INDUCTION AND RECURSION IN n DIMENSIONS

by

Dimitris N. Politis
Purdue University

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Department of Statistics
Purdue University

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Dimitris N. Politis
Department of Statistics
Purdue University
W. Lafayette, IN 47907

Abstract

A generalization of the induction principle from one to n dimensions is presented, with special reference to multidimensional recursions.

Keywords. Induction, recursion.

1. Introduction

Suppose the sequence $X(i)$, $i = 0, 1, \ldots$ is defined recursively by $X(0) = 1$, and $X(i) = aX(i-1)$, for $i = 1, 2, \ldots$, where a is a constant such that $|a| < 1$. Then, invoking the induction principle (see, for example, Apostol (1974), or Birkhoff and MacLane (1977)), it is easy to show that $|X(i)| \leq 1$, for any $i = 0, 1, \ldots$, even without calculating the closed-form expression $X(i) = a^i$.

Now consider the double sequence $X(i,j)$, $i = 0, 1, \ldots$, $j = 0, 1, \ldots$, defined recursively by $X(i,0) = X(0,i) = 1$, for any $i = 0, 1, \ldots$, and $X(i,j) = aX(i-1,j) + bX(i,j-1)$, for $i = 1, 2, \ldots$, $j = 1, 2, \ldots$; here $|a| + |b| < 1$. Here too it should be possible to show that $|X(i,j)| \leq 1$, for any $i = 0, 1, \ldots$, $j = 0, 1, \ldots$, without calculating a closed-form expression for $X(i,j)$. To this effect, a generalization of the induction principle from one to n dimensions is required; such a generalization will be presented in the next sections.

2. Definitions

Let $\mathbf{N}$ be the set of positive integers, $\mathbf{N}_0$ the set of non-negative integers, and let $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbf{N}_0^n$ be a point in the first orthant of the integer lattice in $\mathbf{R}^n$, where the dimension n is some positive integer. Let C_N^+ be the cube consisting of the points $\mathbf{t} \in \mathbf{N}_0^n$ whose coordinates satisfy $0 \leq t_k \leq N$, for $k = 1, \ldots, n$.

Also let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{N}_0^n$ and define a notion of the 'past' of point $\mathbf{t}$ in the 'direction' n by

$$Past_n(\mathbf{t}) = \cup_{j=1}^{n} \{\mathbf{u} \in \mathbf{N}_0^n : u_j < t_j, u_k = t_k, j < k \leq n\}$$

$$= \{\mathbf{u} \in \mathbf{N}_0^n : u_n < t_n\} \cup \{\mathbf{u} \in \mathbf{N}_0^n : u_n = t_n, u_{n-1} < t_{n-1}\} \cup \cdots$$

$$\cdots \cup \{\mathbf{u} \in \mathbf{N}_0^n : u_n = t_n, u_{n-1} = t_{n-1}, \ldots, u_2 = t_2, u_1 < t_1\}$$

Of course, because the labelling and numbering of coordinates is arbitrary, it is straightforward to define the 'past' $Past_j(\mathbf{t})$ in the direction j, for any $j = 1, \ldots, n$.

It is apparent that an ordering '$<<$' of the elements of C_N^+ is induced by defining $\mathbf{s} << \mathbf{t}$ if $|C_N^+ \cap Past_n(\mathbf{s})| < |C_N^+ \cap Past_n(\mathbf{t})|$, for $\mathbf{s}, \mathbf{t} \in C_N^+$, where $|A|$ denotes the cardinality of set A; this is sometimes called a 'lexicographical' ordering (cf., for example, Georgii (1988)). The lexicographical ordering '$<<$' will be crucial in showing that the induction principle in n dimensions soon to be introduced is valid.

3. The induction principle in n dimensions

Let there be associated with each $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbf{N}_0^n$ a proposition $P(\mathbf{t})$ that is either true or false. If $n = 1$, the principle of finite induction (in one of its forms) states:
If $P(0)$ is true, and if, for all $k \in \mathbf{N}$, the assumption that $P(i)$ is true for any $i < k$, implies the conclusion that $P(k)$ is itself true, then $P(k)$ is true for any $k \in \mathbf{N}_0$.

We now propose the following generalization of the principle of finite induction in n dimensions:

Theorem 1. *If $P(t_1, t_2, \ldots, t_n)$ is true whenever $\min_{k=1,2,\ldots,n} |t_k| = 0$, and if, for all $\mathbf{t} \in \mathrm{N}^n$ the assumption that $P(\mathbf{u})$ is true for any $\mathbf{u} \in Past_n(\mathbf{t})$, implies the conclusion that $P(\mathbf{t})$ is true, then $P(\mathbf{t})$ is true for any $\mathbf{t} \in \mathrm{N}_0^n$.*

Proof. Let $S = \{\mathbf{t} \in \mathrm{N}_0^n : P(\mathbf{t}) \text{ is true }\}$, and let $S^* = \{\mathbf{t} \in \mathrm{N}_0^n : P(\mathbf{t}) \text{ is false }\}$; we will show that S^* is the empty set.

So suppose S^* were not empty; then there is an integer N large enough such that $C_N^+ \cap S^*$ is not empty as well. Arrange the elements of C_N^+ using the lexicographical ordering, and let $\mathbf{u}$ be the 'smallest' of the elements of $C_N^+ \cap S^*$ according to this ordering; i.e., $\mathbf{u} << \mathbf{t}$, for any $\mathbf{t} \in C_N^+ \cap S^*$ such that $\mathbf{t} \neq \mathbf{u}$.

Since $P(t_1, t_2, \ldots, t_n)$ is true whenever $\min_{k=1,2,\ldots,n} |t_k| = 0$, it follows that $\min_{k=1,2,\ldots,n} |u_k| > 0$, in other words, $\mathbf{u} \in \mathrm{N}^n$; therefore, the $Past_n(\mathbf{u})$ is not empty. Now since $\mathbf{u}$ is the 'smallest' of the elements of $C_N^+ \cap S^*$ it follows that $Past_n(\mathbf{u}) \subset S$, i.e., that $P(\mathbf{s})$ is true for any $\mathbf{s} \in Past_n(\mathbf{u})$. But by the induction hypothesis $P(\mathbf{u})$ should be true as well, which is a contradiction. □

Theorem 1 is one of many possible generalizations of the induction principle in n dimensions. For a different generalization, consider a different, non-directional notion of 'past' of point $\mathbf{t}$, namely define

$$Past(\mathbf{t}) = \{\mathbf{u} \in \mathrm{N}_0^n : u_j \leq t_j, \forall j\} - \{\mathbf{t}\}.$$

The following theorem is another generalization of the induction principle; its proof is similar to the proof of Theorem 1, bearing in mind that $Past(\mathbf{t}) \subset Past_n(\mathbf{t})$.

Theorem 2. *If $P(t_1, t_2, \ldots, t_n)$ is true whenever $\min_{k=1,2,\ldots,n} |t_k| = 0$, and if, for all $\mathbf{t} \in \mathrm{N}^n$ the assumption that $P(\mathbf{u})$ is true for any $\mathbf{u} \in Past(\mathbf{t})$, implies the conclusion that $P(\mathbf{t})$ is true, then $P(\mathbf{t})$ is true for any $\mathbf{t} \in \mathrm{N}_0^n$.*

Nonetheless, it seems that Theorem 1 is a more useful tool than Theorem 2 since it might be easier to verify that $P(\mathbf{t})$ is true given that $P(\mathbf{u})$ is true for any $\mathbf{u} \in Past_n(\mathbf{t})$, than to verify that $P(\mathbf{t})$ is true given that $P(\mathbf{u})$ is true for any $\mathbf{u} \in Past(\mathbf{t}) \subset Past_n(\mathbf{t})$.

4. Multidimensional recursions

Using either of the proposed generalizations of the principle of finite induction in n dimensions, i.e., Theorem 1 or Theorem 2, it follows immediately that the double sequence $X(i,j)$ defined in the Introduction is always less or equal to 1 in absolute value. However, another interesting point has been left unclear so far, i.e., how to actually carry out the recursion to find the value of $X(i,j)$ for some given pair (i,j). In this simple special two-dimensional case there are many equivalent ways to carry out the recursion, one of which is to order the points $(i,j) \in \mathrm{N}_0^2$ in the same way the (numerator, denominator) pairs corresponding to rational numbers are ordered with the purpose of showing that the set of rational numbers is countable; see, e.g., Birkhoff and MacLane (1977).

We now define a certain general notion of multidimensional recursion and give a way of calculating its values, i.e., carrying out the recursion. Let

$$L(\mathbf{t}) = \{\mathbf{u} : 0 < \sup_j |t_j - u_j| \leq 1\} \cap Past_n(\mathbf{t}),$$

i.e., $L(\mathbf{t})$ consists of the immediate neighbors of point $\mathbf{t}$ (in the sup-distance sense) that are in $Past_n(\mathbf{t})$. So we can define a multidimensional sequence $X(\mathbf{t})$, $\mathbf{t} \in \mathrm{N}_0^n$, by specifying the values of $X(\mathbf{t})$ for $\mathbf{t}$ on the boundary of N_0^n, i.e., for $\mathbf{t}$ such that $\min_{k=1,2,\ldots,n} |t_k| = 0$, and by the recursion $X(\mathbf{t}) = F(L(\mathbf{t}))$, $\mathbf{t} \in \mathrm{N}^n$, where F is some given function. In order to compute the value $X(\mathbf{t})$ for some given $\mathbf{t} \in \mathrm{N}^n$, consider the following scheme. Let $D(\mathbf{t}) = \{\mathbf{u} \in \mathrm{N}_0^n : \sum_j |u_j| \leq \sum_j |t_j|\}$; the points in set $D(\mathbf{t})$ can be ordered using the lexicographical ordering, and the recursion can proceed in that same order until $X(\mathbf{u})$ is known for all $\mathbf{u} \in D(\mathbf{t})$, and hence for point $\mathbf{t}$ as well.

For more general recursions, $X(\mathbf{t})$ could be a function of the r-close neighbors of point $\mathbf{t}$, again in the sup-distance sense, with $r \geq 1$; see Politis (1994) for an elaboration, as well as some discussion on the many different notions of 'past' in n dimensions. Needless to say, a most important tool for analyzing recursions is induction. In addition, it seems plausible that recursions that, as the sequence $X(\mathbf{t})$ defined above, can be carried out in the lexicographical ordering can be handled using the induction that is relative to the directional past $Past_n(\mathbf{t})$, i.e., Theorem 1.

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FIGURE: A TWO-DIMENSIONAL INTEGER LATTICE.