ON THE CONVERGENCE RATE TO NORMALITY OF LATIN HYPERCUBE SAMPLING U-STATISTICS

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This article is concerned with a class of generalized Latin hypercube sampling U-statistics having bounded kernels. In particular it is shown that the rate of normal convergence of these U-statistics is of the order $O(n^{-1/2})$.

1 Introduction

Latin hypercube sampling has generated considerable interest among researchers in the area of computer experiment sampling designs since its introduction in 1979 by McKay, Conover and Beckman [see Chapter 5 of Fang and Wang (1994) for a very nice review]. It has the property of separately stratifying on each input dimension and has been shown by, for example, Stein (1987) and Owen (1992) to be an attractive alternative to simple random sampling especially in high dimensions. More precisely for positive integers d and n let

(i) π_k, $1 \leq k \leq d$, be random permutations of $\{1, \ldots, n\}$ each uniformly distributed over all $n!$ possible permutations,

(ii) $\xi_{i_1,\ldots,i_d,j}$, $1 \leq i_1, \ldots, i_d \leq n$, $1 \leq j \leq d$, be uniform $[0,1]$ random variables and

(iii) that the $\xi_{i_1,\ldots,i_d,j}$'s and π_k's are all stochastically independent.

A Latin hypercube sample of size n (taken from the d-dimensional unit hypercube $[0,1]^d$) is defined to be $\{X(\pi_1(i), \ldots, \pi_d(i)) : 1 \leq i \leq n\}$ where for all $1 \leq i_1, \ldots, i_d \leq n$,

$$\begin{aligned} X_j(i_1, \ldots, i_d) &= (i_j - \xi_{i_1,\ldots,i_d,j})/n, \quad \forall 1 \leq j \leq d, \\ X(i_1, \ldots, i_d) &= (X_1(i_1, \ldots, i_d), \ldots, X_d(i_1, \ldots, i_d))'. \end{aligned}$$

Let $f : [0,1]^d \to \mathcal{R}$ be a measurable function. This article is concerned with the asymptotic behavior of the following class of generalized Latin hypercube sampling U-statistics, namely for $1 \leq k \leq d$ and $n \geq 2$,

$$(1) \qquad U_{k,n} = (2/n) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{\pi_k(i),\pi_k(j)} h(f \circ X(\pi_1(i), \ldots, \pi_d(i)), f \circ X(\pi_1(j), \ldots, \pi_d(j))),$$

where $\{w_{i,j} : 1 \leq i, j \leq n\}$ is a sequence of nonnegative constants satisfying $w_{i,i} = 0$, $w_{i,j} = w_{j,i}$ $\forall 1 \leq i, j \leq n$ and $h : \mathcal{R}^2 \to \mathcal{R}$ is a symmetric kernel, that is $h(s,t) = h(t,s)$ for all $s, t \in \mathcal{R}$.

The rest of the article is organized as follows. Theorem 1 of Section 2 establishes conditions where the rate of normal convergence of the U-statistics is of the order $O(n^{-1/2})$. Section 3 gives an example in which these U-statistics are applicable. A number of conditional characteristic function bounds and asymptotic expansions (Theorem 2) are given in Section 4. These bounds are needed in the proof of Theorem 1. In particular Theorem 2(b) is motivated by a result of von Bahr (1976) and it extends his result in two ways. Firstly it essentially generalizes von Bahr's result from $d = 2$ to arbitary but fixed d. Secondly von Bahr's characteristic function expansion is valid up to $O(n^{-1/2})$ whereas the present expansion is valid up to $O(n^{-1})$. The detailed proof of Theorem 1 can be found in Section 5.

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2 Convergence rate to normality

Let $U_{k,n}$ be as in (1) and for simplicity, we shall from now on write for all $1 \leq i \neq j \leq n$, $\bar{\pi}(i) = (\pi_1(i), \ldots, \pi_d(i))$ and

$$g \circ X(\bar{\pi}(i)) = E[w_{\pi_k(i),\pi_k(j)} h(f \circ X(\bar{\pi}(i)), f \circ X(\bar{\pi}(j))) | X(\bar{\pi}(i))] - E w_{\pi_k(i),\pi_k(j)} h(f \circ X(\bar{\pi}(i)), f \circ X(\bar{\pi}(j))).$$

Then $U_{k,n}$ has the following Hoeffding-type decomposition [see Hoeffding (1948)].

$$U_{k,n} - EU_{k,n} = \frac{2(n-1)}{n} \sum_{i=1}^{n} g \circ X(\bar{\pi}(i)) + \frac{2}{n} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \psi(X(\bar{\pi}(i)), X(\bar{\pi}(j))),$$

where for all $1 \leq i \neq j \leq n$,

$$\begin{aligned} & \psi(X(\bar{\pi}(i)), X(\bar{\pi}(j))) \\ = \; & w_{\pi_k(i),\pi_k(j)} h(f \circ X(\bar{\pi}(i)), f \circ X(\bar{\pi}(j))) - E[w_{\pi_k(i),\pi_k(j)} h(f \circ X(\bar{\pi}(i)), f \circ X(\bar{\pi}(j)))] \\ & - g \circ X(\bar{\pi}(i)) - g \circ X(\bar{\pi}(j)). \end{aligned} \tag{2}$$

We further define for $1 \leq i_1, \ldots, i_d \leq n$,

$$\begin{aligned} \mu_{-j}(i_j) &= (1/n)^{d-1} \sum_{k \neq j} \sum_{i_k=1}^{n} E g \circ X(i_1, \ldots, i_d), \quad \forall 1 \leq j \leq d, \\ Y(i_1, \ldots, i_d) &= \frac{2(n-1)}{n} \{ g \circ X(i_1, \ldots, i_d) - \sum_{j=1}^{d} \mu_{-j}(i_j) \}, \\ \sigma_n^2 &= (1/n)^{d-1} \sum_{1 \leq i_1, \ldots, i_d \leq n} E Y^2(i_1, \ldots, i_d). \end{aligned} \tag{3}$$

The following theorem is the main result of this paper.

Theorem 1 *Let $U_{k,n}$ be as in (1). Suppose there exists a constant γ such that for sufficiently large n,*

$$\max\{ \sup_{1 \leq i < j \leq n} n^{\varepsilon_n} w_{i,j}, \max_{1 \leq i \leq n} \sum_{j=1}^{n} w_{i,j}, \sup_{s,t \in \mathcal{R}} |h(s,t)| \} \leq \gamma, \quad \text{and} \quad \liminf_{n \to \infty} n\sigma_n^2 > 0, \tag{4}$$

where

$$\varepsilon_n = \frac{1}{2} + \frac{2 \log(\log n - 2 \log\log n)}{\log n}. \tag{5}$$

Then as $n \to \infty$,

$$\sup\{ |P((U_{k,n} - EU_{k,n})/\sigma_n \leq t) - \Phi(t)| : -\infty < t < \infty \} = O(n^{-1/2}),$$

where Φ denotes the distribution function of the standard normal distribution.

3 An application

If the observations were replaced by an i.i.d. sample and $w_{i,j} = 1/(n-1)$, $\forall 1 \leq i < j \leq n$ in (1), then $U_{k,n}$ reduces to the type of U-statistics of degree two most commonly encountered [see for example Helmers and van Zwet (1982) and the references cited therein].

The following gives an instance in which the generalized Latin hypercube sampling U-statistics of this article are of interest. Indeed it is this example which first motivated the present work. In many computer experiments, we are interested in estimating $\mu = \int_{[0,1]^d} f(x)dx$. Letting $\{f \circ X(\pi_1(i), \ldots, \pi_d(i)) : 1 \leq i \leq n\}$ be as in (1), we observe that $\hat{\mu}_n = (1/n)\sum_{i=1}^n f \circ X(\bar{\pi}(i))$ is an unbiased estimator of μ. Writing $g_k(x_k) = \int_{[0,1]^{d-1}} f(x)\prod_{j\neq k} dx_j$ and using the ANOVA decomposition

$$f(x) = \sum_{k=1}^{d} g_k(x_k) - (d-1)\mu + f_{rem}(x), \quad \forall x = (x_1, \ldots, x_d)' \in [0,1]^d,$$

Stein (1987) showed that $Var(\hat{\mu}_n) = (1/n)\int_{[0,1]^d} f_{rem}^2(x)dx + o(1/n)$ as $n \to \infty$ if $\int_{[0,1]^d} f^2(x)dx < \infty$. The problem of estimating $\int_{[0,1]^d} f_{rem}^2(x)dx$ was studied by Owen (1992). Observing that

$$\int_{[0,1]^d} f_{rem}^2(x)dx = \int_{[0,1]^d} f^2(x)dx + (d-1)\mu^2 - \sum_{k=1}^{d}\int_0^1 g_k^2(x_k)dx_k,$$

he proposed a class of nearest neighbor estimators for $\theta_k = \int_0^1 g_k^2(t)dt$ and essentially showed that these estimators are $n^{1/2}$ consistent provided the following Lipschitz condition holds: there exists a constant M such that $|g_k(s) - g_k(t)| \leq M|s-t|$ for all $s, t \in [0,1]$. Loh (1994) undertook a more detailed study of estimating θ_k under weaker smoothness assumptions on g_k using observations $\{X_k(\bar{\pi}(i)), f \circ X(\bar{\pi}(i)) : 1 \leq i \leq n\}$. In particular the following class of generalized nearest neighbor estimators $\hat{\theta}_{k,n}$ was proposed for estimating θ_k where

$$\hat{\theta}_{k,n} = (1/n)\sum_{i=1}^{n}\sum_{j\neq i}[2w^*_{\pi_k(i),\pi_k(j)} - \sum_{l\neq i,j}\tilde{w}^*_{\pi_k(l),\pi_k(i)}\tilde{w}^*_{\pi_k(l),\pi_k(j)}]f \circ X(\bar{\pi}(i)) f \circ X(\bar{\pi}(j))$$

and $\{w^*_{i,j}, \tilde{w}^*_{i,j} : 1 \leq i, j \leq n\}$ is a sequence of suitably chosen nonnegative constants. It was shown in Loh (1994) that these estimators have the attractive property that under mild conditions, they possess a smaller asymptotic mean squared error than that of any regular estimator for θ_k based on an i.i.d. sample of the same size. We observe that $\hat{\theta}_{k,n}$ can be written in the form of (1) if we define $h(s,t) = st$, $\forall s, t \in \mathcal{R}$, and $w_{i,j} = w^*_{i,j} + w^*_{j,i} - \sum_{l\neq i,j}\tilde{w}^*_{l,i}\tilde{w}^*_{l,j}$ for all $1 \leq i < j \leq n$.

4 Characteristic function bounds and asymptotic expansions

This section investigates the behavior of the characteristic function of $\sum_{j=1}^n Y(\bar{\pi}(j))/\sigma_n$, with $Y(\bar{\pi}(j))/\sigma_n$ as in Section 2, under conditions in which certain values of $\bar{\pi}(j)$ are held constant. We start with some notations. For $2 \leq m \leq n$, let $A_m = \{1, \ldots, m\}$, $\pi_k(A_m) = \{\pi_k(j) : j \in A_m\}$, $\forall 1 \leq k \leq d$, and $\bar{\pi}(A_m) = \{\bar{\pi}(j) : j \in A_m\}$. Also let $E^{\bar{\pi}(A_m)}$ denote the conditional expectation given $\bar{\pi}(A_m)$ and define for $i_k \in \pi_k(A_m)$, $1 \leq k \leq d$,

$$\hat{\nu} = (1/m)^d \sum_{k=1}^{d}\sum_{i_k\in\pi_k(A_m)} EY(i_1, \ldots, i_d), \tag{6}$$

$$\hat{\nu}_{-j}(i_j) = (1/m)^{d-1} \sum_{k\neq j} \sum_{i_k \in \pi_k(A_m)} EY(i_1, \ldots, i_d), \quad \forall 1 \leq j \leq d, \tag{7}$$

$$\hat{Y}(i_1, \ldots, i_d) = Y(i_1, \ldots, i_d) - \sum_{j=1}^{d} \hat{\nu}_{-j}(i_j) + (d-1)\hat{\nu},$$

$$\hat{\sigma}_m^2 = (1/m)^{d-1} \sum_{k=1}^{d} \sum_{i_k \in \pi_k(A_m)} E\hat{Y}^2(i_1, \ldots, i_d). \tag{8}$$

Theorem 2 *Suppose (4) holds and β is an arbitrary but fixed positive constant. Then there exists a constant $\alpha > 0$ such that*

(a) as $n \to \infty$, we have

$$E\{\exp[it \sum_{j=1}^{m} Y(\bar{\pi}(j))/\sigma_n] | \bar{\pi}(A_m)\} = O(1) \exp(-\beta m t^2/4n),$$

uniformly over $|t| \leq \alpha n^{1/2}$ and over all subsets $\bar{\pi}(A_m)$, $2 \leq m \leq n$, satisfying $\hat{\sigma}_m^2/\sigma_n^2 \geq \beta m/n$.

(b) for $m = n$, we have

$$E \exp[it \sum_{j=1}^{n} Y(\bar{\pi}(j))/\sigma_n]$$

$$= e^{-t^2/2}[1 - \frac{it^3}{6\sigma_n^3 n^{d-1}} \sum_{1 \leq i_1, \ldots, i_d \leq n} EY^3(i_1, \ldots, i_d)] + O((1 \wedge t^6) e^{-t^2/4} n^{-1}), \tag{9}$$

as $n \to \infty$ uniformly over $|t| \leq \alpha n^{1/2}$.

(c) for $m = n-2$, we have

$$E\{\exp[it \sum_{j=1}^{n-2} Y(\bar{\pi}(j))/\sigma_n] | \bar{\pi}(A_{n-2})\}$$

$$= e^{-t^2/2}[1 - \frac{it^3}{6\sigma_n^3 n^{d-1}} \sum_{1 \leq i_1, \ldots, i_d \leq n} EY^3(i_1, \ldots, i_d)] + O(|t| n^{-3/2} + (1 \wedge t^6) e^{-t^2/4} n^{-1}),$$

as $n \to \infty$ uniformly over $|t| \leq \alpha n^{1/2}$ and over all subsets $\bar{\pi}(A_{n-2})$.

PROOF. For $i_k \in \pi_k(A_m)$, $1 \leq k \leq d$, let

$$\hat{\phi}(i_1, \ldots, i_d; t) = E \exp[it\hat{Y}(i_1, \ldots, i_d)], \quad \forall t \in \mathcal{R},$$
$$\hat{b}(i_1, \ldots, i_d) = e^{\hat{\sigma}_m^2 t^2/2m\sigma_n^2} \hat{\phi}(i_1, \ldots, i_d; t/\sigma_n) - 1.$$

Observing that $\sum_{j=1}^{m} [Y(\bar{\pi}(j)) - E^{\bar{\pi}(A_m)} Y(\bar{\pi}(j))] = \sum_{j=1}^{m} \hat{Y}(\bar{\pi}(j))$, we have

$$e^{\hat{\sigma}_m^2 t^2/2\sigma_n^2} E^{\bar{\pi}(A_m)} \exp\{it \sum_{j=1}^{m} [Y(\bar{\pi}(j)) - E^{\bar{\pi}(A_m)} Y(\bar{\pi}(j))]/\sigma_n\}$$

$$= (1/m!)^d \sum_{\pi_1^*, \ldots, \pi_d^*} \prod_{j=1}^{m} e^{\hat{\sigma}_m^2 t^2/2m\sigma_n^2} \hat{\phi}(\pi_1^*(j), \ldots, \pi_d^*(j); t/\sigma_n)$$

$$= 1+\sum_{s=1}^{m}\frac{[(m-s)!]^{d-1}}{s!(m!)^{d-1}}$$

$$\times \sum_{i_{1,1},\ldots,i_{1,s}\in\pi_1(A_m),\text{all distinct}} \cdots \sum_{i_{d,1},\ldots,i_{d,s}\in\pi_d(A_m),\text{all distinct}} \prod_{k=1}^{s} \hat{b}(i_{1,k},\ldots,i_{d,k}), \tag{10}$$

where for each $1 \le k \le d$, π_k^* denotes a permutation of the set $\pi_k(A_m)$.

To proceed, let $\mathcal{P}_s$ denote the set of partitions of the set $\{1,2,\ldots,s\}$, i.e. $Q \in \mathcal{P}_s$ if Q is a class of disjoint subsets (blocks) whose union is $\{1,2,\ldots,s\}$. Q is said to be of type $1^{\lambda_1}2^{\lambda_2}\ldots s^{\lambda_s}$ if Q consists of λ_1 blocks of cardinality 1, λ_2 blocks of cardinality 2, etc. We note that $\sum_{j=1}^{s} j\lambda_j = s$ and $\sum_{j=1}^{s}\lambda_j = r(Q)$, the number of blocks in Q. Now it follows from a lemma of von Bahr (1976), page 134, that the r.h.s. of (10) can be simplified to

$$1+\sum_{s=1}^{m}\frac{[(m-s)!]^{d-1}}{s!(m!)^{d-1}} \sum_{Q_1,\ldots,Q_d\in\mathcal{P}_s} K(Q_1)\ldots K(Q_d)M(Q_1,\ldots,Q_d), \tag{11}$$

where for each $1 \le j \le d$, $K(Q_j) = (-1)^{s-r(Q_j)}(2!)^{\lambda_{j,3}}\ldots((s-1)!)^{\lambda_{j,s}}$ if Q_j is of type $1^{\lambda_{j,1}}\ldots s^{\lambda_{j,s}}$,

$$M(Q_1,\ldots,Q_d) = \sum_{i_{1,1},\ldots,i_{1,s}}{}^{*}\cdots\sum_{i_{d,1},\ldots,i_{d,s}}{}^{*}\prod_{k=1}^{s}\hat{b}(i_{1,k},\ldots,i_{d,k}), \tag{12}$$

and $\sum^*_{i_{j,1},\ldots,i_{j,s}}$ denotes summation over $i_{j,1},\ldots,i_{j,s}\in\pi_j(A_m)$ such that the $i_{j,k}$'s are equal within the same block of Q_j.

To evaluate $M(Q_1,\ldots,Q_d)$, consider an $d\times s$ matrix. For each $1 \le j \le d$, select $\lambda_{j,1}$ elements (without replacement) from row j. Now let $\eta_k(j_1,\ldots,j_k)$, with $1 \le j_1 < \cdots < j_k \le d$ and $0 \le k \le d$, denote the number of columns of the matrix such that elements (in these columns) in rows $j_1,\ldots,j_k$ are not selected and all elements in the remaining rows of those columns are selected. Writing

$$\hat{b}_0 = \sum_{k=1}^{d}\sum_{i_k\in\pi_k(A_m)}\hat{b}(i_1,\ldots,i_d),\quad \hat{b}_{-j}(i_j) = \sum_{k\neq j}\sum_{i_k\in\pi_k(A_m)}\hat{b}(i_1,\ldots,i_d),\quad \forall 1\le j\le d,$$

we observe from (12) that

$$\begin{aligned}|M(Q_1,\ldots,Q_d)| \le\ & |\hat{b}_0|^{\eta_0}\{\prod_{j=1}^{d} m^{r(Q_j)-(d-1)\eta_1(j)-\eta_0}\max_{i_j\in\pi_j(A_m)}|\hat{b}_{-j}(i_j)|^{\eta_1(j)}\}\\ & \times \max_{i_k\in\pi_k(A_m):1\le k\le d}|\hat{b}(i_1,\ldots,i_d)|^{s-\eta_0-\sum_{k=1}^{d}\eta_1(k)}.\end{aligned} \tag{13}$$

Next using Taylor expansions for $\hat{b}$, $\hat{b}_{-j}$ and $\hat{b}_0$, we observe from (3) that there exist positive constants α_1 and C_1 (which depend only on $\liminf_{n\to\infty} n\sigma_n^2$, γ and d) such that for sufficiently large n, $|\hat{b}_0| \le C_1|t|^3m^dn^{-3/2}$,

$$\max_{i_k\in\pi_k(A_m):1\le k\le d}|\hat{b}(i_1,\ldots,i_d)| \le C_1|t|n^{-1/2},\qquad \max_{i_j\in\pi_j(A_m)}|\hat{b}_{-j}(i_j)| \le C_1t^2m^{d-1}n^{-1}, \tag{14}$$

whenever $|t| \le \alpha_1 n^{1/2}$. Substituting (14) into (13), we get

$$|M(Q_1,\ldots,Q_d)| \le m^{r(Q_1)+\cdots+r(Q_d)}(C_1|t|n^{-1/2})^s(|t|n^{-1/2})^{\eta_1(1)+\cdots+\eta_1(d)+2\eta_0}, \tag{15}$$

whenever $|t| \le \alpha_1 n^{1/2}$. Letting $N(\lambda_{j,2}, \dots, \lambda_{j,s-\lambda_{j,1}})$ denote the number of partitions of the set $\{1, 2, \dots, s - \lambda_{j,1}\}$ of type $2^{\lambda_{j,2}} \cdots (s - \lambda_{j,1})^{\lambda_{j,s-\lambda_{j,1}}}$, it follows from (10), (11) and (15) that for sufficiently large n,

$$
\begin{aligned}
&|e^{\hat{\sigma}_m^2 t^2/2\sigma_n^2} E^{\bar{\pi}(A_m)} \exp[it \sum_{j=1}^{m} \hat{Y}(\bar{\pi}(j))/\sigma_n] - 1 - \hat{b}_0 m^{-(d-1)}| \\
&\le \sum_{s=2}^{m} \frac{[(m-s)!]^{d-1}}{s!(m!)^{d-1}} (C_1|t|n^{-1/2})^s \sum_{\eta_0,\dots,\eta_d(1,\dots,d)} \binom{s}{\eta_0, \eta_1(1), \dots, \eta_1(d), \eta_2(1,2), \dots, \eta_d(1,2,\dots,d)} \\
&\times (|t|n^{-1/2})^{\eta_1(1)+\cdots+\eta_1(d)+2\eta_0} \sum_{\lambda_{j,k}:1\le j\le d, 2\le k\le s} \prod_{l=1}^{d} \{|K(Q_l)| N(\lambda_{l,2}, \dots, \lambda_{l,s-\lambda_{l,1}}) m^{\lambda_{l,1}+\cdots+\lambda_{l,s}}\}
\end{aligned} \tag{16}
$$

whenever $|t| \le \alpha_1 n^{1/2}$, where $\sum_{\eta_0,\dots,\eta_d(1,\dots,d)}$, denotes summation over all possible realizations of $\eta_k(j_1, \dots, j_k)$, $0 \le k \le d$, $1 \le j_1 < \cdots < j_k \le d$, Q_l denotes a partition of $\{1, 2, \dots, s\}$ of type $1^{\lambda_{l,1}} 2^{\lambda_{l,2}} \dots s^{\lambda_{l,s}}$ whenever $1 \le l \le d$, and finally $\sum_{\lambda_{j,k}:1\le j\le d, 2\le k\le s}$, denotes summation over all possible nonnegative integers $\lambda_{j,k}$'s satisfying $\sum_{k=2}^{s} k\lambda_{j,k} = s - \lambda_{j,1}$ for all $1 \le j \le d$.

We observe as in von Bahr (1976), page 136, that there exists a constant $C_2 \ge 1$ (which depends only on d) such that

$$
\sum_{\lambda_{j,k}:1\le j\le d, 2\le k\le s-\lambda_{j,1}} \prod_{l=1}^{d} \{|K(Q_l)| N(\lambda_{l,2}, \dots, \lambda_{l,s-\lambda_{l,1}}) m^{\lambda_{l,2}+\cdots+\lambda_{l,s}}\} \le C_2^s \prod_{j=1}^{d} (ms)^{(s-\lambda_{j,1})/2}.
$$

We further observe that $\lambda_{l,1} = \sum_{k=0}^{d} \sum_{j_i \ne l: 1\le i\le k} \eta_k(j_1, \dots, j_k)$ and hence

$$
\sum_{j=1}^{d} \lambda_{j,1} = \sum_{k=0}^{d} (d-k) \sum_{1\le j_1<\cdots<j_k\le d} \eta_k(j_1, \dots, j_k).
$$

Thus it follows that the r.h.s. of (16) is bounded by

$$
\begin{aligned}
&\sum_{s=2}^{m} \frac{[(m-s)!]^{d-1} m^{ds/2}}{s!(m!)^{d-1}} (C_1 C_2 |t| n^{-1/2})^s \sum_{\eta_0,\dots,\eta_d(1,\dots,d)} \binom{s}{\eta_0, \dots, \eta_d(1,\dots,d)} \\
&\quad \times s^{d[s-\eta_0-\eta_1(1)-\cdots-\eta_1(d)-\eta_2(1,2)-\cdots-\eta_d(1,\dots,d)]/2} (|t| s^{1/2} m^{(d-1)/2} n^{-1/2})^{\eta_1(1)+\cdots+\eta_1(d)} \\
&\quad \times (t^2 m^{d/2} n^{-1})^{\eta_0} \prod_{k=2}^{d} [m^{(d-k)/2} s^{k/2}]^{\sum_{1\le j_1<\cdots<j_k\le d} \eta_k(j_1,\dots,j_k)} \\
&\le \sum_{s=2}^{m} \frac{[(m-s)!]^{d-1} m^{ds/2}}{s!(m!)^{d-1}} (C_1 C_2 C_3 |t| n^{-1/2})^s (t^{2s} m^{ds/2} n^{-s} + m^{(d-2)s/2} s^s)
\end{aligned} \tag{17}
$$

whenever $|t| \le \alpha_1 n^{1/2}$, where $C_3 \ge 1$ is a constant depending only on d. Since $(m-s)!/m! \le (e/m)^s$ and $s! \ge (s/e)^s$ for $1 \le s \le m$, the r.h.s. of (17) is bounded by

$$
\sum_{s=2}^{m} \{(C_1 C_2 C_3 e^d |t| n^{-1/2})^s + (C_1 C_2 C_3 e^d |t|^3 m n^{-3/2})^s / s!\}, \quad \forall |t| \le \alpha_1 n^{1/2}. \tag{18}
$$

Let $\beta > 0$ be an arbitrary but fixed constant. Choose $0 < \alpha_2 \le \alpha_1$ such that $C_1 C_2 C_3 e^d |t| n^{-1/2} \le 1/2$ and $C_1 C_2 C_3 e^d |t|^3 m n^{-3/2} \le \beta m t^2 / 8n$ whenever $|t| \le \alpha_2 n^{1/2}$. Observing that $\sum_{s=2}^{m} x^s/s! \le x^2 e^x$ for all $x \ge 0$, we conclude that (18) is bounded by

$$
O(t^2 n^{-1} + t^6 m^2 n^{-3} e^{\beta m t^2/8n}), \tag{19}
$$

as $n \to \infty$ uniformly over $|t| \leq \alpha_2 n^{1/2}$ and over all subsets $\bar{\pi}(A_m)$, $2 \leq m \leq n$.

PROOF OF (a). We observe from (14), (16) and (19) that

$$|E^{\bar{\pi}(A_m)} \exp[it \sum_{j=1}^{m} Y(\bar{\pi}(j))/\sigma_n]| = O(1)(1 + t^6 m^2 n^{-3} e^{\beta m t^2/8n}) e^{-\hat{\sigma}_m^2 t^2/2\sigma_n^2} = O(1) e^{-\beta m t^2/4n},$$

as $n \to \infty$ uniformly over $|t| \leq \alpha_2 n^{1/2}$ and over all subsets $\bar{\pi}(A_m)$, $2 \leq m \leq n$, satisfying $\hat{\sigma}_m^2/\sigma_n^2 \geq \beta m/n$.

PROOF OF (b). Taking $m = n$ and $\beta = 1$, we observe that $\hat{\sigma}_m^2/\sigma_n^2 = 1$. Using a Taylor series expansion, we have

$$|\hat{b}_0 + \frac{it^3}{6\sigma_n^3} \sum_{1 \leq i_1, \ldots, i_d \leq n} EY^3(i_1, \ldots, i_d)| \leq O(1) t^4 n^{d-2},$$

as $n \to \infty$ uniformly over $|t| \leq \alpha_3 n^{1/2}$ for some positive constant $\alpha_3 \leq \alpha_2$. (b) now follows from (16) and (19).

PROOF OF (c). Taking $m = n - 2$, we observe that there exist constants C_4 and $0 < \alpha \leq \alpha_3$ (depending only on $\liminf_{n \to \infty} n\sigma_n^2$, γ and d) such that for sufficiently large n, we have

$$\begin{aligned}
|e^{-\hat{\sigma}_{n-2}^2 t^2/2\sigma_n^2} - e^{-t^2/2}| &\leq C_4(t^2 e^{-t^2/2} n^{-1}), \\
|\hat{b}_0 + \frac{it^3}{6\sigma_n^3} \sum_{1 \leq i_1, \ldots, i_d \leq n} EY^3(i_1, \ldots, i_d)| &\leq C_4(|t|^3 + t^4) n^{d-2}, \\
|\exp[it \sum_{j=1}^{n-2} E^{\bar{\pi}(A_{n-2})} Y(\bar{\pi}(j))/\sigma_n] - 1| &\leq C_4 |t| n^{-3/2},
\end{aligned}$$

whenever $|t| \leq \alpha n^{1/2}$. (c) now follows from (16) and (19) by taking $\beta = 2$. This completes the proof of Theorem 2. □

5 Proof of Theorem 1

Writing

$$(20) \quad T_n = \sum_{a=1}^{n} Y(\bar{\pi}(a)), \quad \Delta_n(m) = (2/n) \sum_{a=1}^{m} \sum_{b=a+1}^{n} \psi(X(\bar{\pi}(a)), X(\bar{\pi}(b))), \quad \forall 2 \leq m < n,$$

and $\Delta_n = \Delta_n(n-1)$ for short, we observe that $U_{k,n} - EU_{k,n} = T_n + \Delta_n$. Let $\phi_n(t) = E\exp[it(U_{k,n} - EU_{k,n})/\sigma_n]$.

CASE I. Suppose $|t| \leq n^{(2\varepsilon_n - 1)/4}$. Using Lemma 1 below, we have

$$(21) \quad \phi_n(t) = Ee^{itT_n/\sigma_n}(1 + it\sigma_n^{-1}\Delta_n) + O(t^2\sigma_n^{-2}\Delta_n^2) = Ee^{itT_n/\sigma_n}(1 + it\sigma_n^{-1}\Delta_n) + O(t^2/n^{\varepsilon_n}),$$

as $n \to \infty$ uniformly over $|t| \leq n^{(2\varepsilon_n - 1)/4}$. We observe from Theorem 2(b) that

$$(22) \quad |Ee^{itT_n/\sigma_n} - e^{-t^2/2}| = O((1 \wedge t^6) e^{-t^2/4} n^{-1/2}),$$

as $n \to \infty$ uniformly over $|t| \leq \alpha n^{1/2}$. Also

$$\begin{aligned}
Eit\sigma_n^{-1}\Delta_n e^{itT_n/\sigma_n} &= (2/n) Eit\sigma_n^{-1} \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \psi(X(\bar{\pi}(a)), X(\bar{\pi}(b))) \\
(23) \quad & \times e^{it[Y(\bar{\pi}(a)) + Y(\bar{\pi}(b))]/\sigma_n} E[e^{it\sum_{l \neq a,b} Y(\bar{\pi}(l))/\sigma_n} | \bar{\pi}(a), \bar{\pi}(b)].
\end{aligned}$$

Writing $B_n(t) = e^{-t^2/2}[1 - it^3(6\sigma_n^3 n^{d-1})^{-1}\sum_{1\le i_1,\ldots,i_d\le n} EY^3(i_1,\ldots,i_d)]$, it follows from (2), (23) and Theorem 2(c) that

$$\begin{aligned} & |Eit\sigma_n^{-1}\Delta_n e^{itT_n/\sigma_n}| \\ \le\ & |2n^{-1}t\sigma_n^{-1}B_n(t)\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} E\psi(X(\bar\pi(a)),X(\bar\pi(b)))e^{it[Y(\bar\pi(a))+Y(\bar\pi(b))]/\sigma_n}| \\ & +O(t^2n^{-1} + (|t|+|t|^7)e^{-t^2/4}n^{-1/2}) \\ =\ & O(t^2n^{-1} + (|t|+|t|^7)e^{-t^2/4}n^{-1/2}), \qquad (24) \end{aligned}$$

as $n\to\infty$ uniformly over $|t|\le \alpha n^{1/2}$. We conclude from (21) to (24) that

$$|\phi_n(t) - e^{-t^2/2}| = O(t^2n^{-\varepsilon_n} + (|t|+|t|^7)e^{-t^2/4}n^{-1/2}),$$

as $n\to\infty$ uniformly over $|t|\le n^{(2\varepsilon_n-1)/4}$ and hence

$$\int_{-n^{(2\varepsilon_n-1)/4}}^{n^{(2\varepsilon_n-1)/4}} |\frac{\phi_n(t)-e^{-t^2/2}}{t}|dt = O(n^{-1/2}), \qquad (25)$$

as $n\to\infty$.

CASE II. Suppose $n^{(2\varepsilon_n-1)/4}\le |t|\le \alpha n^{1/2}$ with α as in Theorem 2. We first observe that for $\hat\sigma_m^2$ as in (8), there exists a constant $C_7\ge 1$ (depending only on $\liminf_{n\to\infty} n\sigma_n^2$, γ and d) such that for sufficiently large n,

$$m|E\frac{n\hat\sigma_m^2}{m\sigma_n^2} - 1| + \frac{n\hat\sigma_m^2}{m\sigma_n^2} \le C_7, \quad \forall 2\le m\le n. \qquad (26)$$

Next take $\beta = 1/2$ in Theorem 2(a). We observe from (26) and Markov's inequality that for all $0<u<1$,

$$\begin{aligned} P(\frac{n\hat\sigma_m^2}{m\sigma_n^2}\le\beta) &\le e^{um[\beta - En\hat\sigma_m^2/(m\sigma_n^2)]}E\exp[-um(\frac{n\hat\sigma_m^2}{m\sigma_n^2} - E\frac{n\hat\sigma_m^2}{m\sigma_n^2})] \\ &\le (1+2C_7um)e^{um(\beta-1)+uC_7}\sum_{s=0}^{\infty}\frac{(um)^{2s}}{(2s)!}E(\frac{n\hat\sigma_m^2}{m\sigma_n^2} - E\frac{n\hat\sigma_m^2}{m\sigma_n^2})^{2s}. \qquad (27) \end{aligned}$$

Using Stirling's formula, we have

$$\sum_{s\ge 2eC_7m}\frac{(um)^{2s}}{(2s)!}E(\frac{n\hat\sigma_m^2}{m\sigma_n^2} - E\frac{n\hat\sigma_m^2}{m\sigma_n^2})^{2s}\le 1. \qquad (28)$$

We further observe that with $\hat\nu$ and $\hat\nu_{-j}(i_j)$ as in (6) and (7),

$$\begin{aligned} & E(\frac{n\hat\sigma_m^2}{m\sigma_n^2} - E\frac{n\hat\sigma_m^2}{m\sigma_n^2})^{2s} \\ \le\ & (d+2)^{2s}\{E[\frac{n}{\sigma_n^2m^d}\sum_{k=1}^{d}\sum_{i_k\in\pi_k(A_m)} EY^2(i_1,\ldots,i_d) - E\frac{n}{\sigma_n^2m^d}\sum_{k=1}^{d}\sum_{i_k\in\pi_k(A_m)} EY^2(i_1,\ldots,i_d)]^{2s} \\ & +(d-1)^{2s}E[\frac{n}{\sigma_n^2}(\hat\nu^2 - E\hat\nu^2)]^{2s} + \sum_{j=1}^{d}E[\frac{n}{\sigma_n^2m}\sum_{i_j\in\pi_j(A_m)}\hat\nu_{-j}^2(i_j) - E\frac{n}{\sigma_n^2m}\sum_{i_j\in\pi_j(A_m)}\hat\nu_{-j}^2(i_j)]^{2s}\}. \qquad (29) \end{aligned}$$

Now it follows from Lemma 2 below that for $0 \le s \le 2eC_7m$ and $m/n \to 0$ as $n \to \infty$, the r.h.s. of (29) is bounded by $C_8^s(2s)!/(s!m^s)$ for some constant $C_8 \ge 1$ which depends only on $\liminf_{n\to\infty} n\sigma_n^2$, γ and d provided n is sufficiently large. Thus we conclude from (27), (28) and (29) that by taking $u = (1-\beta)/(2C_8)$, there exists a constant C_9 (depending only on $\liminf_{n\to\infty} n\sigma_n^2$, γ and d) such that for sufficiently large n,

$$P(\frac{n\hat{\sigma}_m^2}{m\sigma_n^2} \le \beta) \le (1 + 2C_7um)(1 + e^{C_8u^2m})e^{um(\beta-1)+uC_7} \le C_9e^{-m(1-\beta)^2/(5C_8)}, \tag{30}$$

for all $2 \le m \le n$. Let

$$m = \lceil (20\alpha^2C_8 \vee 8)nt^{-2}\log n \rceil. \tag{31}$$

Then for such an m, it follows from (30) that for sufficiently large n,

$$P(\frac{n\hat{\sigma}_m^2}{m\sigma_n^2} < \beta) \le C_9n^{-1}. \tag{32}$$

Finally observing that

$$\phi_n(t) = Ee^{it(T_n+\Delta_n-\Delta_n(m))/\sigma_n}[1 + it\sigma_n^{-1}\Delta_n(m)] + O(t^2m/n^{1+\varepsilon_n}),$$

as $n \to \infty$ uniformly over $|t| \le \alpha n^{1/2}$ and conditioning on whether or not $\hat{\sigma}_m^2/\sigma_n^2 \ge \beta m/n$, it follows from (5), (31), (32), Theorem 2(a) and Lemma 1 that

$$\int_{n^{(2\varepsilon_n-1)/4}\le|t|\le\alpha n^{1/2}} |\frac{\phi_n(t)}{t}|dt = O(n^{-1/2}), \tag{33}$$

as $n \to \infty$. We observe from (5) and (33) that

$$\int_{n^{(2\varepsilon_n-1)/4}\le|t|\le\alpha n^{1/2}} |\frac{\phi_n(t) - e^{-t^2/2}}{t}|dt = O(n^{-1/2}), \tag{34}$$

as $n \to \infty$. Theorem 1 now follows from (25) and (34) by the smoothing lemma of Esseen (see for example, Feller (1971), page 538). □

Lemma 1 *Let $\Delta_n(m)$ be as in (20) such that (4) holds. Then there exists a constant C_5 (depending only on γ and d) such that $E\Delta_n^2(m) \le C_5m/n^{2+\varepsilon_n}$ for all $1 \le m < n$.*

PROOF. Define for $0 \le s, t < 1$,

$$\delta_n(s,t) = \begin{cases} 1 & \text{if } \lfloor ns \rfloor = \lfloor nt \rfloor, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. We observe from (20) that

$$\begin{aligned} E\Delta_n^2(m) &= (4/n^2)E\{\sum_{i_1=1}^{m}\sum_{j_1=i_1+1}^{n} \psi^2(X(\bar{\pi}(i_1)), X(\bar{\pi}(j_1))) \\ &+ \sum_{i_1=1}^{m}\sum_{j_1=i_1+1}^{n}\sum_{1\le i_2\le m\wedge(j_1-1), i_2\ne i_1} \psi(X(\bar{\pi}(i_1)), X(\bar{\pi}(j_1)))\psi(X(\bar{\pi}(i_2)), X(\bar{\pi}(j_1)))\end{aligned}$$

$$+\sum_{i_1=1}^{m}\sum_{j_1=i_1+1}^{n}\sum_{i_1<j_2\le n, j_2\ne j_1}\psi(X(\bar\pi(i_1)),X(\bar\pi(j_1)))\psi(X(\bar\pi(i_1)),X(\bar\pi(j_2)))$$
$$+2\sum_{i_1=1}^{m-1}\sum_{j_1=i_1+1}^{m}\sum_{j_2=j_1+1}^{n}\psi(X(\bar\pi(i_1)),X(\bar\pi(j_1)))\psi(X(\bar\pi(j_1)),X(\bar\pi(j_2)))$$
$$+\sum_{i_1=1}^{m}\sum_{j_1=i_1+1}^{n}\sum_{1\le i_2\le m, i_2\ne i_1,j_1}\sum_{i_2<j_2\le n, j_2\ne i_1,j_1}\psi(X(\bar\pi(i_1)),X(\bar\pi(j_1)))$$
$$\times\psi(X(\bar\pi(i_2)),X(\bar\pi(j_2)))\}. \tag{35}$$

We further observe from (2) that for i_1, i_2, j_1, j_2 all distinct, there exists a constant C_6, depending only on γ and d, such that

$$E\psi^2(X(\bar\pi(i_1)),X(\bar\pi(j_1))) \le C_6 n^{-1-\varepsilon_n}, \tag{36}$$
$$|E\psi(X(\bar\pi(i_1)),X(\bar\pi(j_1)))\psi(X(\bar\pi(i_2)),X(\bar\pi(j_1)))|$$
$$= \frac{n^{2d}}{(n-1)^d(n-2)^d}\Big|\int_{[0,1]^{3d}}\psi(x^{(1)},x^{(3)})\psi(x^{(2)},x^{(3)})$$
$$\times\{\prod_{l=1}^{d}\prod_{1\le k_1<k_2\le 3}[1-\delta_n(x_l^{(k_1)},x_l^{(k_2)})]\}dx^{(1)}dx^{(2)}dx^{(3)}\Big|$$
$$\le C_6 n^{-2-\varepsilon_n}, \tag{37}$$

and in a similar manner,

$$|E\psi(X(\bar\pi(i_1)),X(\bar\pi(j_1)))\psi(X(\bar\pi(i_2)),X(\bar\pi(j_2)))| \le C_6 n^{-3-\varepsilon_n}. \tag{38}$$

Lemma 1 now follows from (35)–(38). □

Lemma 2 *Suppose (4) holds, $0 \le s \le 2eC_7 m$ and $m/n \to 0$ as $n \to \infty$. Then there exists a constant C_{10} depending only on $\liminf_{n\to\infty} n\sigma_n^2$, γ and d such that for sufficiently large n,*

$$E[\frac{n}{\sigma_n^2}(\hat\nu^2 - E\hat\nu^2)]^{2s} \le \frac{C_{10}^s(2s)!}{s!m^s},$$
$$E[\frac{n}{\sigma_n^2 m}\sum_{i_j\in\pi_j(A_m)}\hat\nu_{-j}^2(i_j) - E\frac{n}{\sigma_n^2 m}\sum_{i_j\in\pi_j(A_m)}\hat\nu_{-j}^2(i_j)]^{2s} \le \frac{C_{10}^s(2u)!}{s!m^s},$$

and

$$E[\frac{n}{\sigma_n^2 m^d}\sum_{k=1}^{d}\sum_{i_k\in\pi_k(A_m)}EY^2(i_1,\ldots,i_d) - E\frac{n}{\sigma_n^2 m^d}\sum_{k=1}^{d}\sum_{i_k\in\pi_k(A_m)}EY^2(i_1,\ldots,i_d)]^{2s} \le \frac{C_{10}^s(2s)!}{s!m^s}.$$

Proof. The proofs of the three inequalities of Lemma 2 are similar (though the first two are somewhat longer) and as such we shall only give a proof of the third inequality here. Without loss of generality we assume that $s \ge 1$. For simplicity we write $EY^2(i_1,\ldots,i_d) = W(i_1,\ldots,i_d)$ and C_{11} is some constant (depending only on $\liminf_{n\to\infty} n\sigma_n^2$, γ and d) such that

$$C_{11} > \max\{1, \limsup_{n\to\infty}\{n\sigma_n^{-2}|W(i_1,\ldots,i_d)| : 1 \le i_1,\ldots,i_d \le n\}\}.$$

Also let $\{J_{j,k} : 1 \le j \le d, 1 \le k \le 2s\}$ be a sequence of i.i.d. random variables each uniformly distributed on $\{1, 2, \ldots, n\}$ and are independent of all previously defined random quantities such as $\bar{\pi}(A_m)$. Let $\mathcal{P}_{2s}$ denote the set of partitions of the set $\{1, 2, \ldots, 2s\}$. If $Q_j \in \mathcal{P}_{2s}$, $1 \le j \le d$, let $\chi_{Q_j}(J_{j,1}, \ldots, J_{j,2s})$ denote the indicator of the event that $J_{j,1}, \ldots, J_{j,2s}$ are equal within the same block of Q_j and are distinct in different blocks of Q_j. Then

$$\begin{aligned} & E[\frac{n}{\sigma_n^2 m^d} \sum_{k=1}^{d} \sum_{i_k \in \pi_k(A_m)} EY^2(i_1, \ldots, i_d) - E\frac{n}{\sigma_n^2 m^d} \sum_{k=1}^{d} \sum_{i_k \in \pi_k(A_m)} EY^2(i_1, \ldots, i_d)]^{2s} \\ = & \ E \sum_{Q_1, \ldots, Q_d \in \mathcal{P}_{2s}} [\prod_{j=1}^{d} \chi_{Q_j}(J_{j,1}, \ldots, J_{j,2s})] \\ & \times \prod_{k=1}^{2s} \frac{n^{d+1}}{\sigma_n^2 m^d} W(J_{1,k}, \ldots, J_{d,k})[I\{(J_{1,k}, \ldots, J_{d,k}) \in \bar{\pi}(A_m)\} - (\frac{m}{n})^d], \end{aligned} \tag{39}$$

where $I\{.\}$ denotes the indicator function. To proceed, suppose Q_j is of type $1^{\lambda_{j,1}} 2^{\lambda_{j,2}} \ldots (2s)^{\lambda_{j,2s}}$ and consider an $d \times 2s$ matrix. For each $1 \le j \le d$, select $\lambda_{j,1}$ elements (without replacement) from row j. Let $\eta_l(j_1, \ldots, j_l)$, with $1 \le j_1 < \cdots < j_l \le d$ and $0 \le l \le d$, denote the number of columns of the matrix such that elements (in these columns) in rows $j_1, \ldots, j_l$ are not selected and all elements in the remaining rows of those columns are selected. Also let H_{η_0} denote the set of columns of the $d \times 2s$ matrix which have all its elements selected. We show next by induction that there exists a constant C_{12} (depending only on $\liminf_{n \to \infty} n\sigma_n^2$, γ and d) such that for sufficiently large n,

$$\begin{aligned} 4C_{11}(eC_7)^{1/2} \sum_{l=1}^{d} \binom{d}{l} (8eC_7)^{l-1} & \le C_{12}, \\ \sum_{l=1}^{d} \binom{d}{l} (\frac{C_{11}}{C_{12}})^{l+1} 2^{(3l-1)/2} (2eC_7)^{(l-1)/2} & \le 1, \end{aligned} \tag{40}$$

and for all subsets $H'_{\eta_0} \subseteq H_{\eta_0}$,

$$\begin{aligned} & |E[\prod_{k \in H'_{\eta_0}} \frac{n^{d+1}}{\sigma_n^2 m^d} W(J_{1,k}, \ldots, J_{d,k})[I\{(J_{1,k}, \ldots, J_{d,k}) \in \bar{\pi}(A_m)\} - (\frac{m}{n})^d] | \{I\{J_{a,b} \in \pi_a(A_m)\}, \\ & J_{a,b} : 1 \le a \le d, b \in \{1, \ldots, 2s\} \backslash H'_{\eta_0}\}, \prod_{j=1}^{d} \chi_{Q_j}(J_{j,1}, \ldots, J_{j,2s}) = 1]| \le C_{12}^{\eta'_0} (\frac{2s}{m})^{\eta'_0/2}, \end{aligned} \tag{41}$$

where η'_0 denotes the cardinality of H'_{η_0}. Here we use the convention that the product over an empty set is 1 and hence it follows that (41) holds for $\eta'_0 = 0$. Now suppose that $\eta'_0 \ge 1$ and that (41) holds for all subsets strictly smaller than H'_{η_0}. Let $r(Q_j)$ denote the number of blocks of Q_j and $\{J_{j,k_{j,l}} : 1 \le l \le r(Q_j)\}$ represent the distinct values of $\{J_{j,k} : 1 \le k \le 2s\}$. Without loss of generality we also assume that $\{k_{j,1}, \ldots, k_{j,\eta_0}\} = H_{\eta_0}$ and $\{k_{j,1}, \ldots, k_{j,\eta'_0}\} = H'_{\eta_0}$ for all $1 \le j \le d$, and write $k_l = k_{j,l}$ whenever $1 \le l \le \eta_0$. By conditioning on $\{J_{a,b}, I\{J_{a,b} \in \pi_a(A_m)\} : 1 \le a \le d, b \in \{1, \ldots, 2s\} \backslash \{k_1\}\}$ and $\prod_{j=1}^{d} \chi_{Q_j}(J_{j,1}, \ldots, J_{j,2s}) = 1$, we observe that the l.h.s. of (41) is equal to

$$|E\{\frac{n}{\sigma_n^2} W(J_{1,k_1}, \ldots, J_{d,k_1})[\sum_{j_1=1}^{d} L_{j_1} + \sum_{1 \le j_1 < j_2 \le d} L_{j_1} L_{j_2} + \cdots + L_1 \ldots L_d]$$

$$\times \prod_{k \in H'_{\eta_0} \backslash \{k_1\}} \frac{n^{d+1}}{\sigma_n^2 m^d} W(J_{1,k}, \ldots, J_{d,k})[I\{(J_{1,k}, \ldots, J_{d,k}) \in \bar{\pi}(A_m)\} - (\frac{m}{n})^d]|\{I\{J_{a,b} \in \pi_a(A_m)\},$$

$$(42) \qquad J_{a,b} : 1 \leq a \leq d, b \in \{1, \ldots, 2s\} \backslash H'_{\eta_0}\}, \prod_{j=1}^{d} \chi_{Q_j}(J_{j,1}, \ldots, J_{j,2s}) = 1\}|,$$

where

$$L_j = -\frac{n}{m(n - r(Q_j) + 1)} \sum_{l=2}^{r(Q_j)} [I\{J_{j,k_l} \in \pi_j(A_m)\} - \frac{m}{n}], \quad \forall 1 \leq j \leq d.$$

From the induction hypothesis and (40), we observe that (42) is bounded by

$$\sum_{l=1}^{d} \binom{d}{l} C_{11}(4s/m)^l \leq C_{12}(2s/m)^{1/2}, \quad \text{if } \eta'_0 = 1,$$

or

$$\sum_{l=1}^{d} \binom{d}{l} C_{11}^{l+1} (\frac{4s}{m})^l C_{12}^{\eta'_0 - l - 1} (\frac{2s}{m})^{(\eta'_0 - l - 1)/2} \leq C_{12}^{\eta'_0} (\frac{2s}{m})^{\eta'_0/2}, \quad \text{if } \eta'_0 \geq 2,$$

for sufficiently large n. This proves (41) and we observe by taking $H'_{\eta_0} = H_{\eta_0}$ in (41) that the r.h.s. of (39) is bounded by

$$\sum_{\eta_0, \ldots, \eta_d(1, \ldots, d)} \binom{2s}{\eta_0, \ldots, \eta_d(1, \ldots, d)} C_{12}^{\eta_0} (\frac{2s}{m})^{\eta_0/2} (2C_{11})^{2s - \eta_0}$$

$$(43) \qquad \times \prod_{j=1}^{d} \sum_{\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}}} N(\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}}) (\frac{1}{m})^{\lambda_{j,2} + 2\lambda_{j,3} + \cdots + (2s - \lambda_{j,1} - 1)\lambda_{j,2s-\lambda_{j,1}}},$$

where $\sum_{\eta_0, \ldots, \eta_d(1, \ldots, d)}$ denotes summation over all possible realizations of $\eta_l(j_1, \ldots, j_l)$, $0 \leq l \leq d$, $1 \leq j_1 < \cdots < j_l \leq d$, $\sum_{\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}}}$ denotes summation over all possible nonnegative integers $\lambda_{j,k}$'s satisfying $\sum_{k=2}^{2s-\lambda_{j,1}} k\lambda_{j,k} = 2s - \lambda_{j,1}$ for all $1 \leq j \leq d$ and $N(\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}})$ is the number of partitions of the set $\{1, \ldots, 2s\}$ of type $2^{\lambda_{j,2}} \cdots (2s - \lambda_{j,1})^{\lambda_{j,2s-\lambda_{j,1}}}$. We now observe from the definition of $N(\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}})$, see von Bahr (1976) page 136, that (43) is bounded by

$$\sum_{\eta_0, \ldots, \eta_d(1, \ldots, d)} \binom{2s}{\eta_0, \ldots, \eta_d(1, \ldots, d)} C_{12}^{\eta_0} (\frac{2s}{m})^{\eta_0/2} (2C_{11})^{2s - \eta_0} \prod_{j=1}^{d} \frac{(2s - \lambda_{j,1})!}{s^{s - \lambda_{j,1}/2}} (\frac{1}{m})^{s - \lambda_{j,1}/2}$$

$$(44) \qquad \times \sum_{\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}}} \frac{s^{s - \lambda_{j,1}/2}}{(2!)^{\lambda_{j,2}} \ldots [(2s - \lambda_{j,1})!]^{\lambda_{j,2s-\lambda_{j,1}}} \lambda_{j,2}! \ldots (\lambda_{j,2s-\lambda_{j,1}})!} (\frac{1}{m})^{\sum_{l=3}^{2s-\lambda_{j,1}} (l-2)\lambda_{j,l}/2}.$$

We further observe that

$$\sum_{\lambda_{j,2}, \ldots, \lambda_{j,2s-\lambda_{j,1}}} \frac{s^{s - \lambda_{j,1}/2}}{(2!)^{\lambda_{j,2}} \ldots [(2s - \lambda_{j,1})!]^{\lambda_{j,2s-\lambda_{j,1}}} \lambda_{j,2}! \ldots (\lambda_{j,2s-\lambda_{j,1}})!} (\frac{1}{m})^{\sum_{l=3}^{2s-\lambda_{j,1}} (l-2)\lambda_{j,l}/2}$$

$$(45) \quad \leq \quad [\exp(e^{2eC_7})]^s, \quad \forall 1 \leq j \leq d.$$

Since $s/m \leq 2eC_7$ and $2s(d-1) \geq \sum_{l=1}^{d} (d-l) \sum_{1 \leq j_1 < \cdots < j_l \leq d} \eta_l(j_1, \ldots, j_l)$, using Stirling's formula, we conclude from (45) that (44) is bounded by $C_{10}^s (2s)!/(s! m^s)$. This proves the third inequality of Lemma 2. □

References

[1] FANG, K. T. and WANG, Y. (1994). *Number-theoretic Methods in Statistics.* Chapman and Hall, New York.

[2] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd edition. Wiley, New York.

[3] HELMERS, R. and VAN ZWET, W. R. (1982). The Berry-Esseen bound for U-statistics. *Statist. Decision Theory and Related Topics III* (eds. S. S. Gupta and J. O. Berger) **1** 497-512. Academic Press, New York.

[4] HOEFFDING, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.* **19** 293-325.

[5] LOH, W. L. (1994). Estimating the integral of a squared regression function with Latin hypercube sampling. Tech. Report, Dept. Statist., Purdue University.

[6] MCKAY, M. D., CONOVER, W. J. and BECKMAN, R. J. (1979). A comparison of three methods for selecting values of output variables in the analysis of output from a computer code. *Technometrics* **21** 239-245.

[7] OWEN, A. B. (1992). A central limit theorem for Latin hypercube sampling. *J. R. Statist. Soc.* **54** 239-245.

[8] STEIN, M. L. (1987). Large sample properties of simulations using Latin hypercube sampling. *Technometrics* **29** 143-151.

[9] VON BAHR, B. (1976). Remainder term estimate in a combinatorial limit theorem. *Z. Wahrsch. Verw. Gebiete* **35** 131-139.