proof of Theorem 2: Under condition (i), (3.18) of Berman (1992) holds, and (3.19) in Berman (1992) follows from conditions (iii) and (iv) on a bit of calculations. So by Corollary 3.3 of Berman (1992), Theorem 2 follows, as we make assumption (ii) of regular variation as well.

4 Indeterminacy of Large Powers

In this section, we give a few results that indicate that large powers are often undetermined. In particular, there is something special about the third power and for many logconcave densities, the cube is undetermined. However, we give a more general result on large powers.

Theorem 3 *Let X be a real valued random variable with density $p(x)$ and suppose for some $\alpha \geq 1$, and $a > 0$, $\frac{1}{p(x)}$ satisfies the growth condition $\frac{1}{p(x)} \leq e^{a|x|^\alpha}$. Let $\in$ be any other random variable and suppose both $X, \in$ have all moments. Then T^{2k+1} is undetermined whenever $2k+1 > \alpha$, where T denotes the convolution $X + \in$.*

Corollary 3 *For the normal, double exponential, and logistic densities, the cube is undetermined.*

Proof of Corollary 3: This will follow from Theorem 3 as $\alpha = 2$ works for the normal and $\alpha = 1$ for the double exponential and the logistic (take $\in = 0$).

Proof of Theorem 3: Let $f(t)$ denote the density of the convolution T and G the cdf of $\in$. Then,

$$\begin{aligned} f(t) &= \int p(t- \in) dG(\in) \\ \Rightarrow \log f(t) &= \log E_G p(t- \in) \\ &\geq E_G \log p(t- \in) \quad \text{(by Jensen's inequality)} \\ \Rightarrow -\log f(t) &\leq E_G \log \frac{1}{p(t- \in)} \leq a E_G |t- \in|^\alpha \end{aligned} \tag{12}$$

Since $\alpha \geq 1$, pointwise,

$$\begin{aligned} |t- \in|^\alpha &\leq 2^{\alpha-1}(|t|^\alpha + |\in|^\alpha) \\ \Rightarrow -\log f(t) &\leq a 2^{\alpha-1}(|t|^\alpha + E_G|\in|^\alpha) \end{aligned} \tag{13}$$

Now, the density of $Y = T^{2k+1}$ is

$$\begin{aligned} g(y) &= \text{constant. } y^{-\frac{2k}{2k+1}} f\left(y^{\frac{1}{2k+1}}\right) \\ \Rightarrow -\log g(y) &= \text{constant } + \frac{2k}{2k+1} \log |y| - \log f\left(y^{\frac{1}{2k+1}}\right) \end{aligned} \tag{14}$$

By (13),

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{-\log f\left(y^{\frac{1}{2k+1}}\right)}{1+y^2}\, dy \\ \leq \; & \text{constant.} \left[E_G| \in |^{\alpha} \int_{-\infty}^{\infty} \frac{1}{1+y^2}\, dy + \int_{-\infty}^{\infty} \frac{|y|^{\frac{\alpha}{2k+1}}}{1+y^2}\, dy \right] \\ & < \infty \text{ if } 2k+1 > \alpha. \end{aligned} \tag{15}$$

Since $\int_{-\infty}^{\infty} \frac{\log |y|}{1+y^2} dy < \infty$, it follows from (14) and (15) that $-\int_{-\infty}^{\infty} \frac{\log g(y)}{1+y^2} dy < \infty$ if $2k+1 > \alpha$, and hence $Y = (x+ \in)^{2k+1}$ is undetermined if $2k+1 > \alpha$.

A very similar argument gives the following result for the particular densities $ce^{-a|x|^{\alpha}}$ previously considered in Section 2. Note that if the density is already in this exponential form, then we do not need the restriction $\alpha \geq 1$. The thing worth mentioning is that the result is about the powers T^{2k+1} of the maximum $T = T_n$ of n iid random variables $x_1, \ldots, x_n$ but n has no role in the indeterminacy result.

Proposition 4 *Let $x_1, \ldots, x_n$ be iid with density $f(x)$ as in (1) and let $T = T_n = \max(x_1, \ldots, x_n)$. Then T^{2k+1} is undetermined if $2k+1 > \alpha$.*

We omit the proof.

Remark. From Proposition 4, it will follow for instance, that not only the cube of one normal, but the cube of the maximum (and the minimum) of any number of normals is undetermined.

5 Product of Random Variables

Since the square of a normal is determined but the cube is not, one may surmise that the product of two iid normals is determined but the product of three is not. We close the article by showing that this is indeed the case, but indeterminacy of the product of three needs quite a bit of caculation. It is almost certainly true that the product of four or more iid normals is also undetermined. The density of $Y = Z_1 Z_2$ is $\frac{K_0(y)}{\pi}$ where K_0 is the Bessel K_0 function and the density of $Z_1 Z_2 Z_3$ is a complicated Meijer function. They are plotted in Figure 2 and Figure 3. But first we give a rather curious result. For iid double exponentials, the product of just two is already undetermined.

Proposition 5 *Let X_1, X_2 be iid double exponential with the density $\frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$. Then $X_1 X_2$ is undetermined.*

Remark. One can anticipate Proposition 5 from the fact that the Carleman sufficient conditions fail for $X_1 X_2$.

Proof of Proposition 5. Step 1. By a direct computation using the transformation $(X_1, X_2) \to (u, v)$ where $u = X_1 X_2$ and $v = X_2$, the density of u is

$$\begin{aligned} g(u) &= \text{constant.} \int_0^\infty e^{-\frac{|u|}{v} - v}\, v\, dv \\ &= \text{constant.}\ |u| K_2\left(2\sqrt{|u|}\right) \qquad (16) \end{aligned}$$

(pp 340, Gradshteyn and Ryzhik (1980))

Step 2. For $z > 0$,

$$K_2(z) = \sqrt{\frac{\pi}{2z}}\ e^{-z}\left[1 + \frac{c_1}{z} + \frac{c_2}{z^2} - \frac{c_3(z)}{z^3}\right] \qquad (17)$$

where $c_1, c_2 > 0$, and $c_3(z) > 0$ and bounded. (pp 963, Gradshteyn and Ryzhik (1980); it is necessary to use the fact $0 \le \theta_3 \le 1$ in that page).

Step 3. From (17), for sufficiently large $z > 0$,

$$K_2(z) \ge \sqrt{\frac{\pi}{2z}}\ e^{-z} \cdot \frac{1}{2} = c\,\frac{e^{-z}}{\sqrt{z}}\ \text{(say)} \qquad (18)$$

Step 4. Substituting (18) into (16) and making the change of variable $2\sqrt{|u|} = x$, one sees $\int_{-\infty}^{\infty} \frac{-\log g(u)}{1+u^2}\, du = 2\int_0^\infty \frac{-\log g(u)}{1+u^2}\, du < \infty$ and so by Lemma 1, the Proposition follows.

Theorem 4 *Let Z_1, Z_2, Z_3 be iid $N(0,1)$. Then Z_1Z_2 is determined and $Z_1Z_2Z_3$ is undetermined.*

Proof: From a direct calculation of the moments one can veryify the Carleman condition $\limsup_{k\to\infty} \frac{1}{2k}(E(Z_1Z_2)^{2k})^{\frac{1}{2k}} < \infty$ and so Z_1Z_2 is determined. To see that $Z_1Z_2Z_3$ is undetermined, one needs an evaluation of the order of the tail of $Z_1Z_2Z_3$. This is done by studying the tail of $\log|Z_1Z_2Z_3| = X_1+X_2+X_3$ (say), where $X_i = \log|Z_i|$. The saddlepoint method is used; see Reid (1988) and Jensen (1988) for a description. We will use the notation $\overline{x} = \frac{x_1+x_2+x_3}{3}, \mathcal{S} = x_1+x_2+x_3, \omega = e^{\mathcal{S}} = |Z_1Z_2Z_3|$. The following facts, stated together in Step 1, will be useful in the subsequent steps. $x > 0$ in every statement in Step 1. The basic goal is to show that the tail of the density of $Y = Z_1Z_2Z_3$ is of the order $e^{-9y^{2/9}}$; then one uses Lemma 1.

Step 1. (i) Let $\psi(x) = \frac{d}{dx}\log\Gamma(x)$. Then $\psi(x) = \log x + 0\left(\frac{1}{x}\right)$, $\psi^{-1}(x) = 0(e^x)$, and $\psi'(x) = 0\left(\frac{1}{x}\right)$ (see pp 943 in Gradshteyn and Ryzhik (1980)); (ii) If $f_1(x) = 0(x), g_1(x) = 0(g_2(x))$, and $g_1(x) \to \infty$ as $x \to \infty$, then $f_1(g_1(x)) = 0(g_2(x))$.

Step 2. The moment generating function of $\mathcal{S} = \log|Z_1Z_2Z_3|$ equals

$$M(t) = E\left(e^{t\mathcal{S}}\right) = \frac{1}{\pi^{3/2}}\left(\Gamma\frac{t+1}{2}\right)^3. \tag{19}$$

Step 3. Let $K(t) = \log M(t)$. Then $K'(t) = \frac{3}{2}\psi\left(\frac{t+1}{2}\right)$ and so the solution to $K'(t) = \overline{x}$ is

$$\hat{t} = g(\overline{x}) \triangleq 2\psi^{-1}\left(\frac{2\overline{x}}{3}\right) - 1. \tag{20}$$

Step 4.

$$g'(\overline{x}) = \frac{4}{3\psi'\left(\psi^{-1}\left(\frac{2\overline{x}}{3}\right)\right)} \tag{21}$$

Step 5.

$$k''(t)\Big|_{t=\hat{t}} = \frac{3}{4}\psi'\left(\frac{\hat{t}+1}{2}\right) = \frac{3}{4}\psi'\left(\psi^{-1}\left(\frac{2\overline{x}}{3}\right)\right) \tag{22}$$

Step 6. By employing a saddlepoint approximation (see Reid (1988)), the density of $\overline{x}$ is

$$\begin{aligned} f_{\overline{x}}(\overline{x}) &\sim \left(\frac{1}{k''(\hat{t})}\right)^{\frac{1}{2}} \cdot e^{3(k(\hat{t})-\hat{t}\overline{x})} \\ &\sim \frac{1}{\sqrt{\psi'\left(\psi^{-1}\left(\frac{2\overline{x}}{3}\right)\right)}} e^{3\left\{3\log\Gamma\left(\psi^{-1}\left(\frac{2\overline{x}}{3}\right)\right)-\overline{x}\left(2\psi^{-1}\left(\frac{2\overline{x}}{3}\right)-1\right)\right\}} \end{aligned} \tag{23}$$

Step 7. By using Step 1, the density of $\mathcal{S} = 3\overline{x}$ is

$$f_{\mathcal{S}}(\mathcal{S}) \sim e^{\mathcal{S}/9}\left(\Gamma\left(e^{\frac{2\mathcal{S}}{9}}\right)\right)^9 e^{-2\mathcal{S}e^{\frac{2\mathcal{S}}{9}}} \cdot e^{\mathcal{S}} \tag{24}$$

Step 8. The density of $\omega = e^{\mathcal{S}}$ is

$$\begin{aligned} g(\omega) &\sim \omega^{\frac{1}{9}}\left(\Gamma\left(\omega^{\frac{2}{9}}\right)\right)^9 e^{-2\omega^{\frac{2}{9}}\log\omega} \\ &= \frac{\omega^{\frac{1}{9}}\left(\Gamma\left(\omega^{\frac{2}{9}}\right)\right)^9}{\omega^{2\omega^{\frac{2}{9}}}} \end{aligned} \tag{25}$$

Step 9.

$$\log\Gamma\left(\omega^{\frac{2}{9}}\right) \sim \frac{2}{9}\omega^{\frac{2}{9}}\log\omega - \omega^{\frac{2}{9}} - \frac{1}{9}\log\omega \tag{26}$$

(see pp. 940 in Gradshteyn and Ryzhik (1980))

Step 10. From (25) and (26),

$$\log g(\omega) = -\frac{8}{9}\log\omega - 9\omega^{\frac{2}{9}} + 0(1) \tag{27}$$

Step 11. Since $Y = Z_1Z_2Z_3$ is symmetric, (27) specifies the tail of $Z_1Z_2Z_3$ for both positive and negative values and since $\int_{-\infty}^{\infty}\frac{1}{1+y^2}dy$, $\int_{-\infty}^{\infty}\frac{y^{2/9}}{1+y^2}dy$ and $\int_{-\infty}^{\infty}\frac{\log|y|}{1+y^2}dy$ are all finite real numbers, one has $E_2(f)$ (of Lemma 1) $\not< -\infty$ $>$ and so Y is undetermined.

References

[1] Akhiezer, N.I., *The classical moment problem*, Hafner, New York, (1965).

[2] Berg, C., *On the preservation of determinacy under convolution*, Proc. Amer. Math. Soc., **93**(1985), 351 - 357.

[3] *The cube of a normal is undetermined*, Ann. Prob., **16**(1988), 910 - 913.

[4] *Indeterminate moment problems and the theory of entire functions*, Jour. Comp. and Appl. Math, **65**(1995), 27 - 55.

[5] Berman, S., *The tail of the convolution of densities and its applications to a model of HIV latency time*, Ann.Appl.Prob., **2**(1992),481 - 502.

[6] Brown, L.D., *Fundamentals of statistical exponential families*, IMS, Hayward, California, (1986).

[7] Diaconis, Persi, *Application of the method of moments in probability and statistics*, In Moments in Mathematics, AMS, **37**(1987),125 - 142 .

[8] Gradshteyn,I.S. and Ryzhik, I.M., *Table of integrals, series, and products*, Academic Press, New York, (1980).

[9] Jensen,J.L., *Uniform saddlepoint approximations*, Adv. Appl. Prob., (1988).

[10] Karlin, S. and Studden, W.J., *Tchebycheff systems, with applications in analysis and statistics*, Interscience, New York, (1966).

[11] Kemperman, J.H.B., *The general moment problem, a geometric approach*, Ann.Math.Stat., **39**(1968), 93 - 122.

[12] *On a class of moment problems*, Proc. Sixth Berkeley Symp., **2**(1972), 101 - 126.

[13] *Geometry of the moment problem*, In Moments in Mathematics, AMS, **37**(1987), 16 - 53 .

[14] Korner, T., *Fourier Analysis*, Cambridge University Press, (1990).

[15] Reid, N., *Saddlepoint methods and statistical inference*, Stat. Sc., **3**(1988), 213 - 238.

[16] Shiryayev,S., *Probability*, Springer - Verlag, New York, (1980).

[17] Stieltjes,T.J., *Rescherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Math, **8**(1894),1 - 122.

[18] Stoyanov,J., *Counterexamples in probability*, Wiley, New York, (1987).

FIG 1

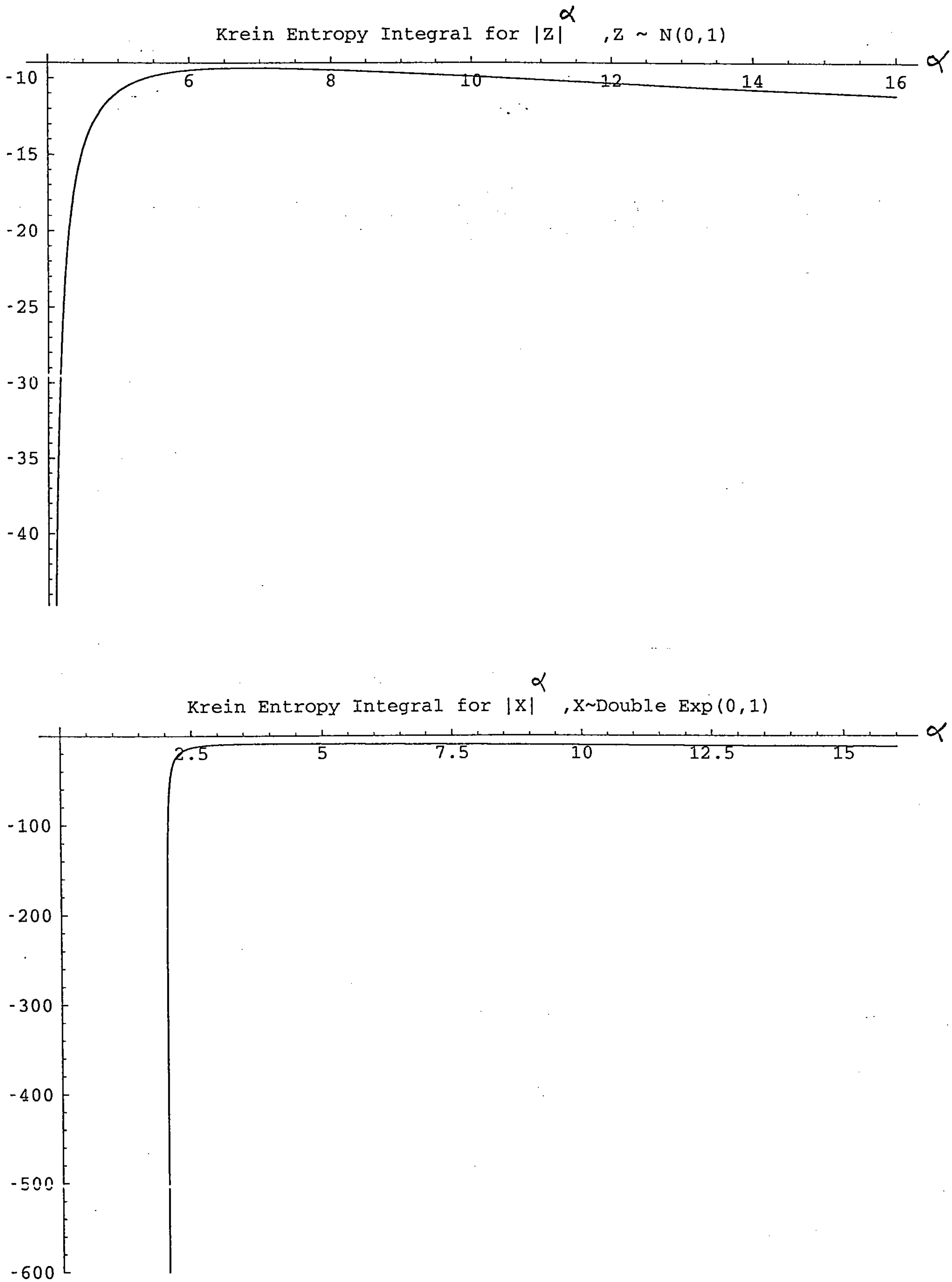

Krein Entropy Integral for |Z|^α ,Z ~ N(0,1)
α
6
8
10
12
14
16
-10
-15
-20
-25
-30
-35
-40
Krein Entropy Integral for |X|^α ,X~Double Exp(0,1)
α
2.5
5
7.5
10
12.5
15
-100
-200
-300
-400
-500
-600

FIG 2

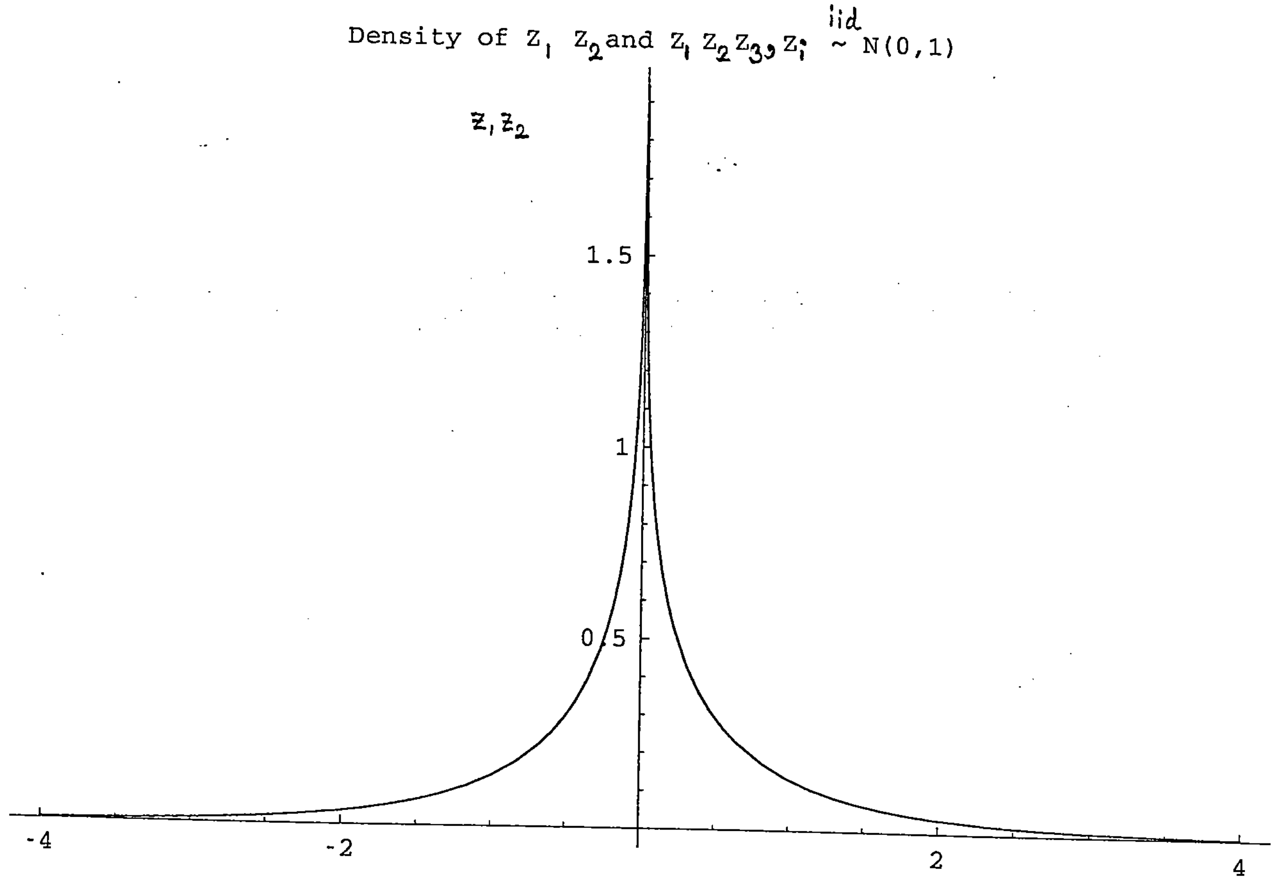
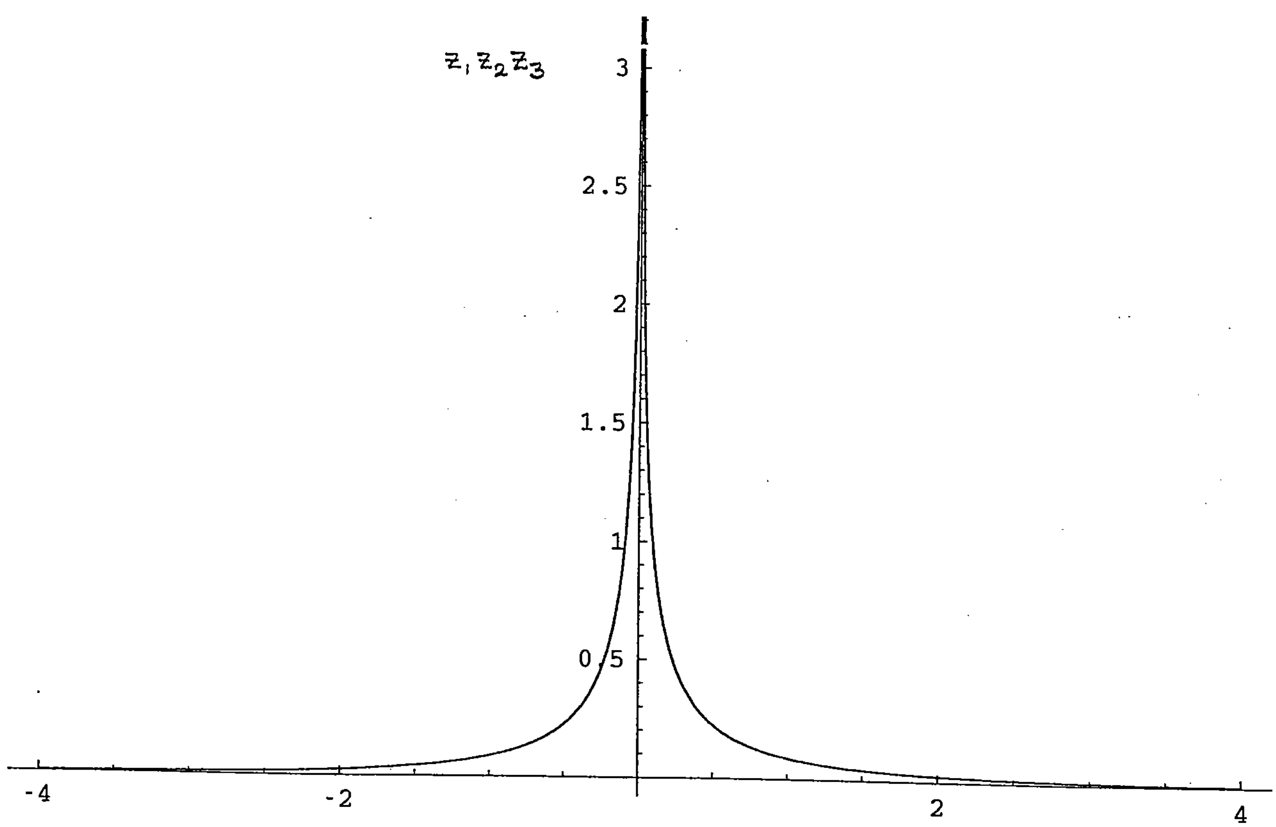

Density of $Z_1 Z_2$ and $Z_1 Z_2 Z_3$, $Z_i \overset{iid}{\sim} N(0,1)$
$Z_1 Z_2$
1.5
1
0.5
-4
-2
2
4

FIG 3


$Z_1 Z_2 Z_3$
3
2.5
2
1.5
1
0.5
-4
-2
2
4