SKOROHOD INTEGRAL OF A PRODUCT OF TWO STOCHASTIC PROCESSES

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1 Introduction

The Skorohod integral introduced in [7] is an extension of the Itô stochastic integral which allows us to integrate processes that are not adapted to the underlying Brownian filtration. A stochastic calculus for the Skorohod integral was developed in [3], and it was proved that this integral has properties analogous to those of the classical Itô integral. In particular, a change of variable formula and a local property were obtained.

The Skorohod integral also generalizes the two-sided stochastic integrals introduced in [6]. In this type of integral one considers integrands of the form $\Phi(u,v)$ where u is a diffusion process and v is a backward diffusion. The two-sided integral is defined as the limit of Riemann sums of the form $\sum_i \Phi(u(t_i), v(t_{i+1}))[W(t_{i+1}) - W(t_i)]$, where W is a Brownian motion.

In this note we are interested in the following problem. Suppose that u and v are respectively an adapted and a backward adapted stochastic processes. Under what conditions on u and v is the product $u\,v$ Skorohod integrable? In the next section we will show that a sufficient condition is the boundedness of the familly of random variables $\{u_t, v_t,\ 0 \le t \le 1\}$ in the Sobolev space $\mathbb{D}^{1/2,2}$. This is the main result of this paper and, as an application, it allows as to deduce the Skorohod integrability of the rough stochastic processes introduced in [1].

In Section 3 following the estimates obtained in [2] we provide a nonsymmetric sufficient condition for the product $u\,v$ to be Skorohod integrable.

2 Skorohod integral of a product of two processes

Let us first introduce the basic notation and some preliminaries. We will assume that $W = \{W_t, t \in [0,1]\}$ is a Wiener process defined on the canonical probability space $(\Omega, \mathcal{F}, P)$. That is, $\Omega = C_0[(0,1)]$, $\mathcal{F}$ is the completed σ-field on Ω and P is the Wiener measure. Set $H = L^2([0,1])$ and for any $h \in H$ put $W(h) = \int_0^1 h_t\, dW_t$.

We will denote by $\mathcal{P}$ the class of random variables F of the form

$$F = f(W(h_1), \ldots, W(h_n))\,, \tag{2.1}$$

where f is a polynomial and $h_i \in H,\ 1 \le i \le n$. For a random variable F of the form (2.1) we define its derivative as the random process given by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))\, h_i(t)\,.$$

The Sobolev space $\mathbb{D}^{1,2}$ is the completion of $\mathcal{P}$ by the norm

$$\|F\|_{1,2}^2 = E(F^2) + E\int_0^1 (D_tF)^2\,dt\,. \tag{2.2}$$

In this way the derivative operator D is an unbounded operator with domain $\mathbb{D}^{1,2} \subset L^2(\Omega)$ and values in $L^2(\Omega \times [0,1])$. We will denote by δ its adjoint. The operator δ is called the Skorohod integral and it has a domain $\mathrm{Dom}\,\delta \subset L^2(\Omega \times [0,1])$. It can be proved that $\mathrm{Dom}\,\delta$ contains the square integrable and predictable processes, and on these processes it coincides with the Itô stochastic integral. On the other hand, the space $\mathbb{L}^{1,2} := L^2([0,1];\mathbb{D}^{1,2})$ is included in $\mathrm{Dom}\,\delta$.

Let us denote by L the generator of the Ornstein-Uhlenbeck semigroup. The operator L can be written as $L = \sum_{n=1}^{\infty} -nJ_n$ where J_n denotes the orthogonal projection on the nth Wiener chaos. For any real numbers s and $p > 1$ we define the space $\mathbb{D}^{s,p}$ as the completion of $\mathcal{P}$ by the norm

$$\|F\|_{s,p} = \left\|(I-L)^{s/2}F\right\|_p\,.$$

For $s = 1$, $p = 2$ this definition is consistent with Eq. (2.2), as it follows from the expression of the operator D in terms of the Wiener chaos expansion (see [5]).

For any Borel subset B of $[0,1]$, $\mathcal{F}_B$ will denote the σ-field generated by the family of random variables $\sigma\{W(\mathbb{1}_C), C \subset B,\, C \in \mathcal{B}([0,1])\}$ and the P-null sets. We will say that a stochastic process $u = \{u_t, t \in [0,1]\}$ is adapted (resp. backward adapted) if u_t is $\mathcal{F}_{[0,t]}$-measurable ($\mathcal{F}_{[t,1]}$-measurable) for any $t \in [0,1]$.

Suppose that $u, v \in L^2(\Omega \times [0,1])$ are square integrable processes such that u is predictable and v is backward predictable. We want to study the following problem:

Under what conditions does it hold that $u\,v \in \mathrm{Dom}\,\delta$?

A sufficient condition would be that the product $u\,v \in \mathbb{L}^{1,2}$ and to have this it is enough to assume that $u, v \in \mathbb{L}^{1,2}$ and that

$$\begin{aligned}
\int_0^1 (E(u_t^2))^2\,dt \quad &+ \int_0^1 \left(\int_0^s E\left(|D_tu_s|^2\right)dt\right)^2 ds < \infty\,,\\
\int_0^1 (E(v_t^2))^2\,dt \quad &+ \int_0^1 \left(\int_0^s E\left(|D_sv_t|^2\right)dt\right)^2 ds < \infty\,.
\end{aligned} \tag{2.3}$$

We would like to impose weaker hypotheses on u and v. Assuming (2.3) we can estimate the L^2-norm of the Skorohod integral of the product $u\,v$ as follows:

$$\begin{aligned} E\Big[\delta(u\,v)^2\Big] &= \int_0^1 E(u_t^2)\,E(v_t^2)\,dt + 2\,E\iint_{\{t<s\}} D_t(u_s v_s)\,D_s(u_t v_t)\,dt \\ &= \int_0^1 E(u_t^2)\,E(v_t^2)\,dt + 2\,E\left(\iint_{\{t<s\}} v_s\,D_t\,u_s u_t\,D_s\,v_t\;ds\,dt\right) \\ &= \int_0^1 E(u_t^2)\,E(v_t^2)\,dt + 2\,E\left(\iint_{\{t<s\}} E[v_s\,D_t\,v_t u_t\,D_t\,u_s|\mathcal{F}_{[t,s]}ds\,dt]\right). \end{aligned}$$

Notice that, for $t < s$, $v_s D_s v_t$ is $\mathcal{F}_{[t,1]}$-measurable, $u_t D_t u_s$ is $\mathcal{F}_{[0,s]}$-measurable and the σ-fields $\mathcal{F}_{[0,s]}$ and $\mathcal{F}_{[t,1]}$ are conditionally independent given $\mathcal{F}_{[t,s]}$. Therefore,

$$E\left[v_s D_s v_t \cdot u_t D_t u_s|\mathcal{F}_{[t,s]}\right] = E\left[v_s D_s v_t|\mathcal{F}_{[t,s]}\right]\cdot E\left[u_t D_t u_s|\mathcal{F}_{[t,s]}\right].$$

So, by the Cauchy-Schwarz inequality we obtain:

$$\begin{aligned} E\left(\delta(u\,v)^2\right) \qquad &\leq \left(\int_0^1 [E(u_t^2)]^2\,dt \int_0^1 [E(v_t^2)]^2\,dt\right)^{1/2} \\ &+ 2\left(\iint_{\{t<s\}} E\Big((E(u_t D_t u_s|\mathcal{F}_{[t,s]}))^2\Big)\,ds\,dt\right)^{1/2} \\ &\times \left(\iint_{\{t<s\}} E\Big((E(v_s D_s v_t|\mathcal{F}_{[t,s]}))^2\Big)\,ds\,dt\right)^{1/2} \end{aligned} \tag{2.4}$$

From the estimate (2.4) we are able to deduce our main result. We will set

$$\mathbb{L} := \{u \in L^2(\Omega\times[0,1]) : \int_0^1 (E(u_t^2))^2\,dt < \infty, sup_{t\in[0,1]}\,\|u_t\|_{1/2,2} < \infty\}.$$

Theorem 2.1 *Suppose that u and v are processes which are adapted and backward adapted, respectively, which belong to the class $\mathbb{L}$. Then $u\,v \in$* Dom δ *and we have*

$$\begin{aligned} E\Big((\delta(u\,v))^2\Big) \leq \left[\int_0^1 \Big(E(u_t^2)\Big)^2 dt \int_0^1 \Big(E(v_t^2)\Big)^2 dt\right]^{1/2} \\ +4\sup_{s,t\in[0,1]} \|u_t\|_{1/2,2}^2\;\|v_s\|_{1/2,2}^2\,. \end{aligned}$$

This theorem is a consequence of the estimate (2.4) and the following lemma.

Lemma 2.2 *Let u be an adapted process belonging to the class $\mathbb{L}$. Then for all $s \in [0,1]$ we have*

$$\int_0^s E\left(\left|E\left[u_t D_t u_s | \mathcal{F}_{[t,s]}\right]\right|^2\right) dt \leq 2 \sup_{t\in[0,s]} \|u_t\|_{1/2,2}^4 . \tag{2.5}$$

In the same way, if v is a backward adapted process in the class $\mathbb{L}$ we have for all $t \in [0,1]$

$$\int_t^1 E\left(\left|E\left[v_s D_s v_t | \mathcal{F}_{[t,s]}\right]\right|^2\right) ds \leq 2 \sup_{s\in[t,1]} \|v_t\|_{1/2,2}^4 . \tag{2.6}$$

Proof:

We will only consider the forward adapted case. Using the Wiener chaos expansion we can write

$$u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) = \sum_{n=0}^{\infty} n! \int_{\{t_1<\ldots<t_n<t\}} f_n(t_1,\ldots,t_n,t)\, dW_{t_1}\ldots dW_{t_n} ,$$

and $D_t u_s = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(\cdot,t,s))$.
Then,

$$\begin{aligned}
&E\left[u_t D_t u_s | \mathcal{F}_{[t,s]}\right] \\
&= E\Bigg(\Big(\sum_{m=0}^{\infty} m! \int_{\{t_1<\ldots<t_m<s\}} f_m(t_1,\ldots,t_m,t,s) dW_{t_1}\ldots dW_{t_m}\Big) \\
&\quad \times \Big(\sum_{n=1}^{\infty} n! \sum_{k=0}^{n-1} \int_{\{t_1<\ldots<t_k<t<t_{k+1}<\ldots<t_{n-1}<s\}} f_n(t_1,\ldots,t_{n-1},t,s) \\
&\quad \times dW_{t_1}\ldots dW_{t_{n-1}}\Big) \Big| \mathcal{F}_{[t,s]}\Bigg) \\
&= \sum_{n=1}^{\infty} n!\, k! \sum_{k=0}^{n-1} \int_{\{t_1<\ldots<t_k<t<t_{k+1}<\ldots<t_{n-1}<s\}} f_k(t_1,\ldots,t_k,t) \\
&\quad \times f_n(t_1,\ldots,t_{n-1},t,s) dt_n \ldots dt_k\, dW_{t_{k+1}}\ldots dW_{t_{n-1}}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} n! \Big\langle f_k(\cdot, t),$$

$$\sum_{n=k+1}^{\infty} \int_{\{t<t_{k+1}<\ldots<t_{n+1}<s\}} f_n(\cdot, t_{k+1}, \ldots, t_{n-1}, t, s)\, dW_{t_{k+1}} \ldots dW_{t_{n-1}} \Big\rangle$$

$$\leq \Big\{ \sum_{k=0}^{\infty} \sqrt{k+1}k! \|f_k(\cdot, t)\|^2 \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}k!} \Big\| \sum_{n=k+1}^{\infty} n!$$

$$\times \int_{\{t<t_{k+1}<\ldots<t_{n-1}<s\}} f_n(\cdot, t_{k+1}, \ldots, t_{n-1}, t, s) dW_{t_{k+1}} \ldots dW_{t_{n-1}} \Big\|^2 \Big\}^{1/2}$$

$$= \left(E\left(|(I-L)^{\frac{1}{4}} u_t|^2 \right) \right)^{1/2} \Big(\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}k!} \Big\| \sum_{n=k+1}^{\infty} n!$$

$$\int_{\{t<t_{k+1}<\ldots<t_{n-1}<s\}} f_n(\cdot, t_{k+1}, \ldots, t_{n-1}, t, s)\, dW_{t_{k+1}} \ldots dW_{t_{n-1}} \Big\|^2 \Big)^{1/2}.$$

Therefore

$$E\left(|E(u_t D_t u_s | \mathcal{F}_{[t,s]})|^2 \right) \leq E\Big(|(I-L)^{1/4} u_t|^2 \Big) \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}k!}$$
$$\times \sum_{n=k+1}^{\infty} (n!)^2 \int_{[0,t]^k \times \{t<t_{k+1}<\ldots<t_{n-1}<s\}} f_n^2(t_1, \ldots, t_{n-1}, t, s)\, dt_1 \ldots dt_{n-1}.$$

The second factor of the above expression can be written as

$$\sum_{n=1}^{\infty} n! \sum_{k=0}^{n-1} \left(\int_{[0,t]^k \times [t,s]^{n-k-1}} f_n^2(t_1, \ldots, t_{n-1}, t, s)\, dt_1 \ldots dt_{n-1} \right)$$

$$\times \frac{n!}{\sqrt{k+1}k!(n-k-1)!}$$

$$= \sum_{n=1}^{\infty} nn! \int_{[0,s]^{n-1}} f_n^2(t_1, \ldots, t_{n-1}, t, s) \frac{1}{\sqrt{k_t(t_1, \ldots, t_{n-1})+1}}\, dt_1 \ldots dt_{n-1},$$

where $k_t(t_1, \ldots, t_{n-1}) = \#\{i : t_i < t\}$.

So,

$$\int_0^s E\Big(|E(u_tD_tu_s)|\mathcal{F}_{[t,s]}|^2\Big)\,dt \;\le\; \sup_{0\le t\le s} E\left(|(I-L)^{\frac{1}{4}}\,u_t|^2\right)$$

$$\times\sum_{n=1}^{\infty} n\,n!\int_{[0,s]^n} f_n^2(t_1,\dots,t_n,s)\frac{1}{\sqrt{k_{t_n}(t_1,\dots,t_{n-1})+1}}dt_1\dots dt_n\,.$$

The symmetrization of $(t_1,\dots,t_n)\longmapsto\dfrac{1}{\sqrt{k_{t_n}(t_1,\dots,t_{n-1})+1}}$ yields the constant

$$\frac{1}{n}\Big(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}\Big)\le\frac{1}{n}\int_0^n\frac{1}{\sqrt{x}}\,dx=\frac{2}{\sqrt{n}}\,.$$

As a consequence, the above summation is bounded by

$$\sum_{n=1}^{\infty} 2n!\,\sqrt{n}\,\|f_n(\cdot,s)\|^2=2\,E\Big(|(-L)^{\frac{1}{4}}u_s|^2\Big)$$

and we obtain

$$\int_0^s E\Big(|E(u_tD_tu_s|\mathcal{F}_{[t,s]})|^2\Big)\,dt\le\;2\sup_{0\le t\le 1}\left(E\left(|(I-L)^{\frac{1}{4}}\,u_t|^2\right)\right)^2,$$

which allows to conclude the proof. □

As an example of an adapted process u_t satisfying the conditions of Theorem 2.1 and which does not belong to $\mathbb{L}^{1,2}$ let us mention the local time $L(x,t)$ at a given point $x\in\mathbb{R}$ (see [4]). Sometimes it is possible to estimate directly the left side of Eq. (2.6) as is the case in the following example.

Example: Suppose that $u_t=f(W_t)$ where $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a bounded measurable function with bounded variation. That is, f' is a finite signed measure μ. In this case we have

$$\begin{aligned}
E\Big(u_tD_tu_s|\mathcal{F}_{[t,s]}\Big) &= E\Big(f(W_t)\,f'(W_s)|\mathcal{F}_{[t,s]}\Big)\\
&= \int_{\mathbb{R}} f(x)f'(x+W_s-W_t)\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx\\
&= \int_{\mathbb{R}} f(y-W_s-W_t)f'(y)\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(y-W_s+W_t)^2}{2t}}\,dy\\
&= \int_{\mathbb{R}} f(y-W_s+W_t)\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(y-W_s+W_t)^2}{2t}}\,\mu(dy)\,.
\end{aligned}$$

The preceding computations are formal and they can be made rigorous using distribution theory. Consequently,

$$\left|E\left(u_t D_t u_s | \mathcal{F}_{[t,s]}\right)\right| \le \|f\|_\infty \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{(y-W_s+W_t)^2}{2t}} |\mu|(dy)$$

and

$$E\left(|E(u_t D_t u_t | \mathcal{F}_{[t,s]})|^2\right) \le \|f\|_\infty^2 \frac{1}{2\pi t} |\mu|(\mathbb{R}) \int_{\mathbb{R}} E\left(e^{-\frac{(y-W_s+W_t)^2}{t}}\right) |\mu|(dy)$$
$$= \frac{\|f\|_\infty^2 |\mu|(\mathbb{R})}{2\pi t\sqrt{2\pi(s-t)}} \int_{\mathbb{R}^2} \exp\left(-\frac{(y-x)^2}{t} - \frac{x^2}{2(s-t)}\right) dx\, |\mu|(dy)$$
$$\le \frac{\|f\|_\infty^2 |\mu|(\mathbb{R})^2}{\pi\sqrt{2}\sqrt{t(s-t)}},$$

which is integrable in $\{t < s\}$.

One can use the above example to show that $\int_0^1 \operatorname{sign}(W_t) \operatorname{sign}(W_1 - W_t) dW_t$ exists and to find estimates for its L^2 norm. This stochastic integral has been studied in [1].

3 Skorohod integrability of uv for an arbitrary adapted process u

In this section we will assume that $u = \{u_t, t \in [0,1]\}$ is an adapted and bounded process. The results of [2] provide a class of processes v such that $u\,v \in \operatorname{Dom}\delta$. The main idea is the following equality which is a consequence of the isometry of the Skorohod integral and the duality relationship:

$$E\left[(\delta(uv))^2\right] = E\int_0^1 u_t^2 v_t^2\, dt + 2\,E\left(\int_0^1 u_t v_t \left(\int_0^t (D_t u_s) u_s dW_s\right)\right) dt$$
$$\le \|uv\|^2 + 2\|uv\|\left(\int_0^1 E\left(|\int_0^t (D_t v_s) u_s dW_s|^2\right) dt\right)^{1/2}, \quad (3.1)$$

where $\|\cdot\|$ denotes the norm in the space $L^2(\Omega \times [0,1])$.
Define

$$\Phi(v) = \sup_{n\ge 0} \left(\frac{1}{n!}\int_0^1 \|D^n v_s\|^2_{L^2(\Omega\times[0,1]^n)} ds\right)^{1/2}.$$

Then iterating the inequality (3.1) one obtains (cf. Proposition 2.2 of [2]):

$$E\left[(\delta(u\,v))^2\right] \leq \left(1+\sqrt{2}\right)^2 \|u\|_\infty^2 \, \Phi(v)^2 . \tag{3.2}$$

As a consequence we obtlain the following result:

Theorem 3.1 *Let $u = \{u_t, t \in [0,1]\}$ be a predictable and bounded process. Then for any process $v \in \cap_{n\geq 1} L^2([0,1];\mathbb{D}^{n,2})$ such that $\Phi(v) < \infty$ we have that the product uv belongs to* $\mathrm{Dom}\,\delta$.

Examples of processes v verifying $\Phi(v) < \infty$ are the processes such that

$$\sup_n \sum_{m=n}^{\infty} \binom{m}{n} \int_0^1 E\left(|J_m v_s|^2\right) ds < \infty .$$

If we assume that the predictable process u satisfies $\int_0^1 (E(u_s^2))^2\, ds < \infty$ then we obtain the following estimation

$$E\left[(\delta(u\,v))^2\right] \leq \left(1+\sqrt{2}\right)^2 \left(\int_0^1 \left(E(u_s^2)\right)^2\right)^{1/2} \Phi_1(v)^2$$

where

$$\Phi_1(v) = \sup_{n\geq 0} \left(\frac{1}{n!}\int_0^1 \|D^n v_s\|^4_{\mathbb{L}^2(\Omega\times[0,1]^n)}\, ds\right)^{1/2} .$$

As a consequence, we have $uv \in \mathrm{Dom}\,\delta$ provided $\Phi_1(v) < \infty$.

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