BROWNIAN MOTION AND RANDOM WALK PERTURBED AT EXTREMA

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Summary

Let b_t be Brownian motion. We show there is a unique adapted process x_t which satisfies $dx_t = db_t$ except when x_t is at a maximum or a minimum, when it receives a push, the magnitudes and directions of the pushes being the parameters of the process. For some ranges of the parameters this is already known. We show that if a random walk close to b_t is perturbed properly, its paths are close to those of x_t.

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1. Introduction

This paper studies walks on the integers which jump to one of the two nearest neighbors according to the rules

$$P(X_{n+1} = X_n + 1 | X_k, k \leq n) = \begin{cases} p, & \text{if } n > 0 \text{ and } X_n = \max_{k \leq n} X_k \\ q, & \text{if } n > 0 \text{ and } X_n = \min_{k \leq n} X_k \\ 0, & \text{otherwise.} \end{cases}$$

The parameters p and q satisfy $0 < p < 1$ and $0 < q \leq 1$. These walks will be called pq walks. A strong version of the invariance principle is proved, completing results in Davis (1996), which did this for only some p and q. The limit processes are shown to be the unique strong solutions of certain equations which intuitively should, and as it turns out, do, define them. This completes results of many authors. Chaumont and Doney have independently, simultaneously, and differently proved this, in the paper Chaumont-Doney (1998) published in this issue of this journal.

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This paper is organized as follows. Only the cases with reflection at zero are treated first. We introduce these following Le Gall-Yor (1992). Then in Section 4, the softly perturbed at both extrema motions are discussed. Random walks are treated in Section 3.

Put $f^*(t) = \sup_{0\le s\le t} f(s)$. Let $r > 0$ and $\theta > -1$. The following equations define, path by path, a process $\gamma^{r,\theta} = \gamma$.

$$(1.1)_{r,\theta} \qquad \begin{array}{ll} \text{i)} & \gamma_0 = r \\ \text{ii)} & \text{If } \gamma_t = \gamma_t^*, \text{ and } \gamma_y > 0,\ t < y < s, \text{ then} \\ & \gamma_s - \gamma_t = b_s - b_t + \theta \max_{t\le y\le s}(b_y - b_t). \\ \text{iii)} & \text{If } \gamma_t = 0, \text{ and } \gamma_y < \gamma_t^*,\ t < y < s, \text{ then} \\ & \gamma_s - \gamma_t = b_s - b_t - \min_{t\le y\le s}(b_y - b_t). \end{array}$$

The argument that γ is determined by these equations is given in Le Gall-Yor (1992): use ii) to determine γ until the first time, call it T, that γ hits zero, then use iii) to define γ until it equals γ_T^*, then use ii), and so on. We will also consider the equations $(1)_{0,\theta}$, where now we insist that γ_t, $t \ge 0$, be continuous at 0, which is no longer implied by the equations. The construction just given fails here, since it is not clear how to start, because $\gamma_0^* = 0$.

Let $\mathcal{F}_t = \sigma(b_s, s \le t)$. We prove the following theorems, which hold for all $\theta > -1$, unless otherwise noted. Throughout this paper α and β stand for strictly positive numbers.

Theorem 1.1. *For each $t > 0$, $\lim_{\alpha,\beta\downarrow 0} |\gamma_t^{\alpha,\theta} - \gamma_t^{\beta,\theta}|^* = 0$ in probability.*

This theorem and the next were known to Le Gall and Yor for the cases $\theta < 1$. See the end of Le Gall-Yor (1992), and Carmona-Petit-Yor (1997). Le Gall and Yor encountered the equations $(1)_{0,\theta}$ while studying the winding of three dimensional Brownian motion. For all the cases $\theta \le 1$, and all $t > 0$, $|\gamma_t^{\alpha,\theta} - \gamma_t^{\beta,\theta}| \le |\beta - \alpha|$, while if $\alpha \ne \beta$ and $\theta > 1$, $|\gamma_t^{\alpha,\theta} - \gamma_t^{\beta,\theta}|$ is not bounded for any $t > 0$. This follows pretty easily from the methods of Davis (1996).

Theorem 1.2. *There is a unique solution of $(1.1)_{0,\theta}$ which is adapted to the filtration $\mathcal{F}_t$. Furthermore, if the filtration $\mathcal{G}_t$, $t \ge 0$, satisfies $\mathcal{G}_t \supseteq \mathcal{F}_t$, and that for each $t < s$, $\mathcal{G}_t$ is independent of $b_s - b_t$, there are no additional $\mathcal{G}_t$ adapted solutions.*

It is easy to prove that $\gamma_t^{r,\theta}$, $t \geq 0$, converges weakly as r decreases to 0: Let $\varepsilon > 0$. If $0 < \alpha, \beta < \varepsilon$, and τ_α, τ_β stand for the first hitting time of ε by $\gamma^{\alpha,\theta}$, $\gamma^{\beta,\theta}$, then $\gamma^{\alpha,\theta}_{\tau_\alpha+t}$, $t \geq 0$, and $\gamma^{\beta,\theta}_{\tau_\beta+t}$, $t \geq 0$, have exactly the same distributions, and if ε is small the distributions of τ_α and τ_β are close to zero. Alternatively, a weak solution of $(1.1)_{0,\theta}$ may be constructed using the excursion methods of Perman-Werner (1997). So processes with the distribution of the unique solution guaranteed by Theorem 1.2 are already known to exist and have been studied. In fact, the distribution of these processes at fixed times t is explicitly found in Carmona-Petit-Yor (1997). At bottom, the results proved in this paper are concerned with the stability of these processes. Theorem 1.2 does not imply that for almost every Brownian path there is a unique solution of $(1.1)_{0,\theta}$, and in fact our method does not show this. It is true, though, and proved in Chaumont-Doney (1998).

In Section 3 we show that if a fair random walk is embedded in b_t in the usual manner, and then perturbed by reflecting at zero and by tossing independent biased coins with probability p of heads to determine the motion at maxima, then this perturbed walk stays close to the unique solution γ^θ with $\theta = (2p-1)/(1-p)$ guaranteed by Theorem 1.2, in the usual sense that if you divide the supremum of the differences of the processes for $0 \leq k \leq n$, by $\sqrt{n}$, the resulting random variable converges to 0 in probability. This is stronger than and immediately implies weak convergence. This is extended to all pq walks in Theorem 4.8. Partial results towards weak convergence are proved in Davis (1996), Toth (1996, 1997), and Toth-Werner (1997), and in this latter paper it is announced that weak convergence for a large class of processes, including pq walks, will be proved in a forthcoming paper. The methods of Toth and Werner are very different from ours.

2. Proof of Theorems 1.1 and 2.1.

Throughout this and the next section we assume $\theta \geq 1$. Our arguments require only minor modifications to handle the $\theta < 1$ cases, just a few of the constants are different. In addition the $\theta < 1$ results are included in the results of Section 4. Throughout this paper, τ_x stands for $\inf\{t : \ b_t = x\}$. The following notation is used in this section. The numbers α and β will satisfy $0 < \alpha < \beta$. We designate $\gamma_t^{\alpha,\theta}$ and $\gamma_t^{\beta,\theta}$ by g_t and h_t respectively, and we put $T_0 = 0$, $T_{2i+1} = \inf\{t > T_{2i}\colon\ g_t = h_t = 0\}$, $i \geq 0$, and $T_{2i} = \inf\{t > T_{2i-1}\colon$ either $g_t = g_t^*$ or $h_t = h_t^*\}$. It is not hard to see that $T_1 < \infty$ a.s., since a large enough decrease in b_t will push both h_t and g_t to 0. Similar considerations show that $T_k < \infty$ a.s. for every k. If this isn't clear to the reader now, it will be after a reading of the proof of Lemma 2.6. We put $\Gamma_n = |g^*_{T_{2n-1}} - h^*_{T_{2n-1}}| / \max(g^*_{T_{2n-1}}, h^*_{T_{2n-1}}), n \geq 1$. Define $\lambda[s,t] = \max\{b_y - b_x\colon\ s \leq x \leq y \leq t\}$.

Lemma 2.1. *For all $0 \leq s \leq t$, and all $r > 0$,*

$$\lambda[s,t] \leq \max\{\gamma^{r,\theta}_x - \gamma^{r,\theta}_y\colon\ s \leq y \leq x \leq t\} \leq (\theta+1)\lambda[s,t]. \tag{2.1}$$

Proof: Shorten $\gamma^{r,\theta}_t$ to γ_t. The left hand side of (2.1) follows from $\gamma_y - \gamma_x \geq b_y - b_x$, which in turn follows from its truth on intervals of the kinds considered in $(1.1)_{r,\theta}$ ii) and iii). To prove the right hand inequality, suppose that $s \leq u \leq v \leq t$ and $\gamma_v - \gamma_u = \max\{\gamma_x - \gamma_y\colon\ s \leq y \leq x \leq t\}$, and that $\gamma_x > 0$, $u < x < v$. Then if $\varphi = \min(v, \inf\{y \geq u\colon\ \gamma_y = \gamma^*_y\})$, we have $b_\varphi - b_u = \gamma_\varphi - \gamma_u \geq 0$, and

$$\begin{aligned}
\gamma_v - \gamma_u &= (\gamma_v - \gamma_\varphi) + (\gamma_\varphi - \gamma_u) \\
&\leq (\theta+1)\max\{b_x - b_\varphi\colon\ \varphi \leq x \leq v\} + b_\varphi - b_u \\
&\leq (\theta+1)\max\{b_x - b_u\colon\ \varphi \leq x \leq v\} \\
&\leq (\theta+1)\lambda[s,t].
\end{aligned}$$

□

Proposition 2.2. *There is a number $\mu = \mu(\theta) > 0$ such that $E(\Gamma_{n+1} | \Gamma_k, k \leq n) \leq \Gamma_n$ on $\{\Gamma_n < \mu\}$.*

This proposition is central to the proofs of the theorems. We do not know whether Γ_n, $n \geq 1$, is a supermartingale. The next few lemmas will be used to prove Proposition 2.2 which is essentially restated as Lemma 2.5. The first of these is easy and essentially known.

Lemma 2.3. *Let $\eta = \inf\{t > 0\colon\ \gamma^{1,\theta}_t = 0\}$. Then*

$$P((\gamma^{1,\theta}_\eta)^* \geq r) = r^{-(\theta+1)^{-1}}, r \geq 1. \tag{2.2}$$

Proof: Designate $\gamma^{1,\theta}_t$ by y_t. Only (1.1), ii), is needed to determine y_t, $0 \leq t \leq \eta$. Scaling and the strong Markov property for b_t give $P(y^*_\eta \geq d | y^*_\eta \geq a) = P(y^*_\eta \geq d/a)$, if $1 < a < d$.

Thus $P(y_\eta^* \geq r) = P(y_\eta^* \geq r^{1/n})^n$, and so what we are trying to prove is equivalent to

$$\text{(2.3)} \qquad \lim_{n\to\infty} n\ \ell n P(y_\eta^* \geq r^{1/n}) = -\ell n\ r/(\theta+1).$$

Let $w = r^{1/n} - 1$ and recall $\tau_x = \inf\{t\colon\ b_t = x\}$. Then $\{\tau_{w/(\theta+1)} < \tau_{-1}\} \subset \{y_\eta^* \geq r^{1/n}\} \subset \{\tau_{w/(\theta+1)} < \tau_{-(w+1)}\}$. The probabilities of the first and last of these events are easily computed, and (2.3) follows. □

Now at any T_{2n}, $n \geq 1$, one of the essentially equivalent events

$$A_n = \{g_{T_{2n}} = g^*_{T_{2n}} = h_{T_{2n}} \leq h^*_{T_{2n}}\},$$
$$B_n = \{h_{T_{2n}} = h^*_{T_{2n}} = g_{T_{2n}} \leq g^*_{T_{2n}}\},$$

occurs. Let $\delta \geq 0$. We define $y_t = \gamma_t^{1,\theta}$, as in the last proof, and $x_t^\delta = x_t$ so that the distribution of (y_t, x_t), $t \geq 0$, is the distribution of $(g_{T_{2n}+t}, h_{T_{2n}+t})$, conditioned on $\{1 = g_{T_{2n}} = g^*_{T_{2n}} = h_{T_{2n}}, h^*_{T_{2n}} = 1+\delta\}$. Precisely, x_t is defined by the equations created by replacing $\gamma_0, \gamma_s, \gamma_t$, and γ_t^* by x_0, x_s, x_t, and $\max\,(x_t^*, 1+\delta)$, respectively, in $(1.1)_{1,\theta}$, and by adding the rule that $x_t - 1 = b_t$ for $t \leq \min(\tau_\delta, \tau_{-1})$. Put $\psi = \inf\{t\colon\ x_t = 0\}$ (note $\psi \leq \eta = \inf\{t\colon\ y_t = 0\}$, recalling that $\theta \geq 1$), and also put $x_t^+ = \max(x_t^*, 1+\delta)$, and

$$M_\delta = \frac{|x_\psi^+ - y_\psi^*|}{\max(x_\psi^+, y_\psi^*)}.$$

Lemma 2.4. *The following inequality holds.*

$$\text{(2.4)} \qquad \limsup_{\delta\downarrow 0} \frac{1+\delta}{\delta} E M_\delta \leq \frac{\theta}{\theta+2}.$$

Proof. Note that $|x_\psi^+ - y_\psi^*| \leq \theta\delta$, with equality exactly when $\tau_\delta < \tau_{-1}$. Now (2.4) is implied by

$$\limsup_{\delta\downarrow 0} \frac{1+\delta}{\delta} E \frac{\theta\delta}{y_\psi^*} \leq \frac{\theta}{\theta+2},$$

which is of course equivalent to

$$\text{(2.5)} \qquad \limsup_{\delta\downarrow 0} E\frac{1}{y_\psi^*} \leq \frac{1}{\theta+2}.$$

Let $\nu = \inf\{t\colon\ y_t = \theta\delta\}$. Since $y_\psi \le \theta\delta$, $\nu \le \psi$ and so $y_\nu^* \le y_\psi^*$. Note also that $y_\nu^* \overset{d}{=} (1-\theta\delta)$ $y_\eta^* + \theta\delta$. Using these, Lemma 2.3 and scaling give

$$\begin{aligned}
E\frac{1}{y_\psi^*} &\le E\frac{1}{y_\nu^*} \\
&= \int_0^1 P(y_\nu^* \le s^{-1})ds \\
&= \int_0^1 1 - [(1-\theta\delta)/(s^{-1}-\theta\delta)]^{1/(\theta+1)}ds \\
&\longrightarrow \int_0^1 (1 - s^{1/(\theta+1)})ds \qquad\qquad \text{as } \delta \downarrow 0 \\
&= \frac{1}{\theta+2}.
\end{aligned}$$

□

Now put

$$\begin{aligned}
\psi_2 &= \inf\{t > \psi\colon\ x_t = x_t^+ \text{ or } y_t = y_t^*\}, \\
\psi_3 &= \inf\{t > \psi_2\colon\ x_t = 0 \text{ or } y_t = 0\}, \\
\psi_4 &= \inf\{t > \psi_3\colon\ x_t = x_t^+ \text{ or } y_t = y_t^*\}, \\
\xi &= \inf\{t > \psi\colon\ x_t = y_t = 0\}, \text{ and} \\
N_\delta &= \frac{|x_\xi^+ - y_\xi^*|}{\max(x_\xi^+, y_\xi^*)}.
\end{aligned}$$

As mentioned earlier, the following lemma is essentially the same as Proposition 2.2.

Lemma 2.5. *For every θ, there is $w(\theta) = w$, $0 < w < 1$, such that if $0 \le \delta < w$ then*

$$EN_\delta \le \delta/(1+\delta). \tag{2.6}$$

Proof: We have $N_\delta = M_\delta I(\xi < \psi_2) + N_\delta I(\psi_2 \le \xi \le \psi_4) + N_\delta I(\psi_4 < \xi) := X + Y + Z$. Now clearly

$$EX \le EM_\delta. \tag{2.7}$$

Furthermore, since $0 = x_\psi \ge y_\psi - \theta\delta$, if $\theta\delta < 1$ we have

$$P(\psi_2 \le \xi) = EP(\psi_2 \le \xi | \mathcal{F}_\psi) \le P(\tau_{1-\theta\delta} < \tau_{-\theta\delta}) = \theta\delta. \tag{2.8}$$

In addition, there is a positive constant $C(\theta)$ such that

$$|x_\xi^+ - y_\xi^*| \le C(\theta)\delta \text{ on } \{\psi_2 \le \xi < \psi_4\}. \tag{2.9}$$

To see this, first note that in $[\psi, \psi_2]$, $|x_t - y_t|$ is monotone and that $|x_{\psi_2}^+ - y_{\psi_2}^*| = |x_\psi^+ - y_\psi^*| \le \theta\delta$. Now in $[\psi_2, \psi_3]$, only (1.1) ii) and its analog for x are needed to define x and y. Considering the three possibilities $x_{\psi_2} \le y_{\psi_2} = y_{\psi_2}^* \le x_{\psi_2}^+$, $x_{\psi_2} \le x_{\psi_2}^+ \le y_{\psi_2} = y_{\psi_2}^*$, and $x_{\psi_2} = x_{\psi_2}^+ \le y_{\psi_2} \le y_{\psi_2}^*$, it is easy to check that there is a $C(\theta)$ such that $|x_t^+ - y_t^*| \le C(\theta)\delta$, $\psi_2 \le t \le \psi_3$. And of course $|x_t^+ - y_t^*| = |x_{\psi_3}^+ - y_{\psi_3}^*|$, if $\psi_3 \le t \le \psi_4$. This, together with the previous sentence gives (2.9), which with (2.8) implies

$$EY \le \theta C(\theta)\delta^2. \tag{2.10}$$

Also, since $|x_{\psi_3} - y_{\psi_3}| \le C(\theta)\delta$, it follows in the same way (2.8) was proved, that

$$P(\psi_4 < \xi | \psi_3 < \xi) \le C(\theta)\delta.$$

Together with $P(\psi_3 < \xi) \le P(\psi_2 < \xi) \le \theta\delta$, this gives $P(\psi_4 < \xi) \le \theta C(\delta)\delta^2$, and since $N_\delta \le 1$, this gives

$$EZ \le \theta C(\theta)\delta^2. \tag{2.11}$$

Inequalities (2.7), (2.10), and (2.11), together with Lemma 2.4, give (2.6). □

Lemma 2.6. *Given $0 < \varepsilon < 1$, there exists $\delta > 0$ such that if $\beta < \delta$ then*

$$P(T_{2n+1} < \varepsilon \text{ and } \Gamma_{n+1} < \varepsilon \text{ for some } n \ge 1) > 1 - \varepsilon.$$

Proof: Consider θ fixed. There are almost surely four numbers $0 < r < s < t < u < \varepsilon$ such that the following eight conditions, which we divide into three groups, hold.

I. i) $b_r = b_s^*$; ii) $b_s - b_r < -(\theta + 2)\lambda[0, r]$; iii) $\lambda[r, s] < \lambda[0, r]$

II. iv) $b_s = \min\{b_y\colon r \le y \le t\}$; v) $b_t - b_s > [(2\theta^2 + 1)/\varepsilon]\lambda[0, s]$

III. vi) $b_t = b_u^*$; vii) $b_u - b_t < -(\theta + 2)\lambda[0, t]$; viii) $\lambda[t, u] < \lambda[0, t]$.

We now show that if $\beta < \lambda[0,r]$, there is an integer n such that both $T_{2n+1} < \varepsilon$ and $\Gamma_{n+1} < \varepsilon$. Since $\lambda[0,r] > 0$ a.s., this will prove the lemma.

The three conditions of I, together with $\beta < \lambda[0,r]$, imply that $T_1 < s$. For the second condition insures that both g and h hit zero in $[r,s]$, since neither g_r nor h_r exceeds $\lambda[0,r] + (\theta+1)\lambda[0,r]$, by Lemma 2.1. Thus condition ii) guarantees both g and h hit 0 in $[r,s]$. Furthermore they must both equal zero at a common time in $[r,s]$, since the only thing that could keep this from happening is that one would rebound after hitting zero to hit its maximum before time s. But this possibility is precluded by Lemma 2.1, which guarantees both g_r^* and h_r^* are not less than $\lambda[0,r]$, and by iii) and Lemma 2.1 again.

Let n be defined by $T_{2n-1} < s < T_{2n}$. We have $\lambda[0,s] \le g_s^*, h_s^* \le (\theta+2)\lambda[0,s]$, and so $|g_s^* - h_s^*| \le 2\theta\lambda[0,2]$. Because of condition iv), only prescription ii) of (1.1) is needed to determine the motion of h and g in $[s,t]$, and it is not hard to show that $|g_t^* - h_t^*| \le 2\theta^2\lambda[0,s]$, and $g_t^*, h_t^* \ge b_t - b_s$. Thus condition v) guarantees $|g_t^* - h_t^*|/\max(g_t^*, h_t^*) < \varepsilon$, and $s < T_{2n} < t$. Finally, the last three conditions guarantee that $t < T_{2n+1} < u$, using an argument like the one which showed $T_{2n-1} < \varepsilon$. $\square$

Lemma 2.7. *Given $\varepsilon > 0$, there is $\varphi = \varphi(\theta, \varepsilon) > 0$, such that if $\beta < \varphi$ there is an N such that $P(T_{2N-1} < \varepsilon) > 1 - \varepsilon$ and $P(\sup_{n \ge N} \Gamma_n > \varepsilon) < \varepsilon$.*

Proof: Let w be as in the statement of Lemma 2.5, and assume that $0 < \varepsilon < w < 1$. Put $N = \inf\{n\colon \Gamma_n < \varepsilon^2\}$, $M = \inf\{n \ge N\colon \Gamma_n > w\}$, and $Z_n = \Gamma_{\min(n+N,M)}$, $n \ge 0$. Then Z_n, $n \ge 1$, is a nonnegative supermartingale, and so by Doob's maximal inequality,

$$\begin{aligned} P(\sup_{n\ge N} \Gamma_n > \varepsilon) &= P(\sup_{k \ge 0} Z_k > \varepsilon) \\ &\le EZ_0/\varepsilon \\ &\le \varepsilon. \end{aligned}$$

And Lemma 2.6 insures that φ can be chosen so small that $P(T_{2N-1} < \varepsilon) > 1 - \varepsilon$. $\square$

Lemma 2.8. *Let $\psi(t) = |g_t - h_t|/\max(g_t^*, h_t^*)$. There exist $\delta = \delta(\theta) > 0$ such that if $0 < \eta < 1/4$ then*

$$\delta P(\sup_{t \ge T_{2n-1}} \psi(t) \ge \eta) \le P(\sup_{k \ge n} \Gamma_k \ge \eta/2), n \ge 0. \tag{2.12}$$

Proof. Fix n and let $\tau = \inf\{t \geq T_{2n-1}\colon\ \psi(t) \geq \eta\}$. Since $\psi(t)$ can only increase when either g_t^* or h_t^* increases, τ cannot lie between T_k and T_{k+1} if k is odd. Define N on $\{\tau < \infty\}$ by $T_{2(N-1)} < \tau \leq T_{2N-1}$. We will show that there exists δ such that

$$P(\Gamma_N \geq \eta/2|\mathcal{F}_\tau) \geq \delta \text{ on } \{\tau < \infty\}, \tag{2.13}$$

which upon integration proves (2.12). If $\tau = T_{2n-1}$, (2.13) is trivial with $\delta = 1$. Assume from now on that $\tau > T_{2n-1}$. Let

$$\nu = \inf\{t > \tau\colon\ h_t \text{ or } g_t = 0\},$$

and $\varphi = \inf\{t > \nu\colon\ h_t = h_t^* \text{ or } g_t = g_t^*\}$.

Now both $|h_t - g_t|$ and $|h_t^* - g_t^*|$ are non-decreasing on $[\tau, \nu]$, and $|h_t^* - g_t^*| \leq |h_t - g_t|$ on this interval. For suppose without loss of generality that $h_\tau > g_\tau$. Then $h_\tau = h_\tau^*$ while $g_\tau \leq g_\tau^*$ so $h_t - g_t$ can only increase on $[\tau, \nu]$. Furthermore, on $\{\varphi > T_{2N-1}\}$, $|h_t^* - g_t^*|$ does not decrease on $[\tau, T_{2N-1}]$.

Now if

$$\max_{\tau \leq t \leq T_{2N-1}} (b_t - b_\tau) \leq (\theta + 1)\max(g_\tau^*, h_\tau^*),$$

by Lemma 2.1 the denominator of $\psi(t)$ can not double before time T_{2N-1}, and thus $\Gamma_N > \psi(\tau)/2$. Thus, by scaling and an argument like the one used in the proof of Lemma 2.6, we may take

$$\delta = P(\tau_{-2} < \tau_{(\theta+1)^{-1}}, \lambda[0, \tau_{-2}] < 3/4). \qquad \square$$

Now we complete the proof of Theorem 1.1. Lemmas 2.7 and 2.8 show that for $s < t$ fixed, $\sup_{s \leq y \leq t} \psi(y)$ approaches zero in probability as β decreases to zero. Together with Lemma 2.1, this implies that $\sup_{s \leq y \leq t} \frac{|h_y - g_y|}{\lambda[0,t]}$ approaches 0 in probability as β decreases to zero, which implies that $\sup_{s \leq y \leq t} |h_y - g_y|$ decreases to 0 in probability as β decreases to 0. Lemma 2.1 also implies that $\sup_{0 \leq y \leq s} |h_y - g_y|$ decreases to zero in probability as both s and β decrease to zero. Theorem 1 follows. $\square$

Proof of Theorem 1.2. This proof is so close to the proof of Theorem 1.1 that we will be very brief. Let $\mathcal{G}_t$ be as in the statement of Theorem 1.2 and suppose that p_t and q_t

are two $\mathcal{G}_t$ adapted solutions of $(1.1)_{0,\theta}$. For $s > 0$, define

$$T_0^s = \inf\{t \geq s\colon\ p_t = q_t = 0\},$$
$$T_1^s = \inf\{t > T_0^s\colon\ \text{either } p_t = p_t^* \text{ or } q_t = q_t^*\},$$

and so on, and put

$$\Gamma_n^s = |p^*_{T^s_{2n-1}} - q^*_{T^s_{2n-1}}| / \max(p^*_{T^s_{2n-1}}, q^*_{T^s_{2n-1}}).$$

Now given $\varepsilon > 0$ there is $\delta < \varepsilon$ such that there is an N such that $P(T^\delta_{2N-1} < \varepsilon$ and $\Gamma_N^\varepsilon < \varepsilon)1 - \varepsilon$. The proof of this strongly parallels the proof of Lemma 2.7. Using the fact that since p and q are continuous at zero, both p_s^* and q_s^* approach zero as s approaches zero. For t fixed, by picking ε small enough we can show $|p_t - q_t|^*$ is arbitrarily close to 0 in probability, which of course implies $p_s = q_s$, $0 \leq s \leq t$. □

3. Perturbed Random Walk

Throughout this section, $\theta \geq 1$ and $\theta = (2p-1)/(1-p)$. Functions defined only on discrete unbounded sets which include zero are identified with their extension to $[0, \infty)$ by linear interpolation. The solution of $(1.1)_{0,\theta}$ guaranteed by Theorem 1.2 is denoted by γ or γ^θ. All functions are assumed without mention to be continuous on $[0, \infty)$ and to vanish at zero. We now give a non-stochastic version of $(1.1)_{0,\theta}$. If f is a function on $[0, \infty)$, we say the function g solves $(3.1)_\theta$ for f if

$(3.1)_\theta$

i) $g(0) = 0$

ii) If $g(t) = g^*(t)$ and $g(y) > 0$, $t < y < s$,
then $g(s) - g(t) = f(s) - f(t) + \theta \max_{t \leq y \leq s}(f(y) - f(s))$.

iii) If $g(t) = 0$ and $g(y) < g^*(t)$, $t < y < s$,
then $g(s) - g(t) = f(s) - f(t) - \min_{t \leq y \leq s}(f(y) - f(t))$.

A function is called piecewise linear on $[a, b]$ if its graph over $[a, b]$ consists of a finite number of line segments. It is not hard to check that if f is piecewise linear on $[0, \varepsilon]$ for some $\varepsilon > 0$, and not zero on $(0, \delta)$ for some $\delta > 0$ (we will define class $\mathcal{L}$ to be all such functions), then there is a unique $g_\theta = g$ which solves $(3.1)_\theta$ for f. For g may be explicitly described on $[0,\varepsilon]$, and then once g gets started the Le Gall-Yor procedure described after $(1.1)_{r,\theta}$ may be used. This unique g will be denoted $S(f)$.

Lemma 3.1. *Let ε, ε_n, $n > 0$, and δ be positive numbers such that $\varepsilon_n \geq \varepsilon$ and $\lim_{n\to\infty} \varepsilon_n = \varepsilon$. Let f_n be functions such that $f_n^*(\varepsilon_n) > \delta$ and such that there is a function f such that $f_n(s) - f_n(t) = f(s) - f(t)$ if $\varepsilon_n \leq s < t$. Suppose that the functions g_n satisfy that $g_n - f_n$ converges uniformly to zero on compact intervals, and that $g_n(s) = f_n(s)$, $0 \leq s \leq \varepsilon_n$, and suppose the functions f_n, $n \geq 1$, and g_n, $n \geq 1$, are all in class $\mathcal{L}$. Then $S(f_n) - S(g_n)$ converges uniformly to zero on compact intervals.*

Proof: Put

$$\tau_{1,n} = \inf\{t \geq \varepsilon_n\colon\ S(f_n(t)) = 0)\},$$

$$\tau_{2,n} = \inf\{t \geq \varepsilon_n\colon\ S(f_n(t)) = S(f_n(t))^*\}$$

$$\tau_{3,n} = \inf\{t \geq \varepsilon_n\colon\ S(g_n(t)) = 0\}$$

$$\tau_{4,n} = \inf\{t \geq \varepsilon_n\colon\ S(g_n(t)) = S(g_n(t))^*\}$$

$$\tau^n = \min\{\max(\tau_{1,n}, \tau_{2,n}), \max(\tau_{3,n}, \tau_{4,n})\}$$

Put $a = \inf\{y\colon \sup\{|f(r) - f(s)|\colon \varepsilon \leq r \leq s \leq \varepsilon + y\} = \delta/3(\theta+1)\}$. Then for large enough n, $\tau^n \geq \varepsilon + a$, since the deterministic analog of the left hand side of (2.1) shows that for either $S(g_n)$ or $S(f_n)$ to hit both a maximum and zero in $[\varepsilon, a+\varepsilon]$, then f_n or g_n respectively must rise or fall at least $(1+\theta)^{-1}$ times the difference between the respective maximums and zero. Furthermore, for n large enough, if $S(f_n)$ hits its maximum on $[\varepsilon, a+\varepsilon]$, $S(g_n)$ cannot hit zero in $[\varepsilon, a+\varepsilon]$, and vice versa. Thus, in $[\varepsilon, a+\varepsilon]$, only (the same) one of $(3.1)_\theta$ ii), iii) is required to determine the behavior of $S(f_n)$ and $S(g_n)$ and now the uniform convergence to zero of $S(f_n) - S(g_n)$ on $[\varepsilon, a+\varepsilon]$ is easily deduced. This procedure may be employed again, to show the uniform convergence of $S(f_n) - S(g_n)$ to zero in $[\varepsilon, \varepsilon + a + a']$, where $a' = \inf\{y > 0\colon \sup\{|f(r) - f(s)|\colon \varepsilon + a \leq r \leq s \leq \varepsilon + a + y\} = \delta/3(\theta+1)\}$, and so on. □

We use $P\lim_{n\to\infty} X_n = X$ to designate that X_n converges to X in probability as n approaches infinity.

Lemma 3.2. *Let A_t^n, $n \geq 1$, and D_t^n, $n \geq 1$, be sequences of stochastic processes defined on the same probability space that b_t is defined on. Let $\varepsilon > 0$ and let ε_n be random variables* *such that $\varepsilon \le \varepsilon_n$ and $P\lim_{n\to\infty} \varepsilon_n - \varepsilon = 0$. Suppose the paths of A^n and D^n are in class $\mathcal{L}$, that $A_t^n = D_t^n$, $t \le \varepsilon_n$, that $P\lim_{n\to\infty} |D_t^n - b_t|^* = 0$, $t > 0$, and that $A_s^n - A_{\varepsilon_n}^n = b_s - b_{\varepsilon_n}$ if $s \ge \varepsilon_n$. Then*

$$P\lim_{n\to\infty} |S(A_t^n) - S(D_t^n)|^* = 0, t > 0. \tag{3.2}$$

Proof: To establish (3.2), it suffices to show that there exists a sequence $n_1 < n_2 < \dots$ such that if $m_k \ge n_k$ then

$$\lim_{k\to\infty} |S(A_t^{m_k}) - S(D_t^{m_k})|^* = 0 \quad \text{a.s..} \tag{3.3}$$

It suffices to pick n_k so that $m_k \ge n_k$ implies both $\lim_{n\to\infty} |D_t^{m_k} - b_t|^* = 0$ a.s., and $\lim_{n\to\infty} |A_{\varepsilon_n}^{m_k} - D_{\varepsilon_n}^{m_k}|^* = 0$ a.s. Now Lemma 3.1 may be applied path by path, upon observing that $\lim_{n\to\infty} D_{\varepsilon_n}^* = b_\varepsilon^* > 0$. □

Lemma 3.3. *Let G_t^n, $0 \le t < \infty$, be a stochastic process with paths in class $\mathcal{L}$. Suppose that for each $s > 0$, a sequence of random variables satisfies $w_{n,s} \ge s$ and $P\lim_{n\to\infty} w_{n,s} = s$. Suppose that $P\lim_{n\to\infty} |G_t^n - b_t|^* = 0$, $t > 0$. Define $H_t^{n,s} = G_t^n$, $t \le w_{n,s}$, and $H_t^{n,s} - H_{w_{n,s}}^{n,s} = b_t - b_{w_{n,s}}$, $t > w_{n,s}$. Then given $T > 0$ and $\delta > 0$ there is an $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then there is an N such that $n \ge N$ implies $P(|S(H_T^{n,\varepsilon}) - \gamma_T|^* > \delta) < \delta$.*

Proof: The proof of this lemma is an easy modification of the proof of Theorem 1.2. In place of T_0^s we use $T_0^{n,s} = \inf\{t \ge w_{n,s}\colon S(H_t^{n,s}) = \gamma_t = 0\}$, and so on. Both $P\lim_{\varepsilon\downarrow 0} \gamma_\varepsilon^* = 0$ and $P\lim_{\varepsilon\downarrow 0} S(H_\varepsilon^{n,\varepsilon})^* = 0$, the last using a deterministic version of Lemma 2.1, and the fact that $P\lim_{n\to\infty} |G_\varepsilon^n - b_\varepsilon|^* = 0$, so that if $\eta > 0$ there is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ there is an $N = N(\varepsilon)$ such that if $n > N$, $P(|G_{w_{n,\varepsilon}}|^* > \eta) < \eta$ and $P(b_{w_{n,\varepsilon}}^* > \eta) < \eta$. □

Now let R_n be a fair random walk with $R_0 = 0$. Let Y_n be a sequence of iid random variables independent of R_n such that $P(Y_n = 1) = 1 - P(Y_n = -1) = p$. Inductively define the process Θ_n, $n \ge 0$, by $\Theta_0 = 0$, $\Theta_1 = 1$, and $\Theta_{k+1} - \Theta_k = R_{k+1} - R_k$ unless $\Theta_k = 0$ or $\Theta_k = \Theta_k^*$, in which case $\Theta_{k+1} - \Theta_k = 1$ or Y_k respectively. Then Θ_n is a $p1$ walk, which we call R_n *perturbed by* Y_n *and by reflection at zero.*

Lemma 3.4. *There is a process Γ_t such that $P\lim_{t\to\infty} |\Gamma_t - R_t|^*/\sqrt{t} = 0$ and $S(\Gamma_n) = \Theta_n$.*

Proof: Define $\Gamma_s = (\theta+1)^{-1}s$, $0 \le s \le 1$, and $\Gamma_{k+s} - \Gamma_k = R_{k+s} - R_k$, if $0 \le s \le 1$ and $\Theta_k \in (0, \Theta_k^*)$. If $\Theta_k = \Theta_k^*$, $k \ge 1$, for $0 \le s \le 1$ define $\Gamma_{k+s} - \Gamma_k = -s$ on $\{Y_k = -1\}$ and $\Gamma_{k+s} - \Gamma_k = (\theta+1)^{-1}s$ on $\{Y_k = +1\}$. If $\Theta_k = 0$, $k \ge 1$, define $\Theta_{k+s} - \Theta_k$, $0 \le s \le 1$, to be s on $\{R_{n+1} - R_n = 1\}$ and to be $-4s$, $0 \le s \le \frac{1}{2}$ and $-2 + 2(s - \frac{1}{2})$, $\frac{1}{2} \le s \le 1$, on $\{R_{k+1} - R_k = -1\}$. Note on $\{\Theta_k = 0\}$, $\Gamma_{k+1} - \Gamma_k = R_{k+1} - R_k$. It is straightforward to show that $S(\Gamma_n) = \Theta_n$.

To establish $P\lim_{t\to\infty} |\Gamma_t - R_t|^*/\sqrt{t} = 0$, it suffices to show

$$P \lim_{n\to\infty} |\Gamma_n - R_n|^*/\sqrt{n} = 0. \tag{3.4}$$

Now $(\Gamma_m - \Gamma_1) - (R_m - R_1) = \sum\limits_{k=2}^{m} (\Gamma_k - R_k) I(\Theta_{k-1} = \Theta_{k-1}^*)$. If J_m is those integers j such that $1 \le j \le m$ and $\Theta_j = \Theta_j^*$, then, conditioned on J_m, $\Gamma_m - R_m$ is the sum of the first $|J_m|$ of iid bounded random variables with zero mean, where $|\ |$ denotes cardinality. Thus, to prove (3.4), it suffices to show that $P\lim_{n\to\infty} |J_n|/n = 0$. This is quite easy. We omit the proof, but note that the proof of Lemma 3.2 of Davis (1996) adapts to the present situation. $\square$

We will call the process Γ_t constructed above the *precursor* of Θ_t.

Now define stopping times $\alpha_{n,k}$, $n \ge 1$, $k \ge 0$, by $\alpha_{n,0} = 0$ and $\alpha_{n,k+1} = \inf\{t \ge \alpha_{n,k}\colon |b_t - b_{\alpha_{n,k}}| = n^{-1/2}\}$. Let Y_n be iid ± 1 with probability p and $1-p$ respectively, and be independent of b_t. Define $H_k^n = \sqrt{n} b_{\alpha_{n,k}}$, $k \ge 0$, $n \ge 1$, let Z_k^n be the perturbation of the random walk H_k^n, $k \ge 0$, by Y_n and by reflection at 0, (so Z^n is a $p1$ walk), and let V_t^n be the precursor of Z_t^n. Define $z_{\alpha_{n,k}}^n = n^{-1/2}\, Z_k^n$, $k \ge 0$, and $v_{\alpha_{n,k}}^n = n^{-1/2} V_k^n$, $n \ge 0$, and extend the domains of z^n and v^n to $[0,\infty)$ by linear interpolation, for t between $\alpha_{n,k}$ and $\alpha_{n,k+1}$. Noting that for any integer M,

$$P \lim_{n\to\infty} \max_{1\le k\le Mn} |\alpha_{n,k} - (k/n)| = 0, \tag{3.5}$$

the following theorem implies the weak convergence of the process φ_t^n, $0 \le t \le 1$, defined by $\varphi_{k/n}^n = Z_k^n/\sqrt{n}$, $0 \le k \le n$ (and by linear interpolation) to γ_t, $0 \le t \le 1$.

Theorem 3.5. $P\lim_{n\to\infty} |z_1^n - \gamma_1|^* = 0$.

Proof: For $s > 0$ define $n(s) = \min\{k\colon\ \alpha_{n,k} > s\}$, and define $\theta_t^{n,s} = v_t^n$, if $0 \le t \le \alpha_{n,n(s)}$, and $\theta_t^{n,s} - \theta_{\alpha_{n,n(s)}}^{n,s} = b_t - b_{\alpha_{n,n(s)}}$ if $t \ge \alpha_{n,n(s)}$.

By Lemma 3.3, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ there is an N such that $n \geq N$ implies

$$P(|S(\theta_1^{n,\varepsilon}) - \gamma_1|^* > \delta) < \delta, \tag{3.6}$$

using Lemma 3.4 to show that v_t^n stays close to the process which is formed by connecting the points $(\alpha_{n,k}, b_{\alpha_{n,k}})$ with straight line segments, which implies $P\lim_{n\to\infty} |v_1^n - b_1|^* = 0$.

Next, Lemma 3.2 implies that for $\varepsilon > 0$

$$P \lim_{n\to\infty} |S(\theta_1^{n,\varepsilon}) - z_1^n|^* = 0, \tag{3.7}$$

recalling that $S(v^n) = z^n$, and again using (3.5). Together (3.6) and (3.7) establish Theorem 3.5. □

4. Soft Perturbation at Both Extrema

We begin with a collection of equations which includes all those defined by $(1.1)_{0,\theta}$. Let $\theta > -1$ and $\lambda \geq -1$, and denote $f^{\#}(t) = \inf\{f(s)\colon\ 0 \leq s \leq t\}$. By a solution γ of the following equation we also mean that γ is continuous at zero.

$(4.1)_{\theta,\lambda}$

i) $\gamma_0 = 0$

ii) If $\gamma_t = \gamma_t^*$ and $\gamma_y > \gamma_t^{\#}$, $t < y < s$,
then $\gamma_s - \gamma_t = b_s - b_t + \theta \max_{t\leq y\leq s}(b_y - b_t)$.

iii) If $\gamma_t = \gamma_t^{\#}$, and $\gamma_y < \gamma_t^*$, $t < y < s$,
then $\gamma_s - \gamma_t = b_s - b_t + \lambda \min_{t\leq y\leq s}(b_y - b_t)$.

The following theorem, which includes Theorem 1.2 as a special case, is new only in the cases $\lambda\theta < -1$, the cases $|\lambda\theta| < 1$, $|\lambda\theta| = 1$, $\lambda\theta > 1$ having been proved in Le Gall-Yor (1992) and Carmona-Petit-Yor (1996), Davis (1996), and Perman-Werner (1997) respectively. Also see Yor (1997).

Theorem 4.1. *There is a unique solution of $(4.1)_{\theta,\lambda}$ which is adapted to $\mathcal{F}_t, t \geq 0$. Furthermore, if $\mathcal{F}_t \subseteq \mathcal{G}_t$ and $b_s - b_t$ and $\mathcal{G}_t$ are independent for each $s > t$, then there are no other solutions of $(4.1)_{\theta,\lambda}$ which are adapted to $\mathcal{G}_t$, $t \geq 0$.*

Once the analog of Proposition 2.2 — Proposition 4.2 below — is in place, the proof of Theorem 4.1 is a close enough copy of the proof of Theorem 1.2 that we are going to omit it. But this analog is significantly harder to prove than Proposition 2.2, and we will elaborate on how the ideas of the proof of Proposition 2.2 need to be extended to accomplish this.

Let p_t and q_t be two $\mathcal{G}_t$ adapted solutions of $(4.1)_{\theta,\lambda}$, with $\mathcal{G}_t$ as in the statement of Theorem 4.1. For $s > 0$, define

$$\begin{aligned}
t_0^s &= \inf\{t \geq s\colon\ p_t = p_t^{\#} \text{ and } q_t = q_t^{\#}\},\\
t_{2i+1}^s &= \inf\{t \geq t_{2i}^s\colon\ p_t = p_t^{*} \text{ and } q_t = q_t^{*}\}, i \geq 0,\\
t_{2i}^s &= \inf\{t \geq t_{2i-1}^s\colon\ p_t = p_t^{\#} \text{ and } q_t = q_t^{\#}\}, i \geq 1,\\
\eta_{2i+1}^s &= \inf\{t \geq t_{2i+1}^s\colon\ p_t = p_t^{\#} \text{ or } q_t = q_t^{\#}\}, i \geq 0,\\
\eta_{2i}^s &= \inf\{t \geq t_{2i}^s\colon\ p_t = p_t^{*} \text{ or } q_t = q_t^{*}\}, i \geq 0,
\end{aligned}$$

Note $t_k^s \leq \eta_k^s \leq t_{k+1}^s$. Put

$$\psi_t = \frac{|p_t^* - q_t^*| + |p_t^{\#} - q_t^{\#}|}{\max(p_t^* - p_t^{\#}, q_t^* - q_t^{\#})},$$

and let $V_k = V_k^s = \psi_{\eta_k^s}, k \geq 0$.

Proposition 4.2. *There is a* $\mu = \mu(\theta, \lambda) > 0$ *such that*

$$E(V_{2n+2}|V_{2k}, k \leq n) \leq V_{2n} \text{ on } \{V_{2n} < \mu\}. \tag{4.2}$$

We sometimes shorten η_{2n}^s to η_{2n}. To prove (4.2) we may and do assume with no loss of generality that $p_{\eta_{2k}}^{\#} \leq q_{\eta_{2k}}^{\#}$, but even so we have two cases to consider. We set things up more or less like we did before (2.4). We use the same notation, x_t and y_t, for both cases, as much of their analysis can be done in common, and we put $\mathcal{H}_t = \sigma((x_s, y_s), s \leq t)$. Always in the following, $0 \leq \delta_1, \delta_2 < .1$ and$-1 \leq a \leq 0$.

Case 1. The distribution of the process (x_t, y_t) is the conditional distribution the process $(p_{\eta_{2n}} + t, q_{\eta_{2n}} + t), t \geq 0$, would have given $p_{\eta_{2n}}^{\#} \leq q_{\eta_{2n}}^{\#} < p_{\eta_{2n}}^{*} \leq q_{\eta_{2n}}^{*}, q_{\eta_{2n}}^{\#} - p_{\eta_{2n}}^{\#} = \delta_1$, $p_{\eta_{2n}}^{*} - q_{\eta_{2n}}^{\#} = 1$, $q_{\eta_{2n}}^{*} - p_{\eta_{2n}}^{*} = \delta_2$, and $q_{\eta_{2n}}^{\#} = a$. Put $x_t^+ = \max(x_t^*, a+1)$, $y_t^+ = \max(y_t^*, a + 1 + \delta_2)$, $x_t^- = \min(x_t^{\#}, a - \delta_1)$, and $y_t^- = \min(y_t^{\#}, a)$. So x_t behaves like

Brownian motion until it hits either $a-\delta_1$ or $a+1$, and y_t behaves like the same Brownian motion until it hits either $a+1+\delta_2$ or a, etc. Note $y_0 - x_0 = \delta_1$, and either $x_0 = a+1$ or $y_0 = a+1+\delta_2$.

Case 2. The distribution of the process (x_t, y_t) is the conditional distribution the process $(p_{\eta_{2n}+t}, q_{\eta_{2n}+t})$ would have given $p^{\#}_{\eta_{2n}} \le q^{\#}_{\eta_{2n}} < q^{*}_{\eta_{2n}} \le p^{*}_{\eta_{2n}}$, $q^{\#}_{\eta_{2n}} - p^{\#}_{\eta_{2n}} = \delta_1$, $q^{*}_{\eta_{2n}} - q^{\#}_{\eta_{2n}} = 1$, $p^{*}_{\eta_{2n}} - q^{*}_{\eta_{2n}} = \delta_2$, and $a = q^{\#}_{\eta_{2n}}$. In this case $x_t^+ = \max(x_t^*, a+1+\delta_2)$, $y_t^+ = \max(y_t, a+1)$, and x_t^- and y_t^- are given by the same formulas as in Case 1.

We don't have to consider the only remaining case since in this case $V_{2n} \ge 1$. Again $y_0 - x_0 = \delta_1$ and now $y_0 = a+1$.

We put $R_0 = 0$,

$$\begin{aligned}
T_1 &= \inf\{t > R_0\colon\ x_t^+ = x_t \text{ and } y_t^+ = y_t\},\\
R_1 &= \inf\{t > T_1\colon\ \text{ either } x_t^- = x_t \text{ or } y_t^- = y_t\},\\
T_2 &= \inf\{t > R_1\colon\ x_t^- = x_t \text{ and } y_t^- = y_t\},\\
R_2 &= \inf\{t > T_2\colon\ \text{ either } x_t = x_t^+ \text{ or } y_t = y_t^+\}.
\end{aligned}$$

Put

$$Z_i = \frac{|x^+_{R_i} - y^+_{R_i}| + |x^-_{R_i} - y^-_{R_i}|}{\max(x^+_{R_i} - x^-_{R_i}, y^+_{R_i} - y^-_{R_i})}, \qquad i = 0, 1, 2.$$

Especially, for Case 1, $Z_0 = (\delta_1 + \delta_2)/\max(1+\delta_1, 1+\delta_2)$, and for Case 2, $Z_0 = (\delta_1 + \delta_2)/(1+\delta_1+\delta_2)$.

Proposition 4.3. *There is $\mu = \mu(\theta, \lambda) > 0$ such that*

$$EZ_2 < Z_0 \text{ if } Z_0 < \mu.$$

Proposition 4.3 proves Proposition 4.2 in the same way that Lemma 2.5 proved Proposition 2.2.

The proof of Proposition 4.3 requires several lemmas. Let $\beta(\theta, \delta_1, \delta_2) = (2+\theta)\delta_1 + (1+\theta)\delta_2$ if $\theta \ge 0$, and $= (2+\theta)\delta_1 + \delta_2$, if $\theta < 0$. Note that if $\varepsilon > 0$ is fixed, $\beta(\theta, \delta_1, \delta_2)/(2+\theta) < Z_0$ if $\delta_2/\delta_1 > \varepsilon$ and $\delta_1 + \delta_2$ is small enough.

Lemma 4.4. *Let $\varepsilon > 0$. There are $\varphi_\theta(\varepsilon) > 0$, $\varphi_\lambda(\varepsilon) > 0$ and $k(\theta,\varepsilon) < 1$, $k(\lambda,\varepsilon) < 1$, such that*

$$EZ_1 < k(\theta,\varepsilon)Z_0 \qquad \text{if } Z_0 < \varphi_\theta(\varepsilon) \text{ and } \delta_2/\delta_1 \geq \varepsilon, \tag{4.3}$$

and

$$E(Z_2|\mathcal{H}_{R_1}) < k(\lambda,\varepsilon)Z_1 \qquad \text{on } \{Z_1 < \varphi_\lambda(\varepsilon), |x^+_{R_1} - y^+_{R_1}|/|x^-_{R_1} - y^-_{R_1}| \geq \varepsilon\}. \tag{4.4}$$

Proof: First note $Z_0 \leq \delta_1 + \delta_2 \leq 2Z_0$. Inequality (4.4) is of course just a conditional version of (4.3). And that $\varphi_\theta(\varepsilon)$ exists so that the left hand side of (4.3) holds follows from

$$|x^+_{R_1} - y^+_{R_1}| + |x^-_{R_1} - y^-_{R_1}| \leq \beta(\theta,\delta_1,\delta_2), \tag{4.5}$$

and calculations almost identical to those which proved (2.4). These are the content of Lemma 4.6 below. The inequality (4.5) is a slightly more complicated version of the first inequality of the proof of Lemma 2.4, which needs to be broken down into four cases, namely, Case 1 or Case 2 above, and $\theta \leq 0$, $\theta \geq 0$. We omit the details.

Lemma 4.5. *Given any constant $K > 1$, there is $q = q(K,\theta) > 0$ such that if $Z_0 < q(K,\theta)$, then $EZ_1 < KEZ_0$. Furthermore, $E(Z_2|\mathcal{H}_{R_1}) < KZ_1$ on $\{Z_1 < q(K,\lambda)\}$.*

Proof: This is even easier to prove than the last lemma and is proved very similarly. Again, though, its proof requires Lemma 4.6 below.

Lemma 4.6 *Let*

$$\alpha_1 = \inf\{t > R_0\colon x^-_t = x_t \text{ or } y^-_t = y_t\}$$
$$\alpha_2 = \inf\{t > \alpha_1\colon x_t = x^+_t \text{ or } y_t = y^+_t\}$$
$$\alpha_3 = \inf\{t > \alpha_2\colon x_t = x^-_t \text{ or } y_t = y^-_t\}$$
$$\alpha_4 = \inf\{t > \alpha_3\colon x_t = x^+_t \text{ or } y_t = y^+t\}.$$

Then there are constants $C_i(\theta)$ such that *i)* $P(\alpha_2 < T_1) \leq C_1(\theta)(\delta_1 + \delta_2)$.

ii) $P(\alpha_4 < T_1) \leq C_2(\theta)(\delta_1 + \delta_2)^2$.

iii) $Z_1 < C_4(\theta)Z_0$ *on* $\{T_1 < \alpha_4\}$.

Proof: All these inequalities are close parallels to inequalities used in the proof of Lemma 2.5. The α_1 above is the counterpart of ψ in that proof, and α_2, α_3, α_4 are the counterparts of ψ_2, ψ_3, and ψ_4. Inequalities i), ii), and iii) above are counterparts of (2.8), the second inequality before (2.11), and essentially the inequalities leading up to (2.10). □

The proof of Lemma 4.6 completes the proof of the preceeding two lemmas.

Lemma 4.7. *There are constants* $C_5(\theta)$, $C_6(\theta) > 0$ *such that* $|x^+_{R_1} - y^+_{R_1}|/|x^-_{R_1} - y^-_{R_1}| > C_5(\theta)$ *on* $\{T_1 < \alpha_2\}$ *if* $\delta_2/\delta_1 < C_6(\theta)$.

Proof: The proof is easy and omitted. The case when $\delta_2 = 0$ is especially easy, and points the way to the entire proof. □

Proof of Proposition 4.3. We first do the proof when $\delta_2/\delta_1 \geq C_6(\theta)$. Recall that $Z_0 \leq (\delta_1 + \delta_2) \leq 2Z_0$ if $\delta_1, \delta_2 < .1$.

Let the constants $k_1(\theta) = k_1 > 0$ and $M = M(\theta) < 1$ satisfy

$$EZ_1 < MZ_0 \text{ if } Z_0 < k_1 \text{ and } \delta_2/\delta_1 \geq C_6(\theta). \tag{4.6}$$

Such M and k_1 exist by Lemma 4.4. Let $J > 1$ satisfy $MJ < 1$, and let $k_2 = k_2(\lambda) > 0$ satisfy

$$E(Z_2|\mathcal{H}_{T_1}) < JZ_1 \text{ on } \{Z_1 < k_2\}. \tag{4.7}$$

Lemma 4.5 permits this. Let $k_3 = k_3(\theta) > 0$ satisfy

$$C_4(\theta)Z_0 < k_2 \text{ if } Z_0 < k_3. \tag{4.8}$$

So k_3 can be any positive number less than $k_2/C_4(\theta)$. Recall $Z_1 < C_4(\theta)Z_0$ on $\{T_1 < \alpha_4\}$. Pick $k_4 = k_4(\theta) > 0$ to satisfy

$$2C_2(\theta)(\delta_1 + \delta_2)^2 < (1 - MJ)Z_0 \text{ if } Z_0 < k_4. \tag{4.9}$$

We may pick any $0 < k_4 < (1-MJ)/4C_2$ since $(\delta_1+\delta_2)/2 \leq Z_0$ by the inequalities stated before Proposition 4.3, and the fact that $\delta_1, \delta_2 < .1$. Then if $Z_0 < \min(k_1, k_3, k_4)$ and $\delta_2/\delta_1 > C_6(\theta)$, we have, using (4.6)–(4.9) and Lemma 4.6 ii), and the fact that $Z_n \leq 2$ for all n, again by the inequalities preceeding Proposition 4.3.

$$\begin{aligned} EZ_2 &= EZ_2 I(T_1 < \alpha_2) + EZ_2 I(\alpha_2 \leq T_1 < \alpha_4) + EZ_2 I(T_1 \geq \alpha_4) \\ &\leq EE(Z_2|\mathcal{H}_{T_1}) I(T_1 < \alpha_2) + EE(Z_2|\mathcal{H}_{T_1}) I(\alpha_2 \leq T_1 \leq \alpha_4) \\ &\qquad\qquad\qquad + 2P(\alpha_4 \leq T_1) \\ &\leq EJZ_1 I(T_1 < \alpha_2) + EJZ_1 I(T_1 \geq \alpha_2) + (1-MJ)Z_0 \\ &< JEZ_1 + (1-MJ)Z_0 < Z_0. \end{aligned}$$

Thus in this case we may take $\mu = \min(k_1, k_3, k_4)$ in Proposition 4.3.

Next we turn to the case $\delta_2/\delta_1 \leq C_6(\theta)$. Let $Q = Q(\theta, \lambda) < 1$ satisfy

$$k(\lambda, C_5(\theta)) < Q, \text{ where } k \text{ is as in (4.4).} \tag{4.10}$$

Now pick k_5 so small that

$$C_4(\theta) Z_0 < \varphi_\lambda(C_5(\theta)) \text{ on } \{T_1 < \alpha_2\} \text{ if } Z_0 < k_5. \tag{4.11}$$

that is, $k_5 < \varphi_\lambda(C_5(\theta))/C_4(\theta)$. Using Lemma 4.7, (4.4), and Lemma 4.6 iii), we have

$$E(Z_2|\mathcal{H}_{R_1}) \leq QZ_1 \text{ on } \{T_1 < \alpha_2\}, \text{ if } Z_0 < k_5. \tag{4.12}$$

Let $\Gamma > 1$ satisfy $\Gamma Q < 1$ and pick k_6 to satisfy both

$$EZ_1 < \Gamma Z_0 \text{ if } Z_0 < k_6, \tag{4.13}$$

possible by Lemma 4.5, and

$$Z_1 \leq C_4(\theta) Z_0 < q(\Gamma, \lambda) \text{ on } \{T_1 < \alpha_4\} \text{ if } Z_0 < k_6, \tag{4.14}$$

possible by Lemma 4.6 iii). Now (4.14) guarantees

$$E(Z_2|\mathcal{H}_{R_1}) \leq \Gamma Z_1 \text{ on } \{T_1 < \alpha_4\}, \text{ if } Z_0 < k_6. \tag{4.15}$$

If $Z_0 < \min(k_5, k_6)$,

$$
\begin{aligned}
EZ_2 &= EZ_2 I(T_1 < \alpha_2) + EZ_2 I(\alpha_2 \le T_1 < \alpha_4) + EZ_2 I(T_1 \ge \alpha_4) \\
&\le EE(Z_2|\mathcal{H}_{R_1}) I(T_1 < \alpha_2) + EE(Z_2|\mathcal{H}_{R_1}) I(\alpha_2 \le T_1 < \alpha_4) + 2P(T_1 \ge \alpha_4) \\
&\le EQZ_1 I(T_1 < \alpha_2) + E\Gamma Z_1 I(\alpha_2 \le T_1 \le \alpha_4) + 2C_2(\theta)(\delta_1 + \delta_2)^2 \\
&\le QEZ_1 + \Gamma C_4(\theta) C_1(\theta)(\delta_1 + \delta_2)^2 + 2C_2(\theta)(\delta_1 + \delta_2)^2 \\
&\le Q\Gamma Z_0 + (\delta_1 + \delta_2)^2 [\Gamma C_4(\theta) C_1(\theta) + 2C_2(\theta)],
\end{aligned}
$$

using Lemma 4.6 i) and iii) for the middle term of the next to last inequality. Since $Z_0 \le \delta_1 + \delta_2 \le 2Z_0$, it is clear that we can pick $\mu > 0$ so small that $Z_0 < \mu$ implies $EZ_2 < Z_0$.

The μ of Proposition 4.3 may thus be chosen to be the smaller of the two μ of the two cases. □

Weak convergence of scaled pq walk to the unique solutions of $(4.1)_{\theta,\lambda}$ guaranteed by Theorem 4.1, and analogs of the considerably stronger result Theorem 3.5, can be proved in an almost identical manner to the proof of Theorem 3.5. Here we perturb our fair random walk with two independent sequences of iid random variables, one of the sequences taking on ± 1 with probabilities p and $1-p$ respectively and used to perturb at maxima exactly as in Section 3, and the other taking on ± 1 with probabilities q and $1-q$ and used to perturb at minima. We have

Theorem 4.8. *The analog for all pq walks of Theorem 3.5 holds.*

Concluding Remarks.

We conclude with a technical comment which elaborates on a remark in the introduction. One way to attempt to prove Theorem 4.1 is to try to prove that almost every Brownian path belongs to the class of those functions f which have a unique solution g of the equations $(3.1)_\theta$. If this can be established, short work can be made of the proof of Theorem 1.2. Furthermore, all the proofs of the already known, restricted parameter, cases of Theorems 1.2 and Theorem 4.1 prove such inequalities-indeed, all but the Perman-Werner (1996) results show that every continuous function is in this class. We do not answer this question here, but it is answered, affirmatively, in Chaumont-Doney (1998).

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