ON THE EQUATION $Y_t = B_t + \alpha \sup_{s \le t} Y_s + \beta \inf_{s \le t} Y_s$

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Technical Report #95-40

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September 1995

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Summary

Let α and β be real numbers and $f \in C_0[0, \infty)$. We study the existence and uniqueness of solutions g of the equation $g(t) = f(t) + \alpha \sup_{0 \le s \le t} g(s) + \beta \inf_{0 \le s \le t} g(s)$. Let $\rho = \alpha\beta/(1-\alpha)(1-\beta)$. Carmona, Petit, and Yor have shown that if $\alpha \ge 1$ or $\beta \ge 1$, there are f with no solutions, and if $\alpha < 1$, $\beta < 1$, and $|\rho| < 1$, every f has a unique solution. We show that if $\alpha < 1$ and $\beta < 1$, a solution exists for each f, but that it is necessarily unique if and only if $|\rho| \le 1$. We show that if $|\rho| < 1$, the processes which result from solving the equation above for Brownian paths are the weak limit of random walks perturbed at their extrema.

1 Partly supported by NSF
MSC 1991 60F05, 60J15, 60J65, 82C41
Keywords: Reinforced random walk, perturbed Brownian motion, weak convergence.

1. Introduction

If f is a real valued function on $[0,\infty)$, we put $f^*(t) = \sup_{0\le s\le t} f(s)$, and $f^\#(t) = \inf_{0\le s\le t} f(s)$, and also we use $*$ and $\#$ to denote maxima and minima of sequences. We study the existence and uniqueness of solutions g of the equation

$$(1.1) \qquad g(t) = f(t) + \alpha g^*(t) + \beta g^\#(t), \quad t \ge 0.$$

Here α and β are real numbers, and f is a continuous function vanishing at 0, an assumption always in force whenever (1.1) is discussed, without further mention. This equation was first studied by Le Gall (1986), and more recently, in a paper that will hereafter be referred to as CPY, by Carmona, Petit, and Yor (1993). Let $\rho = \alpha\beta/(1-\alpha)(1-\beta)$. It is shown in CPY that if either $\alpha \ge 1$ or $\beta \ge 1$ there are f for which (1.1) has no solution, while if $\alpha < 1$, $\beta < 1$, and $|\rho| < 1$ there is a unique solution for every f. In Section 2 we prove results which, when combined with those of CPY, yield the following theorem.

Theorem 1.1. *If $\alpha < 1$, $\beta < 1$, and $|\rho| \le 1$, (1.1) has a unique solution for each f. If $\alpha < 1$, $\beta < 1$ and $|p| > 1$, there is at least one solution of (1.1) for each f, and there are functions $f = f_{\alpha,\beta}$ with more than one solution.*

CPY shows that if $\alpha < 1$, $\beta < 1$, and $|\rho| < 1$, then the solution of (1.1) for Brownian paths, that is, the process $\boldsymbol{Y}^{\alpha,\beta} = \boldsymbol{Y} = Y_t$, $t \ge 0$, defined by

$$(1.2) \qquad Y_t = B_t + \alpha Y_t^* + \beta Y^\#(t),$$

where $\boldsymbol{B} = B_t$, $t \ge 0$, is Brownian motion started at 0, is adapted to the filtration of $\boldsymbol{B}$. It is easy to extend this result to the cases $\alpha < 1$, $\beta < 1$, $|\rho| = 1$, using our proof of the existence and uniqueness of solutions of (1.1) for these α, β.

In Section 3 we show that if $\alpha < 1$, $\beta < 1$, and $|\rho| < 1$, the solution $\boldsymbol{Y}$ of (1.2) can be identified as the weak limit of a discrete process. If $\boldsymbol{Z} = Z_0, Z_1, \ldots$ is a discrete time stochastic process, we identify it with the continuous time process on $[0,\infty)$ which results from linearly interpolating: $Z_t = Z_n + (t-n)\,[Z_{n+1} - Z_n]$ if $n \le t \le n+1$.

Theorem 1.2. *Define the integer valued stochastic process* $\boldsymbol{X}_{\alpha,\beta} = \boldsymbol{X} = X_0, X_1, X_2, \ldots$ *by* $X_0 = 0$, $P(X_{n+1} = X_n + 1 | X_i,\ i \le n) = 1 - P(X_{n+1} = X_n - 1 | X_i, i \le n) = \frac{1}{2}$ *if* $n = 0$ *or* $X_n^{\#} < X_n < X_n^*$; $= \frac{1}{2-\alpha}$ *if* $n > 0$ *and* $X_n = X_n^*$; $= 1 - \frac{1}{2-\beta}$ *if* $n > 0$ *and* $X_n = X_n^{\#}$. *Then if* $|\rho| < 1$, $\alpha < 1$ *and* $\beta < 1$ *the processes* $\frac{1}{\sqrt{n}} \boldsymbol{X}_{nt}$, $t \ge 0$, *converge weakly to* $\boldsymbol{Y}^{\alpha,\beta}$.

The $\alpha = 0$ (and $\beta = 0$) cases of Theorem 1.2 have been proved by Werner (1994). It seems very likely that the analog of Theorem 1.2 for $\alpha < 1$, $\beta < 1$, and $\rho = 1$ holds, but our proof, which uses a strong stability result for solutions of (1.1) when $\alpha < 1$, $\beta < 1$, and $|\rho| < 1$, derived from a theorem of CPY, does not extend to this case. It is also likely that, for all $\alpha < 1$ and $\beta < 1$, the processes $\boldsymbol{X}_{\alpha,\beta}$ converge weakly, but not clear that the limit process can be constructed a.s. path by path by solving (1.2). Several people have suggested that the excursion theory of Perman (1995), for the solutions of (1.2) when $\alpha = 0$, may provide an avenue for an extension of Theorem 1.2 to all the cases $\alpha < 1$, $\beta < 1$.

If $\alpha = \beta < 0$, the processes $\boldsymbol{X}_{\alpha,\alpha}$ can be realized as the simplest of the reinforced random walks: If we assign a weight of 1 to each "bond" $(i, i+1)$ which has not been crossed by $\boldsymbol{X}_{\alpha,\alpha}$, and weight $1 - \alpha$ to bonds which have been crossed, then $\boldsymbol{X}_{\alpha,\alpha}$ may be described as jumping up or down with probabilities proportional to the weights of the connecting bonds. See Davis (1990) for more details. Recent papers at least partly concerned with reinforced random walk include Diaconnis (1988), Pemantle (1988 and 1992), Davis (1989), Sellke (1994a and 1994b), Toth (1994 and 1995), and Othmer and Stevens (1995). Bolthausen (1994) proves weak convergence for a different kind of non-Markovian walk. Harrison and Shepp (1981) prove weak convergence of the (Markovian) walk which behaves like fair random walk except at zero, where it goes up with probability p.

Our study of the processes $\boldsymbol{X}_{\alpha,\beta}$ and $\boldsymbol{Y}^{\alpha,\beta}$ was motivated by Darryl Nester's paper Nester (1993), where stopping times for the processes $\boldsymbol{X}_{\alpha,\beta}$, when $\alpha = \beta$, were studied. Many of Nester's results translate immediately to results about the limiting processes $\boldsymbol{Y}^{\alpha,\alpha}$, of course only in the $\alpha < \frac{1}{2}$ case for now, since Theorem 1.2 does not cover other α. Nester's formulas, in common with the formulas in CPY, are very pretty, and often involve beta densities. One example: Nester's results show the probability $\boldsymbol{Y}^{\alpha,\alpha}$ equals a

before it equals $-b$, a, $b > 0$, is $\int_0^{b/(a+b)} t^{-\alpha}(1-t)^{-\alpha}dt / \int_0^1 t^{-\alpha}(1-t)^{-\alpha}dt$.

2. Proof of Theorem 1.2.

From now on, α and β will both be assumed to be less than 1, often without mention. If g and f satisfy (1.1), that is, g solves (1.1) for f, then our assumptions that f is continuous and vanishes at zero are easily seen to imply that g has these properties. Positive absolute constants which depend on α and β are usually denoted by c and C; subscripts will be used to denote dependence on various quantities. We put $a^+ = \max(a,0)$, $a \vee b = \max(a,b)$, and $a \wedge b = \min(a,b)$. If h is a function on $[a,b]$ we let $h^*[a,b] = \max_{a \le x \le b} h(x)$, and $h^\#[a,b] = \min_{a \le x \le b} h(x)$. $C_0[0,\infty)$ is the continuous functions on $[0,\infty)$ which vanish at 0.

Lemma 2.1. *Let g solve (1.1) for f and let $0 \le a < b < \infty$. Then*

$$g^*[a,b] - g^\#[a,b] \le f^*[a,b] - f^\#[a,b]. \tag{2.1}$$

Proof: Suppose first that g achieves a maximum in $[a,b]$ before it attains a minimum, that is, there exist $a \le s \le r \le b$ such that $g(s) = g^*[a,b]$ and $g(r) = g^\#[a,b]$ and $g^\#[a,b] < g(y) < g^*[a,b]$ if $s < y < r$. This implies $g^\#(s) = g^\#(r)$ and $g^*(s) = g^*(r)$, so subtracting the $t = s$ version of (1.1) from the $t = r$ version gives $g(r) - g(s) = f(r) - f(s)$. Since $|g(r) - g(s)| = g^*[a,b] - g^\#[a,b]$ and $|f(r) - f(s)| \le f^*[a,b] - f^\#[a,b]$, this gives (2.1) in this case. The proof when g achieves a minimum before a maximum is similar. □

Corollary 2.2. *Let $f \in C_0[0,\infty)$. Suppose that f_n, $n \ge 1$, converge uniformly to f on compact subintervals of $[0,\infty)$, and that g_n solves (1.1) for f_n. Then there is a subsequence $n', n \ge 1$, of integers such that $g_{n'}, n \ge 1$, converges uniformly on compact subintervals of $[0,\infty)$. The limit of $g_{n'}$ solves (1.1) for f.*

Proof. Lemma 2.1, and the fact that $f_n, n \ge 1$, is equicontinuous and uniformly bounded on compact subintervals of $[0,\infty)$, imply that g_n, $n \ge 1$, is also. Thus the Arzela-Ascoli theorem, and a diagonalization argument, give the desired sequence $g_{n'}$, $n \ge 1$. It is immediate that the limit of this subsequence solves (1.1) for f. □

Lemma 2.3. *Suppose either of the following two conditions hold.*

a) There are $\delta > 0$, $c \neq 0$, such that $f(x) = cx$, $0 \le x \le \delta$.

b) There are $\delta > 0$, $c \neq 0$, such that $f(x) = 0$, $0 \le x \le \delta/2$, and $f(x) = c(x - (\delta/2))$, $\delta/2 \le x \in \delta$.

Then (1.1) has a solution for f.

Proof: If $g \in C_0[0,\infty)$ solves (1.1) for f then both the following hold.

(2.2) If $[a,b] \subset [0,\infty)$ and if $g(x) \ge g^{\#}(a)$, $x \in [a,b]$, and if $s = \inf\{t\colon g^*(a) = g(t)\}$, then $g(t) - g(a) = f(t) - f(a)$, $a \le t \le b$, if $s \ge b$, while if $s < b$, $g(t) - g(a) = f(t) - f(a)$, $a \le t \le s$, and $g(t) - g(s) = f(t) - f(s) + \alpha(1-\alpha)^{-1} \max_{s \le x \le t}(f(x) - f(s))$, $s \le t \le b$.

(2.3) If $[a,b] \subset [0,\infty)$ and if $g(x) \le g^*(a)$, $x \in [a,b]$, and if $r = \inf\{t\colon g^{\#}(a) = g(t)\}$, then $g(t) - g(a) = f(t) - f(a)$, $a \le t \le b$, if $r \ge b$, while if $r < b$, $g(t) - g(a) = f(t) - f(a)$, $a \le t \le s$, and $g(t) - g(r) = f(t) - f(r) + \beta(1-\beta)^{-1} \min_{r \le x \le t}(f(x) - f(r))$, $r \le t \le b$.

To see (2.2), note that the only nontrivial part concerns the formula for $g(t) - g(s)$ when $s < b$. Now (1.1) gives

$$f(t) - f(s) = (g(t) - g(s)) - \alpha(g^*(t) - g^*(s)),$$

which equals $(1-\alpha)(g^*(t) - g^*(s))$ when $g(t) = g^*(t)$ and is smaller than $(1-\alpha)(g^*(t) - g^*(s))$ when $g(t) < g^*(t)$. Thus $g(t) = g^*(t)$ exactly for those t for which $f(t) = \max_{s \le x \le t} f(x)$. This verifies $g(t) - g(s) = f(t) - f(s) + \alpha(1-\alpha)^{-1} \max_{s \le x \le t}(f(x) - f(s))$, if $s \le t \le b$ and $g(t) = g^*(t)$. To verify it for other $t \in [s,b]$, let $t_0 = \sup\{x < t : \; g(x) = \max_{0 \le y \le x} g(y)\}$, and use its truth for t_0 and the fact that, by (1.1), $f(t) - f(t_0) = g(t) - g(t_0)$. The proof that (1.1) implies (2.3) is similar.

It is also true that (2.2) and (2.3) imply that g solves (1.1) for f, provided $g \in C_0[0,\infty)$. We just sketch this argument: To show (1.1) it suffices to prove (2.4) and (2.5) below.

(2.4) If $[a,b] \subset [0,\infty)$ and $g(x) \ge g^{\#}(a)$, $x \in [a,b]$, then $g(b) - g(a) = f(b) - f(a) + \alpha(g^*(b) - g^*(a))$.

(2.5) If $[a,b] \subset [0,\infty)$ and $g(x) \le g^*(a)$, $x \in [a,b]$, then $g(b) - g(a) = f(x) - f(a) + \beta(g^{\#}(b) - g^{\#}(a))$.

That (2.4) and (2.5) imply (1.1) is not difficult: Fix t, let $0 < \varepsilon < t$, and break $[\varepsilon, t]$ into disjoint intervals $[a, b]$ on which either $g(x) \geq g^{\#}(a)$ on $g(x) \leq g^*(a)$. Take the results of (2.4) and (2.5) on these intervals and add them. Then let $\varepsilon \to 0$. To show (2.2) implies (2.4), let s be as in (2.2) and observe the implication is trivial if $s \geq b$, while if $s < b$, let $\theta = \max\{t \in [s, b] : \ g^*(t) = g(t)\}$. Then $g(s) - g(a) = f(s) - f(a)$, $g(b) - g(\theta) = f(b) - f(\theta)$, and, recalling the discussion after (2.3), $g(\theta) - g(s) = g^*(b) - g^*(a) = (1-\alpha)^{-1} f(\theta) - f(s))$. Adding these three gives (2.4). The proof that (2.3) implies (2.5) is similar.

We prove part a) of Lemma 2.3 first. Suppose $c > 0$. We construct g by putting $g(x) = cx/(1-\alpha)$, $0 \leq x \leq \delta$, and then using (2.2) and (2.3) as a recipe for constructing $g(t)$ for $t > \delta$: Since $g(\delta) = g^*(\delta) > g^{\#}(\delta)$, (2.2) dictates $g(t)$, $\delta \leq t \leq y$, where $y = \inf\{x > \delta \colon g(x) = g^{\#}(x)\}$. Then (2.3) dictates $g(t)$, $y \leq t \leq z = \inf\{x > y : \ g(x) = g^*(x)\}$, and so on. The $c < 0$ case is very similar. Part b) of Lemma 2.3 is established in a similar way, first explicitly exhibiting a solution on $[0, \delta]$, which is 0 on $[0, \delta/2]$ and linear on $[\delta/2, \delta]$.□

Corollary 2.4. *There is at least one solution of (1.1) for every f.*

Proof: Suppose, first, that there is a sequence $t_n \downarrow 0$ such that $f(t_n) \neq 0$. Let $f_n(t) = tf(t_n)/t_n$, $0 \leq t \leq t_n$, and $f_n(t) = f(t)$, $t \geq t_n$. Then Lemma 2.3 guarantees that (1.1) has a solution for f_n, and Corollary 2.2 gives a solution for f. If f is not the 0 function but $f = 0$ on $[0, \delta]$ for some $\delta > 0$, let $\varepsilon = \sup\{s : \ f(t) = 0, \ 0 \leq t \leq s\}$, put $g = 0$ on $[0, \varepsilon]$, and for $t \geq \varepsilon$ mimic the argument above. And, of course, if f is the 0 function, we may take $g = f$. □

The proof of Theorem 1.1 will be completed by proving three propositions, each of which treats some of the α, β not covered by the CPY results. Recall these results settled the issue for $\alpha \geq 1$, or $\beta \geq 1$, and for $|\rho| < 1$. Our propositions consider, respectively, the sets $\{\rho = 1, \alpha < 1, \beta < 1\}$, $\{\rho = -1, \alpha < 1, \beta < 1\}$, and $\{|\rho| > 1, \alpha < 1, \beta < 1\}$.

Our methods will handle the parts of Theorem 1.1 proved in CPY, but for these cases the method of CPY gives much additional information. This will be apparent in Section 3.

Lemma 2.5. *Let g_1 and g_2 be solutions of (1.1) for f, and suppose $t > 0$ and $f^*(t) > f^{\#}(t)$. It cannot happen that both $g_1(t) = g_1^*(t)$ and $g_2(t) = g_2^{\#}(t)$.*

Proof: First note that $g_1^{\#}(t) < g_1^*(t)$, because, since f is not identically zero on $[-,t]$, neither is g_1. Thus if $g_1(t) = g_1^*(t)$ there is $0 < s < t$ such that $g_1(s) < g_1^*(t)$ and $g_1(r) \neq g_1^{\#}(r)$, $s \leq r \leq t$. Then (2.2) implies $f(t) = \max_{s \leq r \leq t} f(r) > f(s)$. Similarly, if $g_2(t) = g_2^{\#}(t)$, there is $0 < y < t$ such that $f(y) > \min_{y \leq r \leq t} f(r) = f(t)$.

Lemma 2.6. *Let $0 < p < \infty$. Let a_k $0 \leq k \leq n$ and b_k, $0 \leq k \leq n$, be sequences of numbers such that $b_{k+1} - b_k = -p(a_{k+1} - a_k)$, $n \geq 0$. Then $pa_k + b_k = pa_0 + b_0$, $0 \leq k \leq n$.*

The proof of Lemma 2.6 is immediate.

Proposition 2.7: *If $0 < \alpha < 1$ and $\beta = 1 - \alpha$, then there do not exist two different solutions of (1.1) for any f.*

Proof: Think of f as fixed. We assume that $f^*(t) - f^{\#}(t) > 0$, $t > 0$. Only minor alterations in our proof are required if this does not hold. Let g_1 and g_2 be solutions of (1.1) for f. We will prove

$$g_1(b) - g_2(b) \leq g_1(a) - g_2(a), \quad 0 < a < b. \tag{2.6}$$

Upon letting a go to zero, (2.6) gives $g_1(b) \leq g_2(b)$, and of course switching the roles of g_1 and g_2 we get $g_1(b) \geq g_2(b)$, verifying the proposition.

For $t \geq 0$, define $\Delta(t) = g_1(t) - g_2(t)$,
${}^*\Delta(t) = g_1^*(t) - g_2^*(t)$,

$$P^+(t) = [(g_2^*(t) - g_2(t)) - (g_1^*(t) - g_1(t))] = \Delta(t) - {}^*\Delta(t),$$

and

$$P^-(t) = [(g_1(t) - g_1^{\#}(t)) - (g_2(t) - g_2^{\#}(t))].$$

We say that an interval $I = [c,d] \subset [0,\infty)$ is *positive* if $g_i(t) > g_i^{\#}(c)$, $c < t < d$, $i = 1,2$, and we say that I is *negative* if $g_i(t) < g_i^*(c)$, $c < t < d$, $i = 1,2$. Equation (2.2) implies

$$g_i^*(d) - g_i^*(c) = (1-\alpha)^{-1}[f^*([c,d]) - f(c) - (g_i^*(c) - g_i(c))]^+, \quad [c,d] \text{ positive.} \tag{2.7}$$

To see this, note that $g_i^*(d) - g_i^*(c) = g_i^*(d) - g_i^*(t)$, where $t = \inf\{s \le d: \ g_i(s) = g_i^*(c)\}$. Then, recalling the argument in the paragraph after (2.3), (2.2) gives $g_i(s) = g_i^*(s)$ if and only if $f(s) = f^*[c,s]$, if $t \le s \le d$, and (2.7) follows. Equation (2.7) implies that ${}^*\Delta(d) - {}^*\Delta(c)$ lies between 0 and $(1-\alpha)^{-1}\ P^+(c)$, inclusive. Now (2.2) gives

$$(2.8) \qquad g_i(d) - g_i(c) = f(d) - f(c)) + \alpha(g_i^*(d) - g_i^*(c)), \quad i = 1,2, \quad [c,d] \text{ positive.}$$

Subtracting the $i = 2$ version of (2.8) from the $i = 1$ version gives

$$(2.9) \qquad \Delta(d) - \Delta(c) = \alpha({}^*\Delta(d) - {}^*\Delta(c)), \quad [c,d] \text{ positive,}$$

which in turn gives

$$(2.10) \qquad P^+(d) - P^+(c) = (\alpha - 1)({}^*\Delta(d) - {}^*\Delta(c)), \quad [c,d] \text{ positive.}$$

Also, (2.9), (2.10), and the fact that neither $g_1^\#$ or $g_2^\#$ changes on a positive interval yield

$$(2.11) \qquad P^-(d) - P^-(c) = \Delta(d) - \Delta(c) = \frac{\alpha}{\alpha - 1}(P^+(d) - P^+(c)), \quad [c,d] \text{ positive.}$$

The sentence before (2.8), together with (2.10), show it cannot happen that both $P^+(c) \ge 0$, $P^+(d) < 0$ or that both $P^+(c) \le 0$, $P^+(d) > 0$. Furthermore,

$$(2.12) \qquad |P^+(d)| \le |P^+(c)|, \quad [c,d] \text{ positive.}$$

A mirror set of equalities and inequalities holds for negative intervals. In particular, we have, recalling $\frac{\beta}{\beta-1} = \frac{\alpha-1}{\alpha}$,

$$(2.13) \qquad P^+(d) - P^+(c) = \Delta(d) - \Delta(c) = \frac{\alpha - 1}{\alpha}(P^-(d) - P^-(c)), \quad [c,d] \text{ negative.}$$

Also we have

$$(2.14) \qquad |P^-(d)| \le |P^-(c)|, \quad [c,d] \text{ negative,}$$

and it cannot happen that both $P^-(c) \ge 0$, $P^-(d) < 0$ or that both $P^-(c) \le 0$, $P^-(d) > 0$, if $[c,d]$ is negative.

If $0 < r < s$, we say $r = a_0 < a_1 < a_2 < \ldots < a_n = s$ is a positive-negative decomposition of $[r,s]$ if each interval $[a_i, a_{i+1}]$, $1 \leq i \leq n$, is either positive or negative, and we let $||[r,s]||$ be the fewest such intervals possible. The following construction, here called the canonical decomposition, not only shows each interval has a positive-negative decomposition, but constructs one which clearly has no more than $2||[r,s]||$ intervals. Call t positive if either $g_1^*(t) = g_1(t)$ or $g_2^*(t) = g_2(t)$, and call t negative if either $g_2^\#(t) = g_2(t)$ or $g^\#(t) = g_1(t)$. Take $a_0 = r$, $a_1 = \min(\inf\{t > a_0 : t \text{ is positive or negative}\}, s)$, if $a_1 < s$ take $a_2 = \min(\inf\{t > a_1 : t \text{ is negative}\}, s)$ if a_1 is positive, and $a_2 = \min(\inf\{t > a_1 : t \text{ is positive}\}, s)$ if a_1 is negative, and if $a_2 < s$ let a_3 be the next negative or positive number, depending on whether a_2 is positive or negative, and so on. This process eventually yields an a_i equal to s, since otherwise Lemma 2.5 would be contradicted, because the limit of positive (negative) numbers is positive (negative). We also observe that if $[u,v] \subset [r,s]$, then the intersection of $[u,v]$ with the intervals in canonical decomposition of $[r,s]$ gives a positive-negative decomposition of $[r,s]$ with at most $2||[u,v]||$ intervals in it.

Let $0 < \varepsilon < a$. We prove

$$\Delta(b) - \Delta(a) \leq C_\alpha(|P^+(\varepsilon)| + |P^-(\varepsilon)|)||[a,b]||. \tag{2.15}$$

Before proving (2.13), we note that both $P^+(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $P^-(\varepsilon) \to 0$, so that (2.15) implies (2.6).

To prove (2.15), first consider the case where both $P^+(\varepsilon) \geq 0$ and $P^-(\varepsilon) \geq 0$. Let $\varepsilon = s_0 < s_1 < \ldots s_n = b$, where $[s_i, s_{i+1}]$, $0 \leq i < n$, are all the intervals which arise by intersecting the intervals in the canonical decomposition of $[\varepsilon, b]$ with both $[\varepsilon, a]$ and $[a,b]$. Then one of the s_i is a, designate it by s_m. The two sequences $P^+(s_i)$, $0 \leq i \leq n$, and $P^-(s_i)$, $0 \leq i \leq n$, are nonnegative by the sentences just before (2.12), and after (2.14), and by (2.11) and (2.13), and by (2.11) and (2.13) they satisfy the conditions of Lemma 2.7, with $p = \frac{\alpha}{1-\alpha}$, $a_k = P^+(s_k)$, and $b_k = P^-(s_k)$. Thus both $P^+(s_i)$ and $P^-(s_i)$ are no larger than $C_\alpha(P^+(s_0) + P^-(s_0)) = C_\alpha(|P^+(\varepsilon)| + |P^-(\varepsilon)|)$, and we have, using (2.11) and (2.13),

$$(2.16) \qquad \begin{aligned} |\Delta(b)-\Delta(a)| &\leq \sum_{k=m}^{n-1} |\Delta(s_{k+1})-\Delta(s_k)| \\ &\leq \sum_{k=m}^{n-1} C_\alpha(|P^+(s_k)|+|P^-(s_k)|) \\ &\leq C_\alpha(n-m)(|P^+(\varepsilon)|+|P^-(\varepsilon)|) \\ &\leq C_\alpha|\|[a,b]\||(|P^+(\varepsilon)|+|P^-(\varepsilon)|). \end{aligned}$$

If $P^+(\varepsilon) \leq 0$ and $P^-(\varepsilon) \leq 0$, then $P^+(s_i) \leq 0$ and $P^-(s_i) \leq 0$, $1 \leq i \leq n$, and so (2.11) and (2.13) imply that $\Delta(s_{k+1})-\Delta(s_k) \leq 0$, implying $\Delta(b)-\Delta(a) \leq 0$. Alternatively, we could mimic the argument just given, to bound $|\Delta(b)-\Delta(a)|$.

Finally, if one of $P^+(\varepsilon)$, $P^-(\varepsilon)$ is positive and one is negative, (2.11)–(2.14), together with the comments before (2.12) and after (2.14), imply that, if $m = \inf\{k : P^+(s_k)$ and $P^-(s_k)$ have the same sign$\}$, then $|P^+(s_{i+1})| \leq |P^+(s_i)|$ and $|P^-(s_{i+1})| \leq |P^-(s_i)|$, $0 \leq i < m-1$. Now if $m = \infty$, $|\Delta(s_{k+1})-\Delta(s_k)| \leq C_\alpha(|P^+(s_k)|+P^-(s_k)|) \leq C_\alpha(|P^+(\varepsilon)|+|P^-(\varepsilon)|)$, and an analysis very similar to (2.16) gives (2.15). And if $m < \infty$, (2.11)–(2.14) imply that

$$\begin{aligned} |P^+(s_m)|+|P^-(s_m)| &\leq C_\alpha(|P^+(s_{m-1})|+P^-(s_{m-1})|) \\ &\leq C_\alpha(|P^+(\varepsilon)|+|P^-(\varepsilon)|). \end{aligned}$$

Furthermore, $|P^+(s_{m+k})|+|P^-(s_{m+k})| \leq C_\alpha(|P^+(s_m)|+|P^-(s_m)|)$, $k > 0$, by the argument that led to the statement just before (2.16). Thus, once again, an analysis similar to (2.16) gives (2.15). □

Lemma 2.8. Let $0 < p < \infty$, and suppose $a_0, a_1, \ldots, a_n$, and $b_0, b_1, \ldots, b_n$, are real numbers which satisfy the following condition. For each k, $0 \leq k < n$, either all of $a_{k+1}-a_k = p(b_{k+1}-b_k)$, $|a_{k+1}| \leq |a_k|$, and $a_{k+1}a_k \geq 0$, or all of $a_{k+1}-a_k = -p(b_{k+1}-b_k)$, $|b_{k+1}| \leq |b_k|$, and $b_{k+1}b_k \geq 0$, hold. Then

$$|a_k|+p|b_k| \leq |a_0|+p|b_0|, \; 1 \leq k \leq n.$$

The proof, by induction, of this lemma is immediate.

Proposition 2.9. *If $\rho = -1$, there do not exist two different solutions of (1.1) for any f.*

Proof: Suppose, with no loss of generality, that $\alpha > 0$, so $\beta < 0$. Let g_1 and g_2 be two solutions for f. Define $\Delta(t)$, $P^+(t), P^-(t)$, and positive and negative intervals symbolically exactly as they were defined in the proof of Lemma 2.8, and define $Q^-(t) = -P^-(t)$. All the equations, inequalities, and discussion appearing between (2.8) and (2.12) inclusive, still holds. In addition, (2.11) gives

$$(2.17) \qquad -[Q^-(d) - Q^-(c)] = \Delta(d) - \Delta(c) = \frac{\alpha}{\alpha - 1}(P^+(d) - P^+(c)), [c, d] \text{ positive.}$$

We also have, by reasoning very similar to that which lead to (2.11),

$$(2.18) \qquad P^+(d) - P^+(c) = \Delta(d) - \Delta(c) = \frac{\beta}{1 - \beta}(Q^-(d) - Q^-(c)), \ [c, d] \text{ negative.}$$

Mirroring the comments before (2.12), if $[c, d]$ is negative it cannot happen that both $Q^-(c) \geq 0$ and $Q^-(d) < 0$ or that both $Q^-(c) \leq 0$ and $Q^-(d) > 0$, and $|Q^-(d)| \leq |Q^-(c)|$.

The rest of the proof of Proposition 2.9 closely models the proof of Proposition 2.7. We fix $[a, b]$ and again make the additional assumption that $f^*(t) - f^\#(t) > 0,\ t > 0$. Let $0 < \varepsilon < a < b$, and let $\varepsilon = s_0 < s_1 < \ldots < s_m = b$ be constructed exactly as they were in the proof of Proposition 2.8. Let $a_i = P^+(s_i)$ and $b_i = Q^-(s_i)$, and $p = \frac{\alpha-1}{\alpha} = \frac{-\beta}{\beta-1}$. If $[s_k, s_{k+1}]$ is negative, the comments after (2.14) imply that either $b_k \geq b_{k+1} \geq 0$ or $b_k \leq b_{k+1} \leq 0$, and (2.17), (2.12) and the comments after (2.14) show that if $[s_k, s_{k+1}]$ is positive, either $a_k \geq a_{k+1} \geq 0$ or $a_k \leq a_{k+1} \leq 0$. Together with (2.17) and (2.18) this shows Lemma 2.8 applies. The remainder of the argument is virtually identical to the proof of Proposition 2.7, and is omitted. □

The following proposition provides the rest of the proof of Theorem 1.1.

Proposition 2.10. *If $|\rho| > |$, there is a function $f_{\alpha,\beta} = f$ for which (1.1) has at least two solutions.*

Proof. We prove the case $\rho < -1$. The case $\rho > 1$ has a very similar proof. We assume, without loss of generality, that $\alpha > 0$. If f is a piecewise linear function on $[0, t]$ it is easy to see there is a unique piecewise linear solution g of (1.1) "on $[0,t]$." These solutions may be found explicitly, as in the proof of Lemma 2.3.

We now construct two functions f_1, f_2 on $[0,\infty)$ by recursively defining them on successively larger intervals. We define $P^+(t)$, $Q^-(t)$, and $\Delta(t)$ symbolically exactly as they were defined in the proof of Proposition 2.8, where g_1 and g_2 are the piecewise linear solutions of (1.1) on these intervals for f_1 and f_2 respectively. Put $f_1(t) = 0$, $0 \le t \le 1$, and $f_2(t) = \gamma t$, $0 \le t \le 1/2$, $f_2(t) = (\gamma/2) - \delta(t-(1/2))$, $1/2 \le t \le 1$, where γ, $\delta > 0$ are chosen so that $g_2(1) = 0$, $g_2^{\#}(1) = 0$, and $g_2^*(1) = 1$. Of course $g_1(t) = 0$, $0 \le t \le 1$, so $P^+(1) = 1$ and $Q^-(1) = 0$. We now define $h(t) := f_1(t) - f_1(1) = f_2(t) - f_2(1)$, thereby defining f_1 and f_2 on the rest of $[0,\infty]$.

Of course $h(1) = 0$, and we put $h'(t) = 1$, $1 < t < t_1$, where $t_1 = \inf\{s : P^+(s) = 0\}$. Note that since h is increasing on $[1,t]$, this interval must be positive, and so (2.17) gives $P^+(t_1) = 0$, $Q^-(t_1) = -\frac{\alpha}{1-\alpha}$. It is worth noting that since $P^+(1) > 0$, $g_2^*(1) - g_2(1) > g_1^*(1) - g_1(1)$ (of course, we knew this anyhow), and thus the increments of both g_1 and g_2 after 1 equal those of h until $g_1 = g_1^*$, after which g_1 increases at a faster rate than g_2 until $g_2 = g_2^*$, which occurs at t_1.

Next define $h'(t) = -1$ on $t_1 < t < t_2$, where $t_2 = \inf\{t \ge t_1 : \quad Q^-(t) = 0\}$. Then $P^+(t_2) = \left(\frac{-\alpha}{1-\alpha}\right)\left(\frac{-\beta}{1-\beta}\right) = \rho$, using (2.18). Then define $h(t) = -1$, $t_2 < t < t_3$, where $t_3 = \inf\{t > t_2 : \ P^+(t_3) = 0\}$ and so on. We have $P^+(t_{2n}) = \rho^n$, $Q^-(t_{2n}) = 0$, and $P^+(t_{2n+1}) = 0$, $Q^-(t_{2n+1}) = \left(-\frac{\alpha}{1-\alpha}\right) P^+(t_{2n})$, $n \ge 0$.

We will show

$$c|\rho|^n < t_{2n} < C|\rho|^n, \quad n \ge 0. \tag{2.19}$$

To prove the left side of (2.19), we first note that

$$|P^+(s) - P^+(t)| \le C|s-t|, \quad 1 \le s < t, \tag{2.20}$$

since, roughly, none of g_1, g_2, g_1^*, or g_2^* changes on $[1,\infty)$ changes at a rate faster than an absolute constant C, since $|h'| = 1$ for all but a discrete set of points. For example, if k is even and $t_k \le s < t \le t_{k+1}$, $h(t) - h(s) = t - s$, and so (2.2) gives $0 \le g_1(t) - g_1(s) \le C$

$(t-s)$, and now (2.4) gives $g_1^*(t) - g_1^*(s) \le C(t-s)$. Thus $t_{2n} - t_{2n-1} \ge c|P^+(t_{2n}) - P^+(t_{2n-1})| = c|\rho^n|$.

The right side of (2.19) follows from

$$t_{k+1} - t_k < C|\rho|^{k/2}. \tag{2.21}$$

To prove (2.21), we first prove

$$(g_1^*(t_k) - g_1^{\#}(t_k)) + (g_2^*(t_k) - g_2^*(t_k)) \le C|\rho|^{k/2}, \quad k \ge 0. \tag{2.22}$$

Suppose first that j is even and $j/2$ is an even integer. Let $y = y_j = \inf\{t > t_j : \ g_1(t) = g_1^*(t)\}$. Then $g_1(s) = g_1^*(s)$, $y \le s \le t_{j+1}$, and $t_{j+1} = \inf\{t > y : \ g_2(s) = g_2^*(s)\}$. Thus $|\rho|^{j/2} = P^+(t_j) - P^+(t_{j+1}) = P^+(y) - P^+(t_{j+1})$, which in turn equals $t_{j+1} - y$, since $g_2(t_{j+1}) - g_2(y) = h(t_{j+1}) - h(y) = t_{j+1} - y$.

Now (2.2) and (2.4) and (2.20) yield

$$\begin{aligned} g_1^*(t_{j+1}) - g_1^*(t_j) = g_1^*(t_{j+1}) - g_1^*(y) &\le C(t_{j+1} - y) \\ &= C(P^+(y) - P^+(t_{j+1})) \\ &= C|\rho|^{j/2}. \end{aligned} \tag{2.23}$$

Since $g_1^{\#}(t_{j+1}) = g_1^{\#}(t_j)$, $g_2^{\#}(t_{j+1}) = g_2^{\#}(t_j)$, and $g_2^*(t_{j+1}) = g_2^*(t_j)$, this gives,

(2.24)

$$(g_1^*(t_{j+1}) - g_1^*(t_j)) + (g_2^*(t_{j+1}) - g_2^*(t_j)) + (g_1^{\#}(t_j) - g_1^{\#}(t_{j+1})) + (g_2^{\#}(t_j) - g_2^{\#}(t_{j+1})) \le C|\rho|^{j/2}.$$

The proof of (2.24) for j odd, and for j even when $j/2$ is not an integer, is similar, and adding these inequalities for $j = 0$ to $k-1$ gives an inequality which immediately implies (2.22). To derive (2.21) from (2.22), let k and $k/2$ be even, as the argument for other k is very similar, and let $y = y_k$ be as defined just after (2.22). Then $t_{k+1} - t_k = (t_{k+1} - y) + (y - t_k)$. Now

$$y - t_k = g_1^*(t_k) - g_1(t_k) \le g_1^*(t_k) - g_1^{\#}(t_k) \le C|\rho|^{k/2},$$

using (2.22). And (2.23) gives $t_{k+1} - y \le C|\rho|^{k/2}$.

Finally we note

$$|\Delta(t_{2n+1}) - \Delta(t_{2n})| \geq C|\rho|^n, \ n \geq 1, \tag{2.25}$$

which follows from (2.11), so that

$$\sup_{0\leq s\leq t_{2n+1}} |g_1(s) - g_2(s)| \geq C|\rho|^n, \ n \geq 1. \tag{2.26}$$

Now define f_n^1 and f_n^2 by $f_n^1(t) = n^{-1}f_1(nt)$ and $f_n^2(t) = n^{-1}f_2(nt)$. Their solutions for (1.1) equal $n^{-1}g_1(nt)$ and $n^{-1}g_2(nt)$ respectively, which we designate g_1^n and g_2^n. Pick a subsequence n^1, $n \geq 1$, of the integers, such that $f_1^{n'}$, $f_2^{n'}$, $g_1^{n'}$, $g_2^{n'}$ converge uniformly on compact subintervals of $[0, \infty]$. This is possible since $\{f_n^1, \ n \geq 1\}$ and $\{f_n^2, \ n \geq 1\}$ are both absolutely continuous and bounded, by their explicit construction, and thus so are $\{g_n^1, \ n \geq 1\}$, and $\{g_n^2, n \geq 1\}$, by Lemma 2.1.

Now $f_{n'}^1$ and $f_{n'}^2$ clearly converge to the same function, again by their constructions. Call this function f. Corollary 2.2 guarantees that the limits of $g_{n'}^1$ and $g_{n'}^2$, call them g_1 and g_2, are both solutions of (1.1) for f. And, finally, (2.26) and (2.15) guarantee that g_1 and g_2 cannot be the same function. □

3. Proof of Theorem 1.2

The basis of our proof of Theorem 1.2 is the following formula of CPY. If g solves (1.1) for f, then

$$g^*(t) = \frac{1}{1-\alpha} \sup_{s\leq t}(f(s) - \frac{\beta}{1-\beta} \sup_{u\leq s}(-f(u) - \alpha g^*(u))). \tag{3.1}$$

Let $||h||_T = \sup_{0\leq s\leq T} |h(s)|$, $T > 0$. Throughout this section we assume $\alpha < 1$, $\beta < 1$, $|\rho| < 1$.

Lemma 3.1. *Let $\alpha < 1$, $\beta < 1$ and $|\rho| < 1$. Then if g_1 and g_2 are solutions of (1.1) for f_1 and f_2 respectively, we have*

$$||g_1 - g_2||_T \leq C||f_1 - f_2||_T, \quad T > 0. \tag{3.2}$$

Proof. Subtracting the version of (3.1) for f_2 from that for f_1,

$$|g_1^*(T) - g_2^*(T)| \leq \frac{1}{1-\alpha}[\sup_{s\leq T}|f_1(s) - f_2(s)| +$$

$$(3.3) \qquad \frac{|\beta|}{1-\beta}\sup_{s\leq T}|\sup_{u\leq s}(-f_1(u) - \alpha g_1^*(u)) - \sup_{u\leq s}(-f_2(u) - \alpha g_2^*(u))|$$

$$\leq \frac{1}{1-\alpha}\left[||f_1 - f_2||_T + \frac{|\beta|}{1-\beta}(\sup_{s\leq T}|\sup_{u\leq s}|f_1(u) - f_2(u)| + |\alpha|\sup_{u\leq s}|g_(^*u) - g_2^*(u)|)\right]$$

$$\leq \frac{1}{1-\alpha}||f_1 - f_2||_T + \frac{|\beta|}{1-\beta}||f_1 - f_2||_T + |\rho|||g_1^* - g_2^*||_T,$$

yielding, upon noticing that $|g_1^*(T) - g_2^*(T)|$ may be replaced in (3.2) by any of $|g_1^*(t) - g_2^*(t)|$, $0 < t < T$, since the right hand side of (3.3) is increasing in T,

$$(3.4) \qquad ||g_1^* - g_2^*||_T(1 - |\rho|) \leq \frac{1 - \beta + |\beta| - \alpha|\beta|}{(1-\alpha)(1-\beta)}||f_1 - f_2||_T.$$

so

$$(3.5) \qquad |g_1^*(t) - g_2^*(t)| \leq C||f_1 - f_2||_T, \; 0 \leq t \leq T.$$

Similarly we have

$$|g_1^\#(t) - g_2^\#(t)| \leq C||f_1 - f_2||_T, \quad 0 \leq t \leq T$$

We claim that the truth of (3.4) and (3.5) for all f_1 and f_2 implies the apparently stronger inequality (3.2).

We show this by showing that if $||f_1 - f_2||_T > 0$ and

$$(3.6) \qquad 4 < K = K_{f_1,f_2,T} = ||g_1 - g_2||_T/||f_1 - f_2||_T,$$

then there are functions $\tilde{f}_1$ and $\tilde{f}_2$, with solutions $\tilde{g}_1$ and $\tilde{g}_2$ respectively, and $S > 0$, such that $||\tilde{f}_1 - \tilde{f}_2||_S = ||f_1 - f_2||_T$, and either $|\tilde{g}_1^*(S) - \tilde{g}_2^*(S)| > \frac{K}{2}||\tilde{f}_1 - \tilde{f}_2||_S$, or $|\tilde{g}_1^\#(S) - \tilde{g}_2^\#(S)| > \frac{K}{2}||\tilde{f}_1 - \tilde{f}_2||_S$.

Suppose, first, that $\alpha > 0$ and $\beta > 0$, and suppose without loss of generality that $|g_1(T) - g_2(T)| = ||g_1 - g_2||_T$, and that $g_1(T) > g_2(T)$. Let $w = \sup\{x \leq T : \; (g_1(T) - g_2(T)) - (g_1(x) - g_2(x)) > (f_1(T) - f_2(T)) - (f_1(x) - f_2(x))\}$.

Note 0 is in the set we are taking the supremum of, since $K > 2$. Now either $g_1(w) = g_1^*(w)$ or $g_2(w) = g_2^\#(w)$, since otherwise, $(g_1(w) - g_2(w)) - (g_1(w-\varepsilon) - g_2(w-\varepsilon)) \leq$

$(f_1(w) - f_1(w-\varepsilon)) - (f_2(w) - f_2(w-\varepsilon))$ for small enough $\varepsilon > 0$, using (2.2) and (2.3). Suppose $g_1(w) = g_1^*(w)$. Define $\tilde{f}_1$ and $\tilde{f}_2$ by $\tilde{f}_1(t) = f_1(t)$, $t \le w$, $\tilde{f}_1(t) - \tilde{f}_1(w) = (t-w)$, $t > w$, and $\tilde{f}_2(t) = f_2(t)$, $t \le w$, $\tilde{f}_2(t) - \tilde{f}_2(w) = (t-w)$, $t > w$. Let $\gamma = \inf\{t \ge w : \tilde{g}_2(w) = \tilde{g}_2^*(w)\}$. Now $\tilde{g}_1(s) = \tilde{g}_1^*(s)$, $w \le s \le \gamma$, and since $\alpha > 0$, $\tilde{g}_1(s) - \tilde{g}_2(s)$ is increasing on (w, γ), and so

$$\begin{aligned}\tilde{g}_1^*(\gamma) - \tilde{g}_2^*(\gamma) = \tilde{g}_1(\gamma) - \tilde{g}_2(\gamma) &\ge \tilde{g}_1(w) - \tilde{g}_2(w) \\ &= g_1(w) - g_2(w).\end{aligned}$$

But

$$\begin{aligned}g_1(w) - g_2(w) &\ge (g_1(T) - g_2(T)) - |(f_1(T) - f_1(w)) - (f_w(T) - f_2(w))| \\ &\ge (g_1(T) - g_2(T)) - 2\|f_1 - f_2\|_T \\ &> \frac{1}{2}(g_1(T) - g_2(T)) \text{ (by (3.6))} \\ &= \frac{K}{2}\|f_1 - f_2\|_T.\end{aligned}$$

Finally, note $\|\tilde{f}_1 - \tilde{f}_2\|_\gamma = \|\tilde{f}_1 - \tilde{f}_2\|_w = \|f_1 - f_2\|_w \le \|f_1 - f_2\|_T$, and so we get $\tilde{g}_1^*(\gamma) - \tilde{g}_2^*(\gamma) \ge \frac{K}{2}\|\tilde{f}_1 - \tilde{f}_2\|_\gamma$, which verifies what we said we were going to, in the sentence containingt (3.6).

The proof if one or both of α, β is not positive is very similar. □

We use $\Rightarrow$ to indicate convergence in distribution of processes, and retain the convention extending discrete time processes to, and identifing them with, continuous time processes, mentioned before the statement of Theorem 1.2. For a process $\boldsymbol{Z}$, we let $\boldsymbol{Z}^n$ be the process $n^{-1/2}Z_{nt}$, $t \ge 0$. We let $\boldsymbol{R}$ be fair random walk, started at 0, $\boldsymbol{B}$ and $\boldsymbol{Y}$ be as in (1.2), and $\boldsymbol{X}$ be as in the statement of Theorem 2.1. It is classical that $\boldsymbol{R}^n \Rightarrow \boldsymbol{B}$. The Continuous Mapping Theorem (see page 70 of Pollard 1984), and Lemma 3.1, now give that if $\boldsymbol{S}$ solves (1.1) for $\boldsymbol{R}$, then $\boldsymbol{S}^n \Rightarrow \boldsymbol{Y}$. If $\boldsymbol{S}$ had the distribution of $\boldsymbol{X}$, this would verify Theorem 1.2, but it does not. To circumvent this problem we find a process $\boldsymbol{U}$ such that $\boldsymbol{U}^n \Rightarrow \boldsymbol{B}$, and such that the solution of (1.1) for $\boldsymbol{U}$ has exactly the distribution of $\boldsymbol{X}$.

$\boldsymbol{U}$ is constructed from $\boldsymbol{R}$. We describe its construction and properties for α, β both nonpositive. The other cases are very similar. We let A_i, $i \ge 1$, be iid indicator variables with $P(A_i = 1) = \frac{-\alpha}{2+\alpha}$ and B_i, $i \ge 1$, be indicator variables independent of the A_i with $P(B_i = 1) = \frac{-\beta}{2-\beta}$. Let $M_0 = R_0$, and $M_1 = R_1$, and if $i \ge 1$ put $M_{i+1} - M_i = R_{i+1} - R_i$

if either $M_i^\# < M_i < M_i^*$, or $M_i = M_i^\#$ and $R_{i+1} - R_i = +1$, or $M_i = M_i^*$ and $R_{i+1} - R_i = -1$. Define $M_{i+1} - M_i = R_{i+1} - R_i - 2A_{J(i)}$ if $M_i = M_i^*$ and $R_{i+1} - R_i = 1$, where $J(i)$ is the number of k, $1 \le k \le i$, such that $M_k = M_k^*$ and $R_{k+1} - R_k = 1$. Define $M_{i+1} - M_i = R_{i+1} - R_i + 2B_{\Theta(i)}$ if $M_i^\# = M_i$ and $R_{i+1} - R_i = -1$, where $\Theta(i)$ is the number of those k, $1 \le k \le i$, such that $M_k = M_k^\#$ and $R_{i+1} - R_i = -1$. Then $\boldsymbol{M}$ has exactly the distribution of $\boldsymbol{X}$. We define the process $\boldsymbol{U}$ as follows: $U_{n+1} - U_n = M_{n+1} - M_n$, except on $\{M_{n+1} - M_n = 1,\ M_n = M_n^*\}$, where we define $U_{n+1} - U_n = (1 - \alpha)$, and on $\{M_{n+1} - M_n = -1,\ M_n = M_n^\#\}$, where we define $U_{n+1} - U_n = -1 + \beta$. Then $\boldsymbol{M}$ is the solution of (1.1) for $\boldsymbol{U}$. Furthermore, we have

$$\begin{aligned} U_{n+1} - U_n &= R_{n+1} - R_n \text{ if } M_n \neq M_n^* \text{ or } M_n^\#, \text{ or } n = 0, \\ U_{n+1} - U_n - (R_{n+1} - R_n) &= [(1-\alpha) - 1] I(R_{n+1} - R_n = 1, A_{J(n)} = 0) \\ &\qquad - 2I(R_{n+1} - R_n = 1,\ A_{J(n)} = 1) \\ &:= \Delta_n^+, \quad \text{if } M_n = M_n^*,\ n > 0. \end{aligned}$$

Also,

$$\begin{aligned} U_{n+1} - U_n - (R_{n+1} - R_n) &= [(1-\beta)] + 1I(R_{n+1} - R_n = -1,\ B_{\Theta(n)} = 0) \\ &\qquad + 2I(R_{n+1} - R_n = -1,\ B_{\Theta(n)} = 1) \\ &:= \Delta_n^-, \quad \text{if } M_n = M_n^\#,\ n > 0. \end{aligned}$$

Thus $U_n - R_n = \sum_{k=0}^{n} \Delta^+(k) + \sum_{k=0}^{n} \Delta^-(k)$. It is easily checked that $\Delta^+(k)$, $k > 0$, and $\Delta^-(k)$, $k > 0$, are both martingale difference sequences, that $|\Delta^+(k)| \le C_\alpha, |\Delta^-(k)| \le C_\beta$, that $\Delta^+(k) = 0$ except on $\{M_k = M_k^*\}$, and that $\Delta^-(k) = 0$ except on $\{M_k = M_k^\#\}$.

Lemma 3.2. *If $\boldsymbol{X}$ is as in the statement of Theorem 1.2, then*

$$n^{-1} \sum_{k=1}^{n} I(X_k = X_k^* \text{ or } X_k^\#) \to 0 \text{ in probability.}$$

Proof: Fix $M > 1 > 0$. Let $\tau_1 = \inf\{k :\ X_k^* - X_k^\# = M\}$. Clearly $\tau_1 < \infty$ a.s. Let

$$\tau_{2k} = \inf\{i \ge \tau_{2k-1} :\ X_i \in (X_i^\#, X_i^*)\}, k \ge 1, \text{ and}$$

$$\tau_{2k+1} = \inf\{i \ge \tau_{2k} :\ X_i = X_i^* \text{ or } X_i^\#\},\ k \ge 1.$$

Let $\mathcal{A}_k = \sigma(X_i,\ i \le k)$.

Now if $i \ge 1$, the conditional distribution of $\tau_{2i} - \tau_{2i-1}$ given $\mathcal{A}_{\tau_{2i-1}}$ is the geometric distribution with parameter $\frac{1}{2-\alpha}$ on $\{X_{\tau_{2i-1}} = X^*_{\tau_{2i-1}}\}$, and it is the geometric distribution with parameter $\frac{1}{2-\beta}$ on $\{X_{\tau_{2i-1}} = X^{\#}_{\tau_{2i-1}}\}$. And the conditional distribution of $\{\tau_{2i+1} - \tau_{2i}\}$ given $\mathcal{A}_{2i}$ is stochastically no smaller than the distribution of the time it takes fair random walk, started at 1, to exit from $(0, M)$. Especially, if E_M is the expected time it takes random walk started at 1 to exit $(0, M)$, we have

$$\overline{\lim}_{n\to\infty} \sum_{k=1}^{n} (\tau_{2k} - \tau_{2k-1}|)/\sum_{k=1}^{n-1}(\tau_{2k+1} - \tau_{2k}) \le C/E_M,$$

where C is the maximum of the expectation of the two geometric variables mentioned above. Since the sum in the denominator is smaller than τ_{2n}, this implies

$$\begin{aligned}\overline{\lim}_{n\to\infty} \sum_{k=1}^{n} I(X_k = X^*_k \text{ or } X^{\#})/n &\le \overline{\lim}_{n\to\infty} \sum_{k=1}^{\tau_{2n}} I(X_k = X^*_k \text{ or } X^{\#}_k)/\tau_{2n} \\ &= \overline{\lim}_{n\to\infty} \sum_{k=1}^{n} (\tau_{2k} - \tau_{2k-1})/\tau_{2n} \le C/E_M.\end{aligned}$$

Since $\sup_m E_M = \infty$, this proves the lemma. □

Note that this lemma is equivalent to

(3.7) $$Q_n/n \to 0 \text{ in probability,}$$

where $Q_n := \sum_{k=I}^{n} I(M_n = M^*_n \text{ on } M^{\#}_n)$ a.s., since $\boldsymbol{X}$ and $\boldsymbol{M}$ have the same distribution.

To complete the proof of Theorem 1.2, we prove the following lemma.

Lemma 3.3. *$\boldsymbol{U}^n \to \boldsymbol{B}$ in distribution as $n \to \infty$.*

Proof. The proof will be accomplished by showing that $\sup_{0\le s\le t} |U^n(s) - R^n(s)| \to 0$ in probability for each fixed t. This follows from

$$
\begin{aligned}
&E \max_{1\le k\le n} |U(k) - R(k)|\sqrt{n} \\
&= E \max_{i\le k\le n} |\sum_{i=1}^{n} \Delta^+(i) + \sum_{i=1}^{n} \Delta^-(i)|/\sqrt{n} \\
&\le E|\max_{1\le k\le n} \sum_{i=1}^{k} \Delta^+(i)|/\sqrt{n} + E|\max_{1\le k\le n} \sum_{i=1}^{b} \Delta^-(i)|/\sqrt{n} \\
&\le [E(\max_{1\le k\le n} \sum_{i=1}^{k} \Delta^+(i))^2]^{1/2}/n \\
&\qquad + [E(\max_{1\le k\le n} \sum_{i=1}^{k} \Delta^-(i))^2]^{1/2}/n \\
&\le 4E(\sum_{i=1}^{n} \Delta^+(i))^2/n + 4E(\sum_{i=1}^{n} \Delta^-(i))^2/n \\
&= C_\alpha EQ(n)/n + C_\beta EQ(n)/n \to 0, \text{ as } n \to \infty,
\end{aligned}
$$

the last inequality by Doob is martingale maximal inequality (p. 317 of Doob 1951) applied to the martingales $\sum_{i=1}^{k} \Delta^+(i)$ and $\sum_{i=1}^{k} \Delta^-(i)$, and the convergence to zero by (3.4). □

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