HOW MANY RANDOM WALKS CORRESPOND TO A GIVEN SET OF RETURN PROBABILITIES TO THE ORIGIN ?

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Abstract. We consider the class of simple random walks or birth and death chains on the nonnegative integers. The set of return probabilities P_{00}^n, $n \geq 0$, uniquely determines the spectral measure of the process. We characterize the class of simple random walks with the same spectral measure or same return probabilities to the origin. The analysis is based on the spectral theory developed by Karlin and McGregor (1959), continued fractions and canonical moments.

1. Introduction. Let $(X_n)_{n \in I\!N_0}$ denote an irreducible random walk with state space $E = \{0, 1, \ldots, N\}$ and one-step transition probabilities $\bar{p}_j, \bar{q}_j, \bar{r}_j$ for jumps down, up and hold. If $E = I\!N_0$ we formally define $N = \infty$ and assume $\bar{q}_j > 0$, $\bar{p}_{j+1} > 0$ $\bar{p}_j + \bar{q}_j + \bar{r}_j \leq 1$ $(j \geq 0)$ and $\bar{p}_0 = 0$. In the case of a finite state space $(N < \infty)$ we assume $\bar{q}_j > 0$ $(j = 0, \ldots, N-1)$, $\bar{q}_N = 0$, $\bar{p}_j > 0$ $(j = 1, \ldots, N)$ and $\bar{p}_0 = 0$. The possible inequality $\bar{p}_i + \bar{q}_i + \bar{r}_i < 1$ is interpreted as a permanent absorbing state i^* which can be reached from state i with probability $1 - \bar{p}_i - \bar{q}_i - \bar{r}_i$. It is shown by Karlin and McGregor (1959) that the n-step transition probabilities of this process can be represented as

$$(1.1) \qquad P_{ij}^n = P(X_n = j | X_0 = i) = \frac{\int_{-1}^1 x^n Q_i(x) Q_j(x) d\psi(x)}{\int_{-1}^1 Q_i^2(x) d\psi(x)} \qquad i, j \in E$$

where ψ is a distribution function on the interval $[-1, 1]$ called the spectral measure of the random walk $(X_n)_{n \in I\!N_0}$ and $Q_j(x)$ is a polynomial of degree j defined recursively by $Q_0(x) = 1$, $Q_{-1}(x) = 0$,

$$(1.2) \qquad xQ_n(x) = \bar{q}_n Q_{n+1}(x) + \bar{r}_n Q_n(x) + \bar{p}_n Q_{n-1}(x) \qquad (0 \leq n \leq N).$$

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When $N < \infty$, $\bar{q}_N = 0$ and $Q_{N+1}(x)$ is left undefined. The last equation then imposes a condition on ψ in that the support of ψ is on those x values for which $xQ_N(x) = \bar{p}_N Q_{N-1}(x) + \bar{r}_N Q_N(x)$.

The distribution ψ has infinite support if and only if the state space of the walk is $\mathbb{N}_0$. If $i = 0$ and $n < j$, it follows from (1.1) that the polynomials $Q_j(x)$ are orthogonal with respect to the spectral measure ψ. Multiplying (1.2) by $Q_{n-1}(x)$ and integrating then shows inductively that

$$\int_{-1}^{1} Q_k^2(x)d\psi(x) = \frac{\bar{p}_1 \cdots \bar{p}_k}{\bar{q}_0 \cdots \bar{q}_{k-1}} \qquad k = 1, \ldots, N\,.$$

Moreover, the distribution ψ is determined by its moments

$$(1.3) \qquad c_n = \int_{-1}^{1} x^n d\psi(x) = P_{00}^n \qquad n = 0, 1, 2, \ldots$$

which coincide with the return probabilities of the random walk from state 0 to state 0. In this sense every probability governing a random walk on the nonnegative integers determines a distribution function ψ on the interval $[-1, 1]$ by its return probabilities from state 0 to state 0. However, a distribution function ψ on the interval $[-1, 1]$ is not necessarily the spectral measure of a random walk for which (1.1) is valid. The problem of characterizing all spectral measures has been investigated by Whitehurst (1982) and Van Doorn (1992). If ψ is in fact the spectral measure of a random walk an important problem is to characterize all random walks on the nonnegative integers with the same spectral measure ψ. Because ψ is determined by its moments it follows from (1.3) that this problem is equivalent to the characterization of all random walks corresponding to a given sequence $(P_{00}^n)_{n=0,1,2,\ldots}$ of return probabilities from state 0 to state 0. A partial answer to this question has been given by Karlin and McGregor (1959). These authors showed that the correspondence between the symmetric distribution functions on the interval $[-1, 1]$ and the random walks on $\{0, 1, 2, \ldots\}$ with reflecting barrier at the origin and one-step up- and downward probabilities $\bar{q}_i$ and $\bar{p}_i$, $\bar{p}_0 = 0$, for which $\bar{p}_j + \bar{q}_j = 1 \quad (j \geq 0)$, is one to one.

In this paper we study this relation in more detail by using certain quantities which are in one to one correspondence with the moments c_n of distribution functions on the interval $[-1, 1]$ and are called canonical moments. A short review of this theory is presented in Section 2. This is used in Section 3 to provide an alternative proof of the Karlin and McGregor representation which does not rely on the L^2-theory of self-adjoint operators. These results are applied in characterizing the distributions on $[-1, 1]$ which are spectral measures of a random walk. Finally, in Section 4 we determine all random walks on the nonnegative integers such that (1.1) is valid for a given spectral measure ψ on $[-1, 1]$. It is shown that a random walk corresponding to a given spectral measure ψ is unique if and only if the process is recurrent. In the transient case all processes corresponding to a given ψ are described in terms of their one-step transition probabilities.

2. Canonical moments, continued fractions, orthogonal polynomials. In this section we briefly provide some background material which is necessary for proving the results in Section 3 and 4. For more details the reader is referred to Skibinsky (1986) or to Dette and Studden (1996). For a distribution function ψ on the interval $[-1,1]$ with moments $c_j = \int_{-1}^{1} x^j d\psi(x)$ $(j = 0,1,2,\ldots)$ let c_j^+ denote the maximum of the jth moment $\int_{-1}^{1} x^j d\mu(x)$ over the set of all probability measures μ having the same given moments $c_1,\ldots,c_{j-1}$ and let c_j^- denote the corresponding minimum. The jth canonical moment of the distribution ψ is defined as

$$p_j = \frac{c_j - c_j^-}{c_j^+ - c_j^-} \qquad j = 1,2,3,\ldots$$

if $c_j^- < c_j^+$ and left undefined if $c_j^+ = c_j^-$. The measure ψ has infinite support if and only if its canonical moments satisfy $0 < p_j < 1$ for all $j \in \mathbb{N}$. If ψ has finite support the canonical moment sequence is of the form $p_1,\ldots,p_{k-1},\ p_k$ for some $k \geq 1$, where $0 < p_i < 1$, $i = 1,\ldots,k-1$ and $p_k = 0$ or 1. There is a one to one correspondence between the ordinary and canonical moment sequences, and consequently a distribution ψ is determined by its canonical moments (see Skibinsky (1986)). More precisely, for every sequence $(p_k)_{k\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ there exists exactly one measure ψ (on $[-1,1]$) with infinite support and canonical moments $(p_k)_{k\in\mathbb{N}}$. Similarly, every "terminating" sequence $(p_1,\ldots,p_k) \in (0,1)^{k-1} \times \{0,1\}$ corresponds to exactly one measure with finite support. Canonical moments are closely related to the orthogonal polynomials with respect to ψ and to the continued fraction expansion of the Stieltjes transform of ψ. More precisely the monic orthogonal polynomials with respect to the measure $d\psi(x)$ can be found recursively by $P_0(x) = 1,\ \ P_1(x) = x + 1 - 2\zeta_1$

$$(2.1) \qquad P_{k+1}(x) = (x + 1 - 2\zeta_{2k} - 2\zeta_{2k+1})P_k(x) - 4\zeta_{2k-1}\zeta_{2k}P_{k-1}(x) \qquad k = 1,2,\ldots$$

where $\zeta_1 = p_1$, $\zeta_j = q_{j-1}p_j$ $(j \geq 2)$ and $q_j = 1 - p_j$ $(j \geq 1)$. The ability to write the recursive equations in (2.1) in terms of the canonical moments is essential in the results to follow. The continued fraction expansion of the Stieltjes transform of ψ is given by

$$(2.2) \qquad \int_{-1}^{1} \frac{d\psi(x)}{z-x} = \frac{1}{|z+1-2\zeta_1} \Big| - \frac{4\zeta_1\zeta_2}{|z+1-2\zeta_2-2\zeta_3} \Big| - \frac{4\zeta_3\zeta_4}{|z+1-2\zeta_4-2\zeta_5} \Big| - \cdots$$

where $z \in \mathbb{C} \setminus [-1,1]$ and the continued fraction on the right hand side converges uniformly on every compact set with positive distance from the interval $[-1,1]$. Note that the continued fraction in (2.2) terminates whenever $p_j \in \{0,1\}$ and that the polynomial in the denominator of the nth approximant is precisely $P_n(x)$ defined by (2.1). The sequence of polynomials $P_0(x), P_1(x),\ldots$ will terminate at $P_{N+1}(x)$ if $\zeta_{2N+1}\zeta_{2N+2} = 0$. This is the case if $p_{2N} = 1$, $p_{2N+1} = 1, p_{2N+1} = 0$, or $p_{2N+2} = 0$ and the zeros of the polynomial $P_{N+1}(x)$ give the support of the measure ψ.

From the Karlin McGregor representation (1.1) we obtain in combination with (2.2), a continued fraction expansion for the generating function of the return probabilities from state 0 to state 0, i.e.

$$
\begin{aligned}
P_0(z) &= \sum_{n=0}^{\infty} P_{00}^n z^n \\
&= \frac{1 \;|}{|\,1+(1-2\zeta_1)z} - \frac{4\zeta_1\zeta_2 z^2 \;|}{|\,1+(1-2\zeta_2-2\zeta_3)z} - \frac{4\zeta_3\zeta_4 z^2 \;|}{|\,1+(1-2\zeta_4-2\zeta_5)z} - \cdots
\end{aligned} \tag{2.3}
$$

For alternative derivations of this continued fraction expansion see Good (1958) or Flajolet (1980) who used probabilistic and combinatorial arguments.

3. The Karlin McGregor representation revisited. In the following let $\bar{p}_n, \bar{q}_n, \bar{r}_n$ denote the one-step down-, up- and holding transition probabilities of a random walk on the nonnegative integers such that $\bar{p}_0 = 0$, $\bar{q}_0 + \bar{r}_0 \leq 1$ and $\bar{p}_j + \bar{r}_j + \bar{q}_j \leq 1$ $(j \in E)$. Similarly, p'_j, q'_j, r'_j will always denote one-step transition probabilities of a process such that $p'_0 = 0$, $q'_0 + r'_0 = 1$, $p'_j + r'_j + q'_j = 1$ $(0 \leq j < N)$.

Definition 3.1. *A distribution ψ on the interval $[-1,1]$ is called a spectral measure or a random walk measure if and only if there exists a random walk $(X_n)_{n\in\mathbb{N}_0}$ on the nonnegative integers with one-step down-, up- and holding transition probabilities $\bar{p}_j, \bar{q}_j, \bar{r}_j$ $(\bar{p}_0 = 0)$ such that one of the following two conditions is satisfied*

(A) The polynomials $Q_n(x)$ defined by (1.2) and the transition probabilities of $(X_n)_{n\in\mathbb{N}}$ satisfy the Karlin McGregor representation (1.1).

(B) The polynomials $Q_n(x)$ defined by (1.2) are orthogonal with respect to $d\psi(x)$.

Note again that (B) is obtained from (A) by putting $i = 0$ and $n < j$ in (1.1). To show the converse the equations (1.2) are written in the form

$$xQ(x) = PQ(x)$$

where Q denotes the column of polynomials $Q_0(x)$, $Q_1(x), \ldots$ and P is the matrix of one step transition probabilities. The sequence of polynomials terminates at $Q_N(x)$ if $N < \infty$. A simple iteration gives

$$x^n Q(x) = P^n Q(x). \tag{3.1}$$

If the polynomials in (1.2) are orthogonal with respect to the measure $d\psi(x)$ then the representation (1.1) is obtained by multiplying the $(i+1)$th row of (3.1) by $Q_j(x)$ and integrating with respect to the measure $d\psi(x)$.

The polynomials in (1.2) are more conveniently put in monic form as $P_0(x) = 1$ $P_1(x) = x - \bar{r}_0$ and for $1 \le n \le N$

$$P_{n+1}(x) = (x - \bar{r}_n)P_n(x) - \bar{q}_{n-1}\bar{p}_n P_{n-1}(x). \tag{3.2}$$

If $N < \infty$, $P_{N+1}(x)$ is formally defined by (3.2).

It is important to note that the measure ψ on the interval $[-1,1]$ uniquely determines the polynomials in monic form and the polynomials are uniquely determined by the coefficients $\bar{r}_n$ and $\bar{q}_{n-1}\bar{p}_n$. Thus the polynomials in (1.2) or (3.2) are orthogonal with respect to some distribution ψ on the interval $[-1,1]$ if and only if for some set of canonical moments we have

$$\begin{aligned} \bar{r}_0 &= -1 + 2\zeta_1 \\ \bar{r}_n &= -1 + 2\zeta_{2n} + 2\zeta_{2n+1} \qquad 1 \le n \le N\,. \\ \bar{q}_{n-1}\bar{p}_n &= 4\zeta_{2n-1}\zeta_{2n} \end{aligned} \tag{3.3}$$

All solutions of these equations for $\{\bar{p}_i, \bar{q}_i, \bar{r}_i\}_{i\in E}$ give the same monic orthogonal polynomials and hence will have the same spectral measure.

These equations are more readily analyzed if there is no absorption or the transition probabilities add to one; in which case the mapping from the set $\{\bar{p}_i, \bar{q}_i, \bar{r}_i\}_{i\in E}$ to the canonical moments turns out to be one to one. This normalization is accomplished by dividing (1.2) by $Q_n(1)$. Define transition probabilities

$$q'_n = \frac{Q_{n+1}(1)}{Q_n(1)}\bar{q}_n, \quad r'_n = \bar{r}_n, \quad p'_n = \frac{Q_{n-1}(1)}{Q_n(1)}\bar{p}_n \quad (n \in E) \tag{3.4}$$

where, in the case of a finite state space ($N < \infty$), $q'_N = 0$. The "standardized" polynomials

$$R_n(x) = \frac{Q_n(x)}{Q_n(1)} \quad (1 \le n \le N), \tag{3.5}$$

then satisfy the recursion

$$xR_n(x) = q'_n R_{n+1}(x) + r'_n R_n(x) + p'_n R_{n-1}(x). \qquad 1 \le n < N\,. \tag{3.6}$$

Clearly $q'_{n-1}p'_n = \bar{q}_{n-1}\bar{p}_n$. Also $R_n(1) = 1$ and (3.6) imply $p'_n + r'_n + q'_n = 1$ for $1 \le n < N$ which proves the following result.

Lemma 3.2 *The transition probabilities $\{p'_n, q'_n, r'_n\}_{n\in E}$ defined in (3.4) satisfy (3.3) and $r'_0 + q'_0 = 1$, $p'_n + r'_n + q'_n = 1$ $(1 \le n < N)$.* Moreover, if the state space is finite, i.e. $N < \infty$, then we have $p'_N + r'_N = 1$ if and only if $x = 1$ is a zero of the polynomial $P_{N+1}(x)$ defined by (3.2).

Note that by the preceding discussion the random walks with one step transition probabilities $\{\bar{p}_n, \bar{q}_n, \bar{r}_n\}_{n\in E}$ and $\{p'_n, q'_n, r'_n\}_{n\in E}$ have the same spectral measure. The system

$$
\text{(3.7)} \qquad \begin{aligned} r'_0 &= -1 + 2\zeta_1 \\ q'_{n-1}p'_n &= 4\zeta_{2n-1}\zeta_{2n} && 1 \le n \le N \\ r'_n &= -1 + 2\zeta_{2n} + 2\zeta_{2n+1} \end{aligned}
$$

can be rewritten as

$$
\begin{aligned} 1 - r'_0 &= 2q_1 \\ q'_{n-1}p'_n &= 4\zeta_{2n-1}\zeta_{2n} = (2q_{2n-2}q_{2n-1})(2p_{2n-1}p_{2n})\ . \\ 1 - r'_n &= 2p_{2n-1}p_{2n} + 2q_{2n}q_{2n+1} \end{aligned}
$$

In this case a simple induction shows that if $p'_0 = 0$ and $p'_n + r'_n + q'_n = 1$ for $0 \le n < N$ then

$$
\text{(3.8)} \qquad \begin{aligned} q'_0 &= 2q_1 \\ q'_n &= 2q_{2n}q_{2n+1} && 1 \le n < N \\ p'_n &= 2p_{2n-1}p_{2n} && 1 \le n \le N\ . \end{aligned}
$$

Thus the solution of (3.7) for p'_n, q'_n, r'_n is unique if $p'_n + q'_n + r'_n = 1$, $0 \le n < N$. Conversely, the solution of (3.8) for the canonical moments p_k, $1 \le k \le 2N$, is clearly unique and an induction argument shows that

$$
\text{(3.9)} \qquad \begin{aligned} 0 < p_{2n} &\le p'_n \\ 1/2 \le p_{2n-1} &< 1 \end{aligned} \qquad 1 \le n \le N
$$

From (3.8) it is easy to see that the canonical moments can be written in continued fraction form as

$$
\text{(3.10)} \qquad \begin{aligned} q_{2n-1} &= \frac{q'_{n-1}/2|}{|\ 1} - \frac{p'_{n-1}/2|}{|\ 1} - \frac{q'_{n-2}/2|}{|\ 1} - \cdots - \frac{q'_0/2|}{|\ 1} \\ p_{2n} &= \frac{p'_n/2|}{|\ 1} - \frac{q'_{n-1}/2|}{|\ 1} - \cdots - \frac{q'_0/2|}{|\ 1}\ . \end{aligned}
$$

for $1 \le n \le N$.

Theorem 3.3. *Let $(X_n)_{n\ge1}$ denote a random walk on the nonnegative integers with transition probabilities $\bar{p}_n$, $\bar{q}_n$, $\bar{r}_n$ with $\bar{p}_0 = 0$, $\bar{p}_n + \bar{r}_n + \bar{q}_n \le 1$ $(n \in E)$. Then there exist a unique spectral measure ψ on the interval $[-1, 1]$ satisfying the representation (1.1).*
Moreover, for every spectral measure ψ there exists a unique random walk $(X_n)_{n\in \mathbb{N}}$ with one step up-, down and holding probabilities p'_n, q'_n, r'_n satisfying $p'_0 = 0$, $p'_n + q'_n + r'_n = 1$ $(0 \le n < N)$.

Proof: The assertion is established by showing that the sequence of polynomials in (1.2) determines a sequence of canonical moments. The corresponding distribution turns out to be the spectral measure of the process. Consider first the case where $N = \infty$. Starting with $\bar{p}_i$, $\bar{q}_i$, $\bar{r}_i$, one calculates the p'_i, q'_i, r'_i from (3.4) and arrives at the canonical moments given in (3.10). Since (3.9) holds there exists a unique measure corresponding to these canonical moments (see the discussion in Section 2). By Lemma 3.2 and the statement following (3.3) this measure is the (unique) spectral measure of the random walk $(X_n)_{n \in \mathbb{N}_0}$. Conversely, for a given spectral measure (in terms of its canonical moments) the preceding discussion shows that the solution of (3.7) is unique if $p'_n + q'_n + r'_n = 1$ for $n \geq 1$.

The case $N < \infty$ requires further elaboration. From (3.1) it follows that ψ must now be supported on the $N+1$ zeros of the polynomial $P_{N+1}(x)$ which is proportional to $(x - \bar{r}_N)Q_N(x) - \bar{p}_N Q_{N-1}(x)$. From the theory in section 2 this will be the case when $\zeta_{2N+1}\zeta_{2N+2} = 0$. This, in turn, is true when either $p_{2N} = 1$, $p_{2N+1} = 1$, $p_{2N+1} = 0$ or $p_{2N+2} = 0$. For the case $n = N$ we have from (3.7) and (3.8) the equations

$$\begin{aligned} 1 - r'_N &= 2p_{2N-1}p_{2N} + 2q_{2N}q_{2N+1} \\ p'_N &= 2p_{2N-1}p_{2N} \end{aligned}$$

which shows that $p'_N + r'_N = 1$ if and only if $q_{2N}q_{2N+1} = 0$. Thus if $p_{2N} = 1$ or $p_{2N} < 1$ and $p_{2N+1} = 1$ the measure ψ is determined by its canonical moment sequence which is obtained from (3.10) (note that (3.9) implies $p_{2N} > 0$). If

$$0 < 1 - \bar{r}_N - \bar{p}_N = 2q_{2N}q_{2N+1}$$

then $p_{2N} < 1$ and $p_{2N+1} < 1$. By (3.9) $p_{2N} > 0$ and we either have $p_{2N+1} = 0$ $(q_{2N+1} = 1)$ or $0 < p_{2N+1}$ $(q_{2N+1} < 1)$ and $p_{2N+2} = 0$. Thus in all cases the measure ψ is defined by its canonical moment sequence which terminates with either $p_{2N} = 1$, $p_{2N+1} = 1$, $p_{2N+1} = 0$ or $p_{2N+2} = 0$. ∎

Note that the second part of Theorem 3.3 extends a result of Karlin and McGregor (1959) who showed a one to one correspondence between the symmetric distribution functions on the interval $[-1, 1]$ and the random walks with infinite state space and transition probabilities satisfying $q'_0 = 1$, $p'_n + q'_n = 1$ $(n \in E)$. The following result characterizes a spectral measure in terms of its canonical moments. The proof follows from the previous discussion.

Proposition 3.4. *Let ψ denote a probability measure on the interval $[-1, 1]$ with canonical moments $p_1, p_2, \ldots$*

a) ψ is the spectral measure of a random walk if and only if

$$2p_{2n-1}p_{2n} + 2q_{2n}q_{2n+1} \leq 1 \tag{3.11}$$

whenever $1 \leq n \leq N$.

b) If ψ is a spectral measure of a random walk, then the canonical moments of odd order satisfy $p_{2n-1} \geq 1/2$ whenever $1 \leq n \leq N$.

Some further remarks when $N < \infty$ are in order. By Lemma 3.2 and the proof of Theorem 3.3 it follows that $r'_N + p'_N = 1$ if and only if $x = 1$ is a support point of ψ or equivalently $q_{2N}q_{2N+1} = 0$. In this case $p_{2N} = 1$ or $p_{2N+1} = 1$. This is in agreement with the general theory of canonical moments since in these cases the corresponding measure must have mass at the upper end point $x = 1$ (see Skibinsky (1986)).

From the equations (3.3) it can be shown inductively that

$$\begin{aligned} &\bar{q}_0 \leq 2q_1 \\ &\bar{q}_n \leq 2q_{2n}q_{2n+1} \qquad 1 \leq n \leq N-1\,. \\ &\bar{p}_n \geq 2p_{2n-1}p_{2n} \qquad 1 \leq n \leq N \end{aligned}$$

If $q_{2N}q_{2N+1} = 0$ an induction in (3.3) from the top end shows the reverse inequalities. Equality must then occur and $\bar{p}_n + \bar{q}_n + \bar{r}_n = 1$, $0 \leq n \leq N$. Thus in the non-absorbing or recurrent case the process is actually unique. Moreover, if there is a strict inequality $\bar{p}_n + \bar{q}_n + \bar{r}_n < 1$ for some n then $\bar{r}_N + \bar{p}_N < 1$. It then follows that all of the processes corresponding to a given ψ (with finite support) are either recurrent (in which case the process is unique) or they are all transient. The recurrent processes correspond to $p_{2N} = 1$ or $p_{2N+1} = 1$ while the transient processes correspond to $p_{2N+1} = 0$ or $p_{2N+2} = 0$.

Finally the case $p_{2N} = 1$ is further specialized. It was shown in (3.9) that $p_{2n-1} \geq 1/2$ for all $1 \leq n \leq N$. If $p_{2N} = 1$ one can again start with the value $n = N$ and show similarly that $p_{2n-1} \leq 1/2$ so that $p_{2n-1} = 1/2$ for $1 \leq n \leq N$. In this situation $\bar{r}_n = r'_n = 0$, $0 \leq n \leq N$, $\bar{q}_0 = q'_0 = 1$ and $\bar{p}_N = p'_N = 1$. The spectral measure ψ is symmetric with $N+1$ support points including -1 and $+1$ (see Skibinsky (1986)).

4. Random walks with the same spectral measure. In this section the set of solutions of (3.3) is described more fully. The random walks with holding probabilities $\bar{r}_n \equiv 0$ for all $n \in E$ are of some special interest. If $\bar{r}_0 = -1 + 2\zeta_1 = 0$ then $p_1 = 1/2$, and $\bar{r}_k = -1 + 2\zeta_{2k} + 2\zeta_{2k+1} = 0$ implies inductively that $p_{2k+1} \equiv 1/2$ whenever the canonical moments are defined. Conversely, if all canonical moments of odd order are 1/2 then $\bar{r}_k \equiv 0$ for all $k \in E$. This, in turn, is equivalent to the spectral measure ψ being symmetric. Equation (3.3) then reduces to

$$\bar{q}_{i-1}\bar{p}_i = q_{2i-2}p_{2i} \qquad i \in E \;\; (q_0 := 1) \tag{4.1}$$

Lemma 4.1. *If $\bar{r}_i \equiv 0$, $\bar{p}_i + \bar{q}_i \equiv 1$, ($\bar{q}_0 = 1$, $\bar{p}_N = 1$ if $N < \infty$) then the canonical moments of the spectral measure ψ satisfy*

$$\bar{p}_{2i-1} = \frac{1}{2}\,, \qquad \bar{p}_i = p_{2i}\,. \tag{4.2}$$

Note that if the measure ψ is symmetric then

$$P_{00}^{2n} = \int_{-1}^{1} x^{2n} d\psi(x) = \int_{0}^{1} y^n d\psi_0(y)$$

where ψ_0 is obtained from ψ via the transformation $y = x^2$. The canonical moments $\tilde{p}_i$ of the measure ψ_0 can be shown to be $\tilde{p}_i = p_{2i}$ (see Dette and Studden (1996)) in which case the canonical moments of the measure ψ_0 are precisely the downward transition probabilities, i.e. $\bar{p}_i = \tilde{p}_i$.

Equations of the form (4.1) are closely related to the theory of chain sequences. See, for example, Wall (1948) or Chihara (1978). A sequence $a_1, a_2, a_3, \ldots$ is called a chain sequence if there exists another sequence $g_0, g_1 \ldots$ such that

$$(4.3) \qquad (1 - g_{i-1})g_i = a_i \qquad i \geq 1$$

where $0 \leq g_i \leq 1$. Here, we will discuss only the case where $0 < g_i < 1$. The sequence $(g_i)_{i \geq 0}$ is called a parameter sequence for the chain sequence. Any chain sequence has a maximal and minimal parameter sequence. The minimal sequence $(m_i)_{i \geq 0}$ is clearly given by choosing $m_0 = 0$ and recursively calculating the other m_i. The quantity g_0 cannot be chosen too large, otherwise the remaining g_i will not be in the interval $(0, 1)$. The maximal parameter sequence $(M_i)_{i \geq 0}$ is given by

$$(4.4) \qquad M_i = 1 - \frac{a_{i+1}|}{|1} - \frac{a_{i+2}|}{|1} - \cdots$$

In the following discussion this expression will be shown to be equal to

$$(4.5) \qquad M_i = 1 - (1 - g_i)(1 - \frac{1}{T_{i+1}})$$

where T_{i+1} is defined as

$$(4.6) \qquad T_{i+1} = 1 + \sum_{\ell=i+1}^{\infty} \prod_{k=i+1}^{\ell} \frac{g_k}{1 - g_k}$$

with the convention to be $+\infty$ if the series diverges. Using (4.5) and observing $T_i = 1 + g_i(1 - g_i)^{-1}T_{i+1}$ it is then easily seen that $(1 - M_{i-1})M_i = a_i$ and that $g_i \leq M_i$; in which case the sequence $(M_i)_{i \geq 0}$ is the maximal parameter sequence. Since the results from the theory of continued fractions, needed to show that (4.4) and (4.5) are the same, are relatively neat and simple we include them here. They are also used in verifying parts of Theorem 4.4 below.

Lemma 4.2. *If $\rho_i > 0$, $1 \le i \le n$, then*

$$(4.7) \qquad \frac{1|}{|1} - \frac{\rho_1|}{|1+\rho_1} - \frac{\rho_2|}{|1+\rho_2} - \cdots - \frac{\rho_n|}{|1+\rho_n} = 1 + \sum_{\ell=1}^{n} \rho_1 \cdots \rho_\ell$$

Proof: The identity follows by noting that

$$s_k(w) = 1 + \rho_k w = \frac{1|}{|1} - \frac{\rho_k|}{|\rho_k} + \frac{1|}{|w}$$

and that both sides of (4.7) are given by $s_1 \circ s_2 \circ \cdots \circ s_n(1)$. ∎

Note that since $\rho_i > 0$, the expression in (4.7) is increasing with n. Therefore, if $n \to \infty$, the limit is well defined if we include the possibility $+\infty$. The expression for the maximal sequence $(M_i)_{i \ge 0}$ in (4.5) is now an immediate consequence of the following corollary.

Corollary 4.3. *If $0 < g_i < 1$, $i \ge 1$, then*

$$(4.8) \qquad S = \frac{1|}{|1} - \frac{g_1|}{|1} - \frac{(1-g_1)g_2|}{|1} - \cdots = 1 + \sum_{\ell=1}^{\infty} \prod_{i=1}^{\ell} \frac{g_i}{(1-g_i)}$$

Proof: Equation (4.8) follows from (4.7) by writing the continued fraction for S as

$$\frac{1|}{|1} - \frac{\rho_1|}{|1+\rho_1} - \frac{\rho_2|}{|1+\rho_2} \cdots$$

where $\rho_i = g_i/(1-g_i)$. ∎

The equations

$$(4.9) \qquad \begin{aligned} \bar{r}_0 &= -1 + 2\zeta_1 \\ \bar{r}_k &= -1 + 2\zeta_{2k} + 2\zeta_{2k+1} \qquad 1 \le k \le N \\ \bar{q}_{k-1}\bar{p}_k &= 4\zeta_{2k-1}\zeta_{2k} \end{aligned}$$

resemble the chain sequence except $\bar{r}_i \ge 0$ and instead of $\bar{p}_i + \bar{q}_i = 1$ we have $\bar{q}_i + \bar{p}_i + \bar{r}_i \le 1$. In order to describe the set of all transition probabilities $\{\bar{p}_n, \bar{q}_n, \bar{r}_n\}_{n \in E}$ satisfying the system of equations in (4.9) define (for a given set of canonical moments)

$$(4.10) \qquad h_j := \frac{2\zeta_{2j+1}\zeta_{2j+2}|}{|1 - \zeta_{2j+2} - \zeta_{2j+3}} - \frac{\zeta_{2j+3}\zeta_{2j+4}|}{|1 - \zeta_{2j+4} - \zeta_{2j+5}} - \cdots$$

$(j = 0, 1, \ldots, N-1)$ and

$$S_k = 1 + \sum_{\ell=k}^{\infty} \prod_{i=k}^{\ell} \frac{p_i}{q_i} \qquad k = 1, 2, \ldots. \tag{4.11}$$

Note that in the case of a finite state space ($N < \infty$) some care is necessary in these definitions. If the sequence of canonical moments terminates with $p_l = 0$ for $l \in \{2N+1, 2N+2\}$ then the continued fraction and the series terminate. If $p_l = 1$ for $l \in \{2N, 2N+1\}$ a simple induction shows for the continued fraction $h_j = 2q_{2j}q_{2j+1}$. In this case we formally define $S_j = \infty$ $(j = 1, 2, \ldots)$ and $1/\infty = 0$. The reason for this convention becomes clear from the following result.

Theorem 4.4. *Let ψ denote a spectral measure on the interval $[-1, 1]$ with canonical moments satisfying (3.11). A random walk on the nonnegative integers with one-step down-, up, and holding transitions probabilities $\bar{p}_n$, $\bar{q}_n$, $\bar{r}_n$ has ψ as its spectral measure if and only if $\{\bar{p}_n, \bar{q}_n, \bar{r}_n\}_{n \in E}$ satisfies (4.9) and*

$$2(q_1 - \frac{1}{S_1}) \;=\; h_0 \le \bar{q}_0 \;\le\; 2q_1 \tag{4.12}$$

$$\begin{aligned} 2(q_{2j}q_{2j+1} - \frac{q_{2j}}{S_{2j+1}}) \;=\; h_j \;\le\; \bar{q}_j \;&\le\; 2\{1 - \zeta_{2j} - \zeta_{2j+1} - \frac{2\zeta_{2j-1}\zeta_{2j}}{\bar{q}_{j-1}}\} \\ &\le\; 2q_{2j}q_{2j+1} \end{aligned} \tag{4.13}$$

holds for all $0 \le j < N$. Moreover, if $\bar{q}_{j_0} = h_{j_0}$ for some $j_0 \in E$, then $\bar{q}_j = h_j$ for all $j_0 \le j < N$.

Proof: We first verify the right hand inequality for $\bar{q}_j$. Note that the first two equations in (4.9) imply $\bar{q}_0 \le 1 - \bar{r}_0 = 2q_1$ and

$$\bar{p}_j + \bar{q}_j \le 1 - \bar{r}_j = 2(1 - \zeta_{2j} - \zeta_{2j+1})$$

so that by the third equation in (4.9)

$$\bar{q}_j \le 2(1 - \zeta_{2j} - \zeta_{2j+1} - \frac{2\zeta_{2j-1}\zeta_{2j}}{\bar{q}_{j-1}}) \qquad j = 1, \ldots, N-1 \tag{4.14}$$

The choice $\bar{q}_0 = 2q_1$ and successively using equality in this bound produces the maximal solution for $\bar{q}_j$ which is $2q_{2j}q_{2j+1}$ for all $1 \le j < N$. Note that, by Proposition 3.4 the odd canonical moments of the spectral measure satisfy $p_{2i+1} \ge 1/2$ which shows $\bar{q}_j \le 1$.

To prove the left hand inequality on $\bar{q}_j$ we use (4.9) and write

$$\bar{q}_j = \frac{4\zeta_{2j+1}\zeta_{2j+2}}{\bar{p}_{j+1}} \ge \frac{4\zeta_{2j+1}\zeta_{2j+2}}{1 - \bar{r}_{j+1}} = \frac{2\zeta_{2j+1}\zeta_{2j+2}}{1 - \zeta_{2j+2} - \zeta_{2j+3}} \;=:\; h_{j,1}. \tag{4.15}$$

Combining (4.15) with (4.14) and replacing j by $j+1$ yields

$$(4.16)\qquad \frac{2\zeta_{2j+3}\zeta_{2j+4}}{1-\zeta_{2j+4}-\zeta_{2j+5}} \le \bar{q}_{j+1} \le 2\{1-\zeta_{2j+2}-\zeta_{2j+3}-\frac{2\zeta_{2j+1}\zeta_{2j+2}}{\bar{q}_j}\}.$$

Looking at the extremes in (4.16) produces

$$\bar{q}_j \ge 2\left\{\frac{\zeta_{2j+1}\zeta_{2j+2}\,|}{|1-\zeta_{2j+2}-\zeta_{2j+3}} - \frac{\zeta_{2j+3}\zeta_{2j+4}\,|}{|1-\zeta_{2j+4}-\zeta_{2j+5}}\right\} =: h_{j,2}$$

Repeating this argument shows that for all $0 \le j < N$ and $k \ge 1$ with $2j+2k \le N$

$$(4.17)\qquad \bar{q}_j \ge 2\left\{\frac{\zeta_{2j+1}\zeta_{2j+2}\,|}{|1-\zeta_{2j+2}-\zeta_{2j+3}} - \cdots - \frac{\zeta_{2j+2k-1}\zeta_{2j+2k}\,|}{|1-\zeta_{2j+2k}-\zeta_{2j+2k+1}}\right\} := h_{j,k}.$$

It is easy to see that $h_{j,k}$ is increasing with k which proves that in the case $N=\infty$ the continued fraction in (4.10) converges and its limit h_j is also a lower bound for $\bar{q}_j$. If the state space is finite the continued fraction in (4.10) terminates because $\zeta_{2N+1}\zeta_{2N+2}=0$. To verify the value for h_j in (4.12) and (4.13) we restrict ourselves to the case of an infinite state space. The result for the finite state space follows by the same argument observing the convention in (4.10) and (4.11) for $N<\infty$. Note that by a standard contraction formula

$$(4.18)\qquad \frac{1\,|}{|1-\zeta_1} - \frac{\zeta_1\zeta_2\,|}{|1-\zeta_2-\zeta_3} - \cdots = \frac{1|}{|1} - \frac{\zeta_1|}{|1} - \frac{\zeta_2|}{|1} - \cdots$$

(see Wall (1948)) so that by Corollary 4.3

$$\frac{1}{1-\zeta_1-h_0/2} = S_1$$

which gives the equality in (4.12). To complete the proof we verify the value for h_1, the other cases are treated similarly by induction. From (4.10) and (4.12) we have

$$h_1 = 2\{1-\zeta_2-\zeta_3-\frac{\zeta_1\zeta_2}{h_0/2}\} = 2\{q_2q_3+p_1p_2-\frac{p_1q_1p_2}{q_1-\frac{1}{S_1}}\},$$

and $S_j = 1+\frac{p_j}{q_j}S_{j+1}$ implies

$$p_1p_2-\frac{p_1q_1p_2}{q_1-\frac{1}{S_1}} = p_1p_2-\frac{p_1p_2}{1-\frac{1}{q_1+p_1S_2}} = p_1p_2\left(1-\frac{q_1+p_1S_2}{p_1(S_2-1)}\right)$$

$$= -\frac{p_2}{(S_2-1)} = -\frac{p_2}{(\frac{p_2}{q_2}S_3)} = -\frac{q_2}{S_3}\,.$$

Combining these identities yields

$$h_1 = 2\{q_2 q_3 - \frac{q_2}{S_3}\}$$

which is the required representation in the case $j = 1$.

Finally, if $h_j = \bar{q}_j$ for $j = j_0$, then (4.10) and (4.13) yields for $j = j_0 + 1$

$$h_{j_0+1} \le \bar{q}_{j_0+1} \le 2\left[1 - \zeta_{2j_0+2} - \zeta_{2j_0+3} - \frac{2\zeta_{2j_0+1}\zeta_{2j_0+2}}{h_{j_0}}\right] = h_{j_0+1}$$

which shows that there is equality also for $j = j_0 + 1 < N$. ∎

Theorem 4.5. *Let $(X_n)_{n\in\mathbb{N}}$ denote a random walk on the nonnegative integers and let ψ denote the corresponding spectral measure of $(X_n)_{n\in\mathbb{N}}$. $(X_n)_{n\in\mathbb{N}}$ is the unique random walk with spectral measure ψ if and only if $(X_n)_{n\in\mathbb{N}}$ is recurrent.*

Proof: If there is to be a unique random walk then the two bounds for $\bar{q}_0$ in (4.12) must be equal. Otherwise one can construct an infinite class of processes with the same spectral measure by choosing $\bar{q}_0 \in [h_0, 2q_1]$ arbitrarily, solving successively (4.9) for $\bar{p}_j$ and putting $\bar{q}_j = 2\{1 - \zeta_{2j} - \zeta_{2j+1} - 2\zeta_{2j-1}\zeta_{2j}/\bar{q}_{j-1}\}$ $(j = 1, 2, \ldots, N-1)$. If $h_0 = 2q_1$ then $S_1 = \infty$ and the proof of Theorem 4.4 shows $\bar{q}_0 = 2q_1$, and $\bar{q}_j = q_{2j}q_{2j+1}$ $(j = 1, 2, \ldots, N-1)$, $\bar{p}_j = 2p_{2j-1}p_{2j}$ $(j = 1, 2, \ldots, N)$. On the other hand $\{\bar{p}_j, \bar{q}_j, \bar{r}_j\}_{j\in E}$ satisfies (4.9) which implies $\bar{p}_j + \bar{q}_j + \bar{r}_j = 1$ $(j = 1, 2, \ldots, N-1,\ \bar{p}_0 = 0)$. Now let $F_0(z)$ denote the generating function of the first return to zero, then $P_0(1) = 1/(1 - F_0(1))$ and recurrence is equivalent to $P_0(1) = \infty$. From (2.3), the contraction (4.18) and Corollary 4.3 we observe $P_0(1) = S_1$ which shows recurrence and proves the assertion. ∎

Corollary 4.6. *A random walk $(X_n)_{n\in\mathbb{N}}$ on the nonnegative integers is determined by its return probabilities $\{P_{00}^n\}_{n=0,1,\ldots}$ from state 0 to state 0 if and only if it is recurrent.*

Remark 4.7. Note that in the case of an infinite state space and no absorption $(p'_j + q'_j + r'_j = 1)$ the condition of recurrence in terms of S_1 can easily be rewritten into the commonly used criterion

$$\text{the random walk is transient if and only if} \qquad \sum_{k=0}^{\infty} \frac{1}{q'_k \pi'_k} < \infty$$

where $\pi_k = (q'_0 \cdots q'_{k-1})/(p'_1 \cdots p'_k)$ $(k \ge 1)$, $\pi_0 = 1$ To see this assume that the random walk is transient $(S_1 < \infty)$ then

$$S_1 = 1 + \sum_{k=1}^{\infty} \frac{p_1 \cdots p_k}{q_1 \cdots q_k} = 1 + \frac{p_1}{q_1} + \sum_{k=1}^{\infty} \left(\frac{p_1 \cdots p_{2k}}{q_1 \cdots q_{2k}} + \frac{p_1 \cdots p_{2k+1}}{q_1 \cdots q_{2k+1}} \right)$$

$$= \frac{2}{q'_0} + 2\sum_{k=1}^{\infty} \frac{p'_1 \cdots p'_k}{q'_0 q'_1 \cdots q'_k} (q_{2k+1} + p_{2k+1}) = 2\sum_{k=0}^{\infty} \frac{1}{q'_k \pi'_k}$$

where we have used (3.8) to go from the first to the second line.

Example 4.8. Consider the (p,q,r) random walk $(X_n)_{n\in I\!N}$ on the nonnegative integers, that is

$$
\begin{aligned}
&p'_j = p \quad (j \geq 1), \quad p'_0 = 0\\
&r'_j = r \quad (j \geq 1), \quad r'_0 = 1-q\\
&q'_j = q \quad (j \geq 0)
\end{aligned}
\tag{4.19}
$$

where p, q, r are positive with $p+q+r=1$. If $p \geq q$ the walk is recurrent and there exists no other random walk with the same return probabilities P^n_{00} from state 0 to state 0. If $q > p$, there exist an infinite class of random walks with these return probabilities. By Theorem 4.4 and (4.9) the upward transition probabilities satisfy for all $j \geq 0$

$$\overline{q}_j \;\geq\; \frac{pq\ |}{|1-r} - \frac{pq\ |}{|1-r} - \frac{pq\ |}{|1-r} - \cdots \;=\; \frac{p+q-|p-q|}{2} = p,$$

while the holding probabilities are given by

$$\overline{r}_j = -1 + 2\zeta_{2j} + \zeta_{2j+1} = r'_j = \begin{cases} 1-q & \text{if } j=0\\ r & \text{if } j \geq 1. \end{cases}$$

Thus we obtain

$$
\begin{cases}
p \leq \overline{q}_0 \leq 2[1-\zeta_1] = 1 - r'_0 = q\\
p \leq \overline{q}_j \leq 2[1-\zeta_{2j} - \zeta_{2j+1} - 2\zeta_{2j-1}\zeta_{2j}/\overline{q}_{j-1}] = p + q - pq/\overline{q}_{j-1}\\
\overline{p}_j = pq/\overline{q}_{j-1} \qquad (j \geq 1)\\
\overline{r}_0 = 1-q, \quad \overline{r}_j = r, \quad (j \geq 1)
\end{cases}
\tag{4.20}
$$

and every random walk $(Y_n)_{n\in I\!N}$ with one-step transition probabilities $(\overline{p}_n, \overline{q}_n, \overline{r}_n)$ satisfying (4.20) has the same return probabilities P^n_{00} from state 0 to state 0 as $(X_n)_{n\in I\!N}$. For example, a two parametric class of such processes is obtained by putting $\overline{q}_2 = p$ which implies by Theorem 4.4 that $\overline{q}_j = p \;\; (j \geq 2)$ and $\overline{p}_j = q \;\; (j \geq 3)$. The holding probabilities are $\overline{r}_0 = 1-q$, $\overline{r}_j = r$, the downward transition probabilities $(\overline{p}_1, \overline{p}_2)$ are determined by $\overline{p}_j = pq/\overline{q}_{j-1}$ $(j = 1, 2)$, while $(\overline{q}_0, \overline{q}_1)$ vary in the two dimensional set

$$\{(s,t) | p \leq s \leq q, \; p \leq t \leq p + q - \frac{pq}{s}\}.$$

Similarly, by putting $q_{j_0} = p$, we obtain a j_0-dimensional class of random walks with the same return probabilities from state 0 to state 0 as $(X_n)_{n\in I\!N}$.

The "extremal" random walks when $q > p$ should also be singled out. The maximal value for $\bar{q}_j$ is q and the resulting random walk is given by (4.19). The minimal value is given by $\bar{q}_j = p$ and the resulting process has

$$\begin{aligned} \bar{p}_j &= q \quad (j \geq 1) \\ \bar{r}_j &= r \quad (j \geq 1) \quad \bar{r}_0 = 1 - q \\ \bar{q}_j &= p \quad (j \geq 0) \end{aligned} \tag{4.21}$$

Note that the process in (4.19) is transient if $q > p$ and has no absorbing state, while the process in (4.21) would be recurrent except for absorption from zero since $\bar{q}_0 + \bar{r}_0 = 1-(q-p) < 1$. The absolutely continuous part of the spectral measure ψ for these processes has the density

$$\frac{1}{2\pi p} \frac{\sqrt{4pq - (x-r)^2}}{1-x} \quad \text{if } (x-r)^2 < 4pq$$

while there is an additional jump of size $1 - q/p$ at the point $x = 1$ if $p \geq q$.

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