SELECTING THE BEST WEIBULL POPULATION BASED ON TYPE-I CENSORED DATA: A BAYESIAN APPROACH

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Abstract

This paper deals with the problem of selecting the "best" Weibull population in terms of the largest value of the scale parameters, or the largest value of the means, or the largest value of the variance. A Bayes rule based on type-I censored data is derived for selecting the "best" Weibull population. A monotone property of this rule is discussed. A selection rule based on an earlier termination is also proposed and investigated. Both our results and approaches are similar to those of Gupta and Liang (1993) for selecting the best exponential population.

Selecting The Best Weibull Population Based On Type-I Censored Data: A Bayesian Approach

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1 Introduction

Let π_1, π_2,...,π_k denote $k(k \geq 2)$ independent Weibull populations. Suppose π_i have density function

$$h(x|\theta_i) = \frac{\lambda_i x^{\lambda_i - 1}}{\theta_i^{\lambda_i}} \exp\left(-\left(\frac{x}{\theta_i}\right)^{\lambda_i}\right), i = 1, 2, ..., k, \tag{1.1}$$

where the values of the form parameters λ_i, $i = 1, 2, ..., k$ are supposed to be positive and known, and the scale parameters θ_i, $i = 1, 2, ..., k$ are positive but unknown. Let $m_i = m_i(\theta_i)$, $i = 1, 2, ..., k$, be increasing functions of θ_i, $i = 1, 2, ..., k$. In what follows, it is assumed that the population associated with the largest value $m_{[k]} \equiv \max\{m_i\}$ is the best population. It is clear that the problem of selecting the Weibull population with the largest scale parameter, the largest mean and the largest variance can be studied by taking $m_i = \theta_i$, $m_i = \theta_i\Gamma(1+\lambda_i^{-1})$ and $m_i = \theta_i^2[\Gamma(1 + 2\lambda_i^{-1}) - \Gamma^2(1 + \lambda_i^{-1})]$, respectively.

In an application situation of industrial life-testing experiment, m items from each of the k population $\pi_1, ..., \pi_k$ are independently put on test at the outset and are not replaced on failure. Due to the time restriction, the experiment terminates at a prespecified time T. The failure time of an item is observable only if it fails before time T. Otherwise the item is said to be censored at time T. This type of time censoring is known as type-I censoring.

In this paper, we give a Bayes selection rule based on type-I censored data to select the best weibull population. A monotone property of this rule is discussed and an early selection rule is studied. Both our results and approachs are similar to that of Gupta and Liang (1987).

The problem of selecting the largest scale parameter $\theta_{[k]}$ for exponential population was studied by Sobel (1956). Gupta (1963) studied some selection rules for gamma population via subset selection approach. His selection rules can be applied for exponential population case. Huang & Huang (1980) and Berger & Kim (1985) studied this selection problem for Type-II censored data. Gupta & Liang (1987) derived a Bayes selection rule to select the largest scale parameters for exponential population based on type-I censored data. They proved a monotone property for this Bayes selection rule and gave an early selection rule.

2 A Bayes Selection Rule

Let X_{ij}, $1 \leq j \leq m$, denote the failure times of the m items taken from population π_i. According to the type-I censoring scheme, we only observe $\min(X_{ij}, T)$ and $\delta_{ij} = I[X_{ij} < T]$ which is the indicator of the event $[X_{ij} < T]$. So, $N_i = \sum_{j=1}^{m} \delta_{ij}$ is the number of uncensored failure times up to time T.

Let $Y_{i1} \leq Y_{i2} \leq ... \leq Y_{iN_i}$ be the ordered values of the N_i observable failure times. Then $(Y_{i1}, Y_{i2}, ..., Y_{iN_i}, N_i)$ has a joint probability density

$$(2.1) \qquad f_i(\vec{y}_i, n|\theta_i) = \frac{m!}{(m-n)!} \left(\frac{\lambda_i}{\theta_i^{\lambda_i}}\right)^n \prod_{j=1}^{n} y_{ij}^{\lambda_i - 1} \exp\left(-\frac{1}{\theta_i^{\lambda_i}} y_i\right),$$

where $\vec{y}_i = (y_{i1}, y_{i2}, ..., y_{in})$, $0 \leq n \leq m$, $0 < y_{i1} \leq y_{i2} \leq ... \leq y_{in} < T$ and

$$(2.2) \qquad y_i = \sum_{j=1}^{n} y_{ij}^{\lambda_i} + (m-n)T^{\lambda_i}.$$

Let $\mathcal{N}$ be the sample space generated by $\vec{N} = (N_1, N_2, ..., N_k)$ and conditional on $\vec{N} = \vec{n} = (n_1, n_2, ..., n_k)$, let $\mathcal{Y}_{\vec{n}}$ be the sample space generated by $\vec{Y} = (\vec{y}_1, ..., \vec{y}_k)$. Thus, for $\vec{y} = (\vec{y}_1, ..., \vec{y}_k) \in \mathcal{Y}_{\vec{n}}$, $(m - n_i)T^{\lambda_i} \leq y_i \leq mT^{\lambda_i}$, $1 \leq i \leq k$.

Let $\vec{\theta} = (\theta_1, ..., \theta_k)$ and $\Omega = \{\vec{\theta} | \theta_i > 0, 1 \leq i \leq k\}$ the parameter space. Let $\mathcal{A}$ be the action space. Action i corresponds to the selection of population π_i as the best population. For a given $\vec{\theta} \in \Omega$ and an action i, the associated loss function is defined by

$$(2.3) \qquad L^*(\vec{m}, i) = L(m_{[k]} - m_i)$$

where $L(x)$ is a nonnegative and nondecreasing function of x, $x \geq 0$, such that $L(0) = 0$.

Let $g(\vec{\theta}) = \prod_{j=1}^{k} g_j(\theta_j)$ be the prior density over the parameter space Ω. It is assumed that $\int L(m_{[k]}) g(\vec{\theta}) d\vec{\theta} < \infty$.

A selection rule $\vec{\delta} = (\delta_1, \delta_2, ..., \delta_k)$ is defined to be a measurable mapping from the sample space $(\mathcal{N}, (\mathcal{Y}_{\vec{n}})_{\vec{n} \in \mathcal{N}})$ to $[0,1]^k$ such that $0 \leq \delta_i(\vec{n}, \vec{y})) \leq 1$ and $\sum_{j=1}^{k} \delta_i(\vec{n}, \vec{y}) = 1$ for all $\vec{y} \in \mathcal{Y}_{\vec{n}}, \vec{n} \in \mathcal{N}$. The value of $\delta_i(\vec{n}, \vec{y})$ is the probability of selecting population π_i as the best population based on the observation $(\vec{n}, \vec{y})$.

Let $R(\vec{\delta}, g)$ denote the Bayes risk associated with the selection rule $\vec{\delta}$. Then by Fubini's theorem we have

$$(2.4) \qquad R(\vec{\delta}, g) = \sum_{\vec{n} \in \mathcal{N}} \int_{\mathcal{Y}_{\vec{n}}} \sum_{i=1}^{k} \delta_i(\vec{n}, \vec{y}) \int_{\Omega} L(m_{[k]} - m_i) f(\vec{n}, \vec{y} | \vec{\theta}) g(\vec{\theta}) d\vec{\theta} d\vec{y},$$

where $f(\vec{n}, \vec{y} | \vec{\theta}) = \prod_{i=1}^{k} f_i(\vec{y}_i, n_i | \theta_i)$. Let

$$f(\vec{n}, \vec{y}) = \int_{\Omega} f(\vec{n}, \vec{y} | \vec{\theta}) g(\vec{\theta}) d\vec{\theta}$$

and

$$g(\vec{\theta} | \vec{n}, \vec{y}) = \frac{f(\vec{n}, \vec{y} | \vec{\theta}) g(\vec{\theta})}{f(\vec{n}, \vec{\theta})}.$$

Then, we have

$$R(\vec{\delta}, g) = \sum_{\vec{n}\in\mathcal{N}} \int_{\mathcal{Y}_{\vec{n}}} \sum_{i=1}^{k} \delta_i(\vec{n}, \vec{y}) \int_{\Omega} L(m_{[k]} - m_i) g(\vec{\theta}|\vec{n}, \vec{y}) f(\vec{n}, \vec{y}) d\vec{\theta} d\vec{y}$$
$$= \sum_{\vec{n}\in\mathcal{N}} \int_{\mathcal{Y}_{\vec{n}}} \sum_{i=1}^{k} \delta_i(\vec{n}, \vec{y}) \Delta_i(\vec{n}, \vec{y}) f(\vec{n}, \vec{y}) d\vec{y}$$

with

$$\Delta_i(\vec{n}, \vec{y}) = \int_{\Omega} L(m_{[k]} - m_i) g(\vec{\theta}|\vec{n}, \vec{y}) d\vec{\theta}. \tag{2.5}$$

Let

$$A(\vec{n}, \vec{y}) = \left\{ i | \Delta_i(\vec{n}, \vec{y}) = \max_{1\le j\le k} \Delta_j(\vec{n}, \vec{y}) \right\}. \tag{2.6}$$

Then, an uniformly randomized Bayes rule is $\vec{\delta}_G = (\delta_{G_1}, ..., \delta_{G_k})$, where

$$\vec{\delta}_G(\vec{n}, \vec{y}) = \begin{cases} |A(\vec{n}, \vec{y})|^{-1} & \text{if } i \in A(\vec{n}, \vec{y}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

3 A Monotonicity Property of $\vec{\delta}_G$

For the problem of selecting the best exponential population, Gupta and Liang (1987) proved the monotonicity property of $\vec{\delta}_G$. It what follows we prove the same result for Weibull population. Firstly, we have to show that $\vec{\delta}_G$ is a function of $(\vec{n}, y_1, ..., y_k)$ only.

Define

$$f_i(\vec{y}_i, n_i) = \int_0^\infty f_i(\vec{y}_i, n_i|\theta_i) g_i(\theta_i) d\theta_i.$$

$f_i(\vec{y}_i, n_i)$ is positive and from which we get

$$g_i(\theta_i|\vec{y}_i, n_i) \equiv \frac{f_i(\vec{y}_i, n_i|\theta_i) g_i(\theta_i)}{f_i(\vec{y}_i, n_i)}$$
$$= \frac{\theta_i^{-\lambda_i n_i} \exp\left\{-\theta_i^{-\lambda_i} y_i\right\} g_i(\theta_i)}{\int_0^\infty \theta_i^{-\lambda_i n_i} \exp\left\{-\theta_i^{-\lambda_i} y_i\right\} g_i(\theta_i) d\theta_i}.$$

Hence $g_i(\theta_i|\vec{y}_i, n_i)$ is a function of θ_i, n_i and y_i only. We write $g_i(\theta_i|y_i, n_i)$ for $g_i(\theta_i|\vec{y}_i, n_i)$ in what follows. Note that

$$g(\vec{\theta}|\vec{n}, \vec{y}) = \prod_{i=1}^{k} g_i(\theta_i|y_i, n_i).$$

So, we can write $g(\vec{\theta}|\vec{n}, \tilde{y})$ for $g(\vec{\theta}|\vec{n}, \vec{y})$, here $\tilde{y} = (y_1, ..., y_k)$. From (2.5) it is seen that $\Delta_i(\vec{n}, \vec{y})$, and hence $\vec{\delta}_G$ is a function of $\vec{n}$ and $\tilde{y}$. We use $\Delta_i(\vec{n}, \tilde{y})$ and $\vec{\delta}_G(\vec{n}, \tilde{y})$ for $\Delta_i(\vec{n}, \vec{y})$ and $\vec{\delta}_G(\vec{n}, \vec{y})$ in what follows.

For each fixed (y_i, n_i), $g_i(\theta_i|y_i, n_i)$ is a probability density in θ_i such that $g_i(\theta_i|y_i, n_i) = 0$ if and only if $g_i(\theta_i) = 0$. Hence we can define a likelihood ratio by

$$r(\theta|y_i^*, n_i^*, y_i, n_i) = \begin{cases} \dfrac{g_i(\theta|y_i^*, n_i^*)}{g_i(\theta|y_i, n_i)} & \text{if } g_i(\theta|y_i, n_i) > 0 \\ 0 & \text{if } g_i(\theta|y_i, n_i) = 0. \end{cases} \tag{3.1}$$

Simple calculation shows that for some nonnegative function W

$$r(\theta|y_i^*, n_i^*, y_i, n_i) = W(y_i^*, n_i^*, y_i, n_i) \left(\frac{\lambda_i}{\theta^{\lambda_i}}\right)^{n_i^* - n_i} \exp\left(-\frac{1}{\theta^{\lambda_i}}(y_i^* - y_i)\right).$$

From which we get the following lemma.

Lemma 3.1 *Let $r(\theta|y_i^*, n_i^*, y_i, n_i)$ be defined by (3.1). Then,*

(a) as $y_i^* > y_i$, then $r(\theta|y_i^*, n_i, y_i, n_i)$ is nondecreasing in θ, and

(b) as $n_i^* > n_i$, then $r(\theta|y_i, n_i^*, y_i, n_i)$ is nonincreasing in θ.

The following lemma is used in the proof of lemma 3.3.

Lemma 3.2 *If $g(\theta)$ and $h(\theta)$ are probability density functions such that $g(\theta)/h(\theta)$ being nondecreasing, then for any nonnegative and nonincreasing function $f(\theta)$*

$$\int f(\theta)g(\theta)d\theta \leq \int f(\theta)h(\theta)d\theta.$$

Lemma 3.3 *Let $\Delta_i(\vec{n}, \tilde{y})$ be defined in (2.5). For each $i (1 \leq i \leq k)$, $\Delta_i(\vec{n}, \tilde{y})$ is nonincreasing in y_i and also in n_j, $j \neq i$ when all the other variables are kept fixed, and nondecreasing in n_i and also in y_j, $j \neq i$, when all other variables are kept fixed.*

Proof. We prove that $\Delta_i(\vec{n}, \tilde{y})$ is nondecreasing in n_i. The others can be proved in a similar way.

Define

$$\begin{aligned}\vec{\theta^i} =&(\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_k)\\ \Omega^i =&\{\vec{\theta}_i | \theta_j > 0, j = 1, 2, ..., k, j \neq i\}\\ \tilde{y} =&(y_1, ..., y_k) \text{ and } \tilde{y}^* = (y_1, ..., y_{i-1}, y_i^*, y_{i+1}, ..., y_k).\end{aligned}$$

We have

$$\Delta_i(\vec{n}, \tilde{y}) = \int_{\Omega^i} \left(\int_0^\infty L(m_{[k]} - m_i) g_i(\theta_i | y_i, n_i) d\theta_i \right) \prod_{j \neq i} g_j(\theta_j | y_j, n_j) d\vec{\theta^i}.$$

Since for each fixed $\vec{\theta^i}$ and $\vec{n}$, $L(m_{[k]} - m_i)$ is nonincreasing in θ_i and by lemma 3.1 the likelihood ratio $r(\theta_i | y_i^*, n_i, y_i, n_i)$ is nondecreasing in θ_i, for $y_i^* > y_i$. So, Lemma 3.2 implies that

$$\int_0^\infty L(m_{[k]} - m_i) g_i(\theta_i | y_i, n_i) d\theta_i \geq (\int_0^\infty L(m_{[k]} - m_i) g_i(\theta_i | y_i^*, n_i) d\theta_i$$

and hence $\Delta_i(\vec{n}, \tilde{y}) \geq \Delta_i(\vec{n}, \tilde{y}^*)$.

From Lemma 3.3 we can get the monotone property for $\vec{\delta}_G$.

Theorem 3.4 *For each $i = 1, 2, ..., k$, $\vec{\delta}_G(\vec{n}, \tilde{y})$ is nondecreasing in y_i and also in n_j, $j \neq i$ when all other variables are kept fixed, and nonincreasing in n_i and also in y_j, $j \neq i$ when all other variables are kept fixed.*

Proof. For $y_i^* > y_i$ and $j \neq i$, by Lemma 3.3, we have $\Delta_i(\vec{n}, \tilde{y^*}) \leq \Delta_i(\vec{n}, \tilde{y})$ and $\Delta_j(\vec{n}, \tilde{y^*}) \geq \Delta_j(\vec{n}, \tilde{y})$, for $j \neq i$. Now, $i \in A(\vec{n}, \tilde{y^*})$ implies that

$$\Delta_i(\vec{n}, \tilde{y}) \geq \Delta_i(\vec{n}, \tilde{y^*}) = \max_j \Delta_j(\vec{n}, \tilde{y^*}) \geq \Delta_{j \neq i} \Delta_j(\vec{n}, \tilde{y})$$

Hence, $\Delta_i(\vec{n}, \tilde{y}) \geq \max_j \Delta_j(\vec{n}, \tilde{y})$ and it follows that

$$A(\vec{n}, \tilde{y^*}) \subset A(\vec{n}, \tilde{y})$$

. So, $\delta_{G_i}(\vec{n}, \tilde{y^*}) \geq \delta_{G_i}(\vec{n}, \tilde{y})$. Proofs for the rest are similar.

4 An Early Selection Rule

In this section, we consider the following linear loss function: $L(m_{[k]} - m_i) = m_{[k]} - m_i$, the difference between the parameters of the best and the selected populations. Thus the set $A(\vec{n}, \tilde{y})$ given by (2.6) turns out to be:

$$A(\vec{n}, \tilde{y}) = \left\{ i | \int m_i g_i(\theta_i | y_i, n_i)\, d\theta_i = \min_{1 \le j \le k} \int m_j g_j(\theta_j | y_j, n_j)\, d\theta_j \right\}. \tag{4.1}$$

Similar to the proof of Lemma 3.3, we can proof the following result.

Lemma 4.1 *For each fixed i, $\int m_i g_i(\theta_i | y_i, n_i)\, d\theta_i$ is increasing in y_i and decreasing in n_i.*

We will use Lemma 4.1 to derive an earlier selection rule.

At time t, $0 < t < T$, let $N_i(t)$ denote the number of failures from population π_i. That is , $N_i(t)$=number of $\{X_{ij} | 1 \le j \le t, X_{ij} < t\}$. Also, we use $Y_{i1} \le ... \le Y_{iN_i(t)}$ for the $N_i(t)$ failure times up to time t. At time t, exclude population π_i as a nonbest population if there exists some population π_h such that

$$N_h(t) < m \text{ and } \int m_h g_h(\theta_h | y_h(t), m)\, d\theta_h \ge \int m_i g_i(\theta_i | y_i(t, T), N_i(t))\, d\theta_i \tag{4.2a}$$

or

$$N_h(t) = m \text{ and } \int m_h g_h(\theta_h | y_h(t), m)\, d\theta_h > \int m_i g_i(\theta_i | y_i(t, T), N_i(t))\, d\theta_i \tag{4.2b}$$

where

$$y_h(t) = \sum_{j=1}^{N_h(t)} y_{hj}^{\lambda_h} + (m - N_h(t)) t^{\lambda_h}, \quad y_i(t, T) = \sum_{j=1}^{N_i(t)} y_{ij}^{\lambda_i} + (m - N_i(t)) T^{\lambda_i}. \tag{4.3}$$

Let $S(t)$ denote the indices of the contending populations at time t. That is,

$$\begin{aligned} S(t) = \{ i | N_h(t) < (=) m \text{ and } & \int m_i g_i(\theta_i | y_i(t, T), N_i(t))\, d\theta_i \\ & > (\ge) \int m_h g_h(\theta_h | y_h(t), m)\, d\theta_h, h \ne i \}. \end{aligned} \tag{4.4}$$

The following lemma shows that for any t, $0 < t < T$, $S(t)$ is not empty.

Lemma 4.2 *For any $0 < t < T$, the set $S(t)$ defined by (4.4) is not empty.*

Proof. Let

$$S'(t) = \left\{ i \Big| \int m_i g_i(\theta_i | y_i(t,T), N_i(t))\, d\theta_i = \max_{1\le h\le k} \int m_h g_h(\theta_h | y_h(t,T), N_h(t))\, d\theta_h \right\}.$$

Then $S'(t)$ is not empty. We prove that $S'(t)$ is a subset of $S(t)$. For $i \in S'(t)$ and any $h \neq i$, if $N_h(t) < m$, we have $y_h(t) < y_h(t,T)$. Hence, by Lemma 4.1

$$\begin{aligned}\int m_h g_h(\theta_h | y_h(t), m)\, d\theta_h &< \int m_h g_h(\theta_h | y_h(t,T), m)\, d\theta_h \\ &< \int m_h g_h(\theta_h | y_h(t,T), N_h(t))\, d\theta_h \\ &\le \int m_i g_i(\theta_i | y_i(t,T), N_i(t))\, d\theta_i.\end{aligned}$$

If $N_h = m$, we have $y_h(t) = y_h(t,T)$ and it follows that

$$\begin{aligned}\int m_h g_h(\theta_h | y_h(t), m)\, d\theta_h &= \int m_h g_h(\theta_h | y_h(t,T), m)\, d\theta_h \\ &\le \int m_i g_i(\theta_i | y_i(t,T), N_i(t))\, d\theta_i.\end{aligned}$$

Hence, $i \in S(t)$.

Now, the life-testing experiment terminates as soon as there is a time t, $0 < t < T$, such that $|S(t)| = 1$ and in this case, we select the population with its index in $S(t)$ as the best population. Otherwise, the experiment goes until time T. At time T, Let

$$S(T) = \left\{ i \Big| \int m_i g_i(\theta_i | y_i, N_i)\, d\theta_i = \max_{j\in S(T^-)} \int m_j g_j(\theta_j | y_j, N_j)\, d\theta_j \right\}, \tag{4.5}$$

where $S(T^-)$, which is not empty by Lemma 4.2, denotes the set of the indices of those populations having not been eliminated before time T. Then, an uniformly random selection is made over $S(T)$.

From the above description, we see that the early selection rule can make selection earlier than the termination time T. Denote this early selection rule by $\vec{\delta}^*_G = (\delta^*_{G_1}, ..., \delta^*_{G_k})$. Note $\delta^*_{G_i}$, $1 \le i \le k$ are functions of the data during the time interval $(0, T]$.

Theorem 4.3 *Under the loss function $L(\theta) = \theta$, $\delta^*_{G_i} = \delta_{G_i}(\vec{n}, \tilde{y})$ for all $1 \le i \le k$, $\vec{y} \in \mathcal{Y}_{\vec{n}}$ and $\vec{n} \in \mathcal{N}$, where $\delta_{G_i}(\vec{n}, \tilde{y})$ is defined by (4.1) and (2.7).*

Let $B = \{0 < y \le T \,|\, |S(t)| = 1\}$ and

$$t_1 = \begin{cases} \inf B & \text{if } B \ne \phi, \\ T & \text{if } B = \phi, \end{cases} \tag{4.6}$$

where ϕ denotes an empty set. Note that if $B \ne \phi$, then $B = [t_1, T]$. By an uniformly randomized selection over the set $S(T)$ when $t_1 = T$, Theorem 4.3 is equivalent to the following theorem.

Theorem 4.4 *$S(t_1) = A(\vec{n}, \tilde{y})$ for all $(\vec{n}, \tilde{y})$.*

Proof. Case 1. As $t_1 < T$, then $|S(t_1)| = 1$. Without loss of generality, we let π_k be the population with index in the set $S(t_1)$. Since $A(\vec{n}, \tilde{y})$ contains at least one element, it suffices to show that $i \notin A(\vec{n}, \tilde{y})$ for all $i \ne k$. Since $i \notin S(t_1)$, it means that population π_i is eliminated at some time, say t_0, by some population, say π_h. That is, at time t_0 either

$$N_h(t_0) < m \text{ and } \int m_h g_h(\theta_h | y_h(t_0), m)\, d\theta_h \ge \int m_i g_i(\theta_i | y_i(t_0, T), N_i(t_0))\, d\theta_i \tag{4.7a}$$

or

$$N_h(t_0) = m \text{ and } \int m_h g_h(\theta_h | y_h(t_0), m)\, d\theta_h > \int m_i g_i(\theta_i | y_i(t_0, T), N_i(t_0))\, d\theta_i. \tag{4.7b}$$

Now, note that $N_i(t)$ is an nondecreasing function of $t \in (0, T]$ and $N_i(t) \le m$. Also, by (4.3), $y_h(t)$ is nondecreasing in t and $Y_i(t, T)$ is nonincreasing in t. Especially we have

$$\begin{aligned} &N_h = N_h(T) \le m,\ N_i(t) \le N_i(T) = N_i,\ y_i(t_0, T) \ge y_i(T, T) \equiv y_i \\ &y_h \equiv y_h(T) > (=) y_h(t_0) \text{ if } N_h(t_0) < (=) m. \end{aligned}$$

Thus, when $N_h(t_0) = m$, then $N_h = N_h(T) = m$. Then by Lemma 4.1 and (4.7b),

$$\int m_h g_h(\theta_h|y_h, N_h)\,d\theta_h = \int m_h g_h(\theta_h|y_h(t_0), m)\,d\theta_h$$
$$> \int m_i g_i(\theta_i|y_i(t_0,T), N_i(t_0))\,d\theta_i$$
$$\geq m_i g_i(\theta_i|y_i, N_i)\,d\theta_i. \tag{4.8}$$

When $N_h(t_0) < m$, then $y_h \equiv y_h(T) > y_h(t_0)$ and $N_h = N_h(T) \leq m$. Therefore, by Lemma 4.1 and (4.7a),

$$\int m_h g_h(\theta_h|y_h, N_h)\,d\theta_h > \int m_h g_h(\theta_h|y_h(t_0), m)\,d\theta_h$$
$$\geq \int m_i g_i(\theta_i|y_i(t_0,T), N_i(t_0))\,d\theta_i$$
$$\geq m_i g_i(\theta_i|y_i, N_i)\,d\theta_i. \tag{4.9}$$

In either situations, we see that $i \notin A(\vec{n}, \tilde{y})$.

Case 2. As $t_1 = T$, we need to prove that

(a) $i \notin S(T)$ implies $i \notin A(\vec{n}, \tilde{y})$, and

(b) $i \in S(T)$ implies $i \in A(\vec{n}, \tilde{y})$.

We prove (a) first. Suppose $i \notin S(T)$. Then, π_i is eliminated at a time $t_0 \leq T$ by some other π_h.

If $t_0 < T$, this reduces to the situation discussed in Case 1.

If $t_0 = T$, then by (4.5), $\int \theta_h g_h(\theta_h|y_h, N_h)\,d\theta_h > \int \theta_i g_i(\theta_i|y_i, N_i)\,d\theta_i$. Therefore, by the definition of $A(\vec{n}, \tilde{y})$, $i \notin A(\vec{n}, \tilde{y})$.

Note that the statement in part (a) is equivalent to that

$$A(\vec{n}, \tilde{y}) \subset S(T). \tag{4.10}$$

Now, part (b) is a direct consequence of (4.5) and (4.10). Therefore, we complete the proof.

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