A PURELY PROBABILISTIC METHOD FOR FINDING DETERMINISTIC SUMS

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Abstract

Consider an infinite series with positive terms which is summable. We give a method for finding the numerical value of the sum, to any degree of accuracy, by using only a simulated sample from an appropriate multivariate normal distribution. This is done by identifying the required sum as the trace of a kernel of an integral operator and using the Karhunen-Loeve expansion of an associated Gaussian process. An appropriate SLLN is proved which guarantees that the partial sums of the Gaussian process converge almost surely to the trace of the kernel. Examples are provided for the Brownian bridge and motion, the Riemann zeta function and for the eigenvalues of the logarithm of the Poisson kernel for Harmonic functions.

1. Introduction.

The purpose of this note is to point out a method of solving a purely deterministic problem by using purely probabilistic methods. It should therefore be viewed as a result on connections between various branches of mathematics, some of which are far from clear until pointed out. The problem we look at is that of finding the sum of an infinite series of positive numbers. As such, it is a problem purely in the domain of analysis. We use facts from the theory of Fredholm integral equations, from the theory of Gaussian processes and from the theory of numerical analysis to give a method for finding this sum by only simulating from a finite dimensional multivariate normal distribution. Our method essentially shows that a certain partial sum of samples from appropriate normal distributions converges almost surely to the deterministic sum. The dimension of the appropriate multivariate normal distribution depends on the accuracy with which one wants to find the sum and since in real life only finite simulations can be done, the deterministic sum can only be found numerically (to any degree of accuracy) by this method.

Section 2 gives the heuristic of the proof and then the formal proof itself; as is clear, the result is essentially a strong law. Section 3 gives a number of examples. Some brief remarks are made in Section 4 and Section 5 is a technical appendix.

2. 2.1. Notation.

$\{a_k\}$ will denote a summable positive sequence; note that it follows that $\{a_k\}$ is also square summable. X_t will denote a real valued Gaussian process, and $C(s,t)$ its covariance function. The variance $C(t,t)$ of X_t will also be denoted by $\sigma^2(t)$. The time interval is assumed to be $[0,1]$. For $m, n \geq 1$, we will also use the notation

$$Y_{i,m} = \frac{1}{m}\sum_{j=1}^{m} X^2(i, \frac{j}{m})$$

where $\{X(i, j/m)\}$, $i = 1, 2, \ldots, n$ are independent replications of the time evolution $(X_{\frac{1}{m}}, X_{\frac{2}{m}}, \ldots, X_1)$. Finally, we will use the notation

$$\mu_m = \frac{1}{m}\sum_{j=1}^{m} C(j/m, j/m).$$

2.2. The Heuristic.

The heuristic comes from recognizing that the sequence $\{a_k\}$ forms the eigenvalues of a kernel $C(s,t)$ defined as

$$C(s,t) = \Sigma a_k p_k(s) p_k(t)$$

with orthonormal eigenfunctions $\{p_k\}$.

Therefore, $\int C(t,t)dt$ should equal the infinite sum Σa_k. If now X_t is a real Gaussian process with covariance kernel $C(s,t)$, then, by the usual strong law $1/n \sum_{i=1}^{n} Y_{i,m}$ should be approximately equal to μ_m for large n, and this should be approximately equal to the Riemann integral of $C(t,t)$ for large m, and hence approximately equal to the infinite sum we want to evaluate. The proof rigorizes this heuristic. Notice that the Riemann integral of $C(t,t)$ can be approximated by many other numerical methods such as the Simpson rule, and these would correspond to a different kind of partial sum constituted from the process X_t in order to approximate the required infinite sum Σa_k; see Powell (1981) for a catalog of such numerical integration procedures.

2.3. Main Result.

Theorem 1. Let X_t be a real Gaussian process with covariance kernel $C(s,t)$. Assume $\sigma^8(t) = C^4(t,t)$ is integrable on $[0,1]$. Let $\{1/a_k\}$ be the eigenvalues of $C(s,t)$ with corresponding eigenfunctions $\{p_k\}$. Then,

$$\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} X^2(i,\frac{j}{m}) \to \Sigma a_k \text{ a.s., as } m,n \to \infty.$$

Remark. In actual applications, one would start with the sequence $\{a_k\}$ and construct a kernel $C(s,t) = \Sigma a_k p_k(s) p_k(t)$; the orthonormal sequence $\{p_k\}$ should be judiciously chosen such that $C(s,t)$ can be found in closed form. A closed form is necessary because the subsequent simulation will require the covariance structure exactly specified. We will see a number of examples.

Proof of Theorem 1. First, note that since $C(t,t)$ is Riemann integrable,

$$E(Y_{i,m}) = \frac{1}{m}\sum_{j=1}^{m} C(\frac{j}{m},\frac{j}{m}) = \mu_m \to \mu = \Sigma a_k \text{ as } m \to \infty.$$

We assume m is such that $|\mu_m - \mu| \leq \varepsilon$, when $\varepsilon > 0$ is arbitrary but fixed. The proof of the theorem rests on showing that $P(|\frac{1}{n}\sum_{i=1}^{n} Y_{i,m} - \mu| > 2\varepsilon)$ is $O(\frac{1}{n^2})$ and therefore, with probability 1, $|\frac{1}{n}\sum_{i=1}^{n} Y_{i,m} - \mu| \leq 2\varepsilon$ for all but finitely many values of n. Since $\varepsilon > 0$ is arbitrary, that will complete the proof. In the following, also assume m is such that $\frac{1}{m}\sum_{j=1}^{m} C^4(\frac{j}{m},\frac{j}{m}) \leq \int_0^1 C^4(t,t)dt + 1 (< \infty)$. Now,

$$\begin{aligned}
P(\frac{1}{n}\sum_{i=1}^{n} Y_{i,m} - \mu > 2\varepsilon) &= P(\frac{1}{n}\sum_{i=1}^{n}(Y_{i,m} - \mu_m) + \mu_m - \mu > 2\varepsilon) \\
&\leq P(\frac{1}{n}\sum_{i=1}^{n}(Y_{i,m} - \mu_m) > \varepsilon) \\
&\leq \frac{E\{\sum_{i=1}^{n}(Y_{i,m} - \mu_m)\}^4}{n^4\varepsilon^4}
\end{aligned}$$

$$= \frac{E\sum_{i=1}^{n}(Y_{i,m}-\mu_m)^4 + 6\sum\sum_{i<j=2}^{n}(Y_{i,m}-\mu_m)^2(Y_{j,m}-\mu_m)^2}{n^4\varepsilon^4}$$

$$= \frac{nE(Y_{1,m}-\mu_m)^4 + 3n(n-1)(E(Y_{1,m}-\mu_m)^2)^2}{n^4\varepsilon^4}$$

$$\leq \frac{(3n^2-2n)E(Y_{1,m}-\mu_m)^4}{n^4\varepsilon^4}$$

$$= \frac{(3n^2-2n)E\left[\frac{1}{m}\sum_{j=1}^{m}\{X^2(\frac{j}{m})-\sigma^2(\frac{j}{m})\}\right]^4}{n^4\varepsilon^4}$$

$$\leq \frac{(3n^2-2n)E\sum_{j=1}^{m}\{X^2(\frac{j}{m})-\sigma^2(\frac{j}{m})\}^4}{mn^4\varepsilon^4}$$

$$= \frac{60(3n^2-2n)\sum_{j=1}^{m}\sigma^8(\frac{j}{m})}{mn^4\varepsilon^4}$$

$$= \frac{60(3n^2-2n)}{n^4\varepsilon^4}\frac{1}{m}\sum_{j=1}^{m}C^4\left(\frac{j}{m},\frac{j}{m}\right)$$

$$\leq \frac{1}{\varepsilon^4}60\left(\int_0^1 C^4(t,t)dt+1\right)\frac{(3n^2-2n)}{n^4};$$

$P(\frac{1}{n}\sum_{i=1}^{n}Y_{i,m}-\mu<-2\varepsilon)$ is handled similarly. This completes the proof.

A moment argument as above for the usual strong law can be seen in Ross (1994).

3. Examples. 1. Brownian Bridge. Consider the summable sequence $a_k = \frac{1}{(k\pi)^2}$; these are the reciprocals of the eigenvalues of the Triangular kernel

$$\begin{aligned} C(s,t) &= t(1-s), \quad && \text{if } t\leq s \\ &= s(1-t), && \text{if } s\leq t, \end{aligned}$$

with associated eigenfunctions $p_k(s) = \sqrt{2}\ \sin(k\pi s)$; see Tricomi (1985). The infinite sum Σa_k equals 1/6.

The following is a report of simulations for approximating this sum by the partial sum of Theorem 1.

m	n	Partial Sum	Error
10	1000	.167436	.4616%
50	1000	.167127	.2762%
50	2000	.164292	−1.4248%

Notice the interesting fact that increasing n (the number of simulations) did not do any good if $1/m$ (the fineness of the grid) was already small.

2. The Riemann Zeta Function. Consider the sequence $a_k = 1/k^4$; the infinite sum Σa_k equals $\zeta(4) = \pi^4/90$, where $\zeta(\cdot)$ denotes the Riemann Zeta function. The choice of 4 in the argument is completely arbitrary.

Choose the specific orthonormal sequence $p_k(s) = \sqrt{2}\sin(k\pi s)$. For this choice of an orthonormal sequence, the kernel

$$C(s,t) = \Sigma a_k p_k(s) p_k(t)$$

can be evaluated in a closed form. Indeed,

$$C(s,t) = \frac{\pi^4}{6}[2st(1-s)(1-t) - (s-t)^2(1-s \vee t)s \wedge t]$$

where '$\wedge$' and '$\vee$' denote minimum and maximum respectively; see the appendix for a derivation. Again, some simulations give the following:

m	n	Partial Sum	Error
5	5000	1.0932	1.005%
10	3000	1.07823	−.3778%
10	5000	1.11138	2.6854%

3. The Poisson Kernel. Consider the sequence $a_k = r^k/k$, where $0 < r < 1$. The infinite sum Σa_k equals $\log(1/(1-r))$.

Consider the orthonormal sequence

$$p_k(s) = \sqrt{2}\ \sin(k\pi s)$$

With this choice, the kernel $C(s,t)$ equals

$$C(s,t) = \frac{1}{2}\log\frac{1 - 2r\cos(\pi(s+t)) + r^2}{1 - 2r\cos(\pi(s-t)) + r^2};$$

again see the appendix for a derivation. In particular, for $r = \frac{1}{2}$, one has

$$C(s,t) = \frac{1}{2}\log\frac{\frac{5}{4} - \cos(\pi(s+t))}{\frac{5}{4} - \cos(\pi(s-t))}.$$

Here are some simulations:

m	n	Partial Sum	Error
6	5000	.68503	−1.17%
10	2000	.703330	1.47%
20	5000	.690885	−.326%

4. Standard Brownian Motion. Consider the covariance kernel $C(s,t) = \min(s,t)$ having eigenvalues (proportional to) $1/a_k = (k-1/2)^2$ and associated eigenfunctions $p_k(s) = \frac{\sqrt{2}}{\pi} \sin\left((k-\frac{1}{2})\pi s\right)$. Note that the sum Σa_k equals .5.

Some simulation results follow:

m	n	Partial Sum	Error
20	100	.542293	8.4586%
50	1000	.502123	.4246%
100	10000	.509721	1.9442%

4. Concluding Remarks. We are not recommending our method for serious practical use; it is the probabilistic connection that is interesting. Finally,all of our simulation results indicate that if m is about 20 and n about 3000, then one already gets a fairly accurate approximation to the exact sum.

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5. Appendix.

i.

$$\begin{aligned} C(s,t) &= \Sigma a_k p_k(s) p_k(t) \\ &= 2\Sigma \frac{1}{k^4} \sin(k\pi s) \sin(k\pi t) \\ &= \Sigma \frac{1}{k^4} [\cos k\pi(s-t) - \cos k\pi(s+t)] \\ &= \frac{\pi^4}{12}(s+t)^2 - \frac{\pi^4}{12}(s+t)^3 + \frac{\pi^4}{48}(s+t)^4 \\ &- \frac{\pi^4}{12}(s-t)^2 + \frac{\pi^4}{12}(s-t)^3 - \frac{\pi^4}{48}(s-t)^4, \end{aligned}$$

(using $\Sigma \frac{\cos\ kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{n^4}{48}$ for $0 \leq x \leq 2\pi$; see Gradshteyn and Ryzhik (1980)) which on algebra reduces to

$$C(s,t) = \frac{\pi^4 (1-s)t}{6}(2s - s^2 - t^2),$$

for $s \geq t$.

ii.

$$\begin{aligned} C(s,t) &= 2\Sigma \frac{r^k}{k} \sin\ (k\pi s) \sin\ (k\pi t) \\ &= \Sigma \frac{r^k}{k}[\cos\ k\pi(s-t) - \cos\ k\pi(s+t)] \\ &= \log \frac{1}{\sqrt{1 - 2r\ \cos\ \pi(s-t) + r^2}} - \log \frac{1}{\sqrt{1 - 2r\ \cos\ \pi(s+t) + r^2}} \end{aligned}$$

(using $\Sigma \frac{r^k\ \cos\ kx}{k} = \log \frac{1}{\sqrt{1 - 2r\ \cos x + r^2}}$; see Gradshteyn and Ryzhik (1980))

$$= \frac{1}{2} \log \frac{1 - 2r\ \cos\ \pi(s+t) + r^2}{1 - 2r\ \cos\ \pi(s-t) + r^2},$$

which equals

$$C(s,t) = \frac{1}{2} \log \frac{5 - 4\ \cos\ \pi(s+t)}{5 - 4\ \cos\ \pi(s-t)}$$

for $r = \frac{1}{2}$.

References

Gradshteyn, I. S. and Ryzhik, I. M. (1980). Table of integrals, series, and products, Academic Press, New York.

Powell, M. J. D. (1981). Approximation theory and methods, Cambridge University Press, New York.

Ross, S. M. (1994). A first Course in Probability, Macmillan, New York.

Tricomi, F. G. (1985). Integral Equations, Dover Publications, New York.