# The Solvability of the Convolution Equation $X \stackrel{d}{=} Z + Y$

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## ABSTRACT

We consider the convolution equation $X \stackrel{d}{=} Z + Y$ where $X, Z$ have given distributions $F, G$ on the Real line and ask if there exists $Y$ such that the convolution equation holds. Three types of cases are considered: when each of $X, Z$ is a normal scale mixture, when $Z$ has a symmetric stable law $G$ and $X$ is a scale mixture of $G$, and when $Z$ has a distribution $G$ with a characteristic function of regular variation and $X$ is a scale mixture of $G$. Each result is illustrated by examples. We also derive, in the second case, an interesting tail property of the convolution, which follows as an amusing consequence of a result in Integral transforms theory.

Key Words: Convolution, scale mixture, normal, symmetric, stable, characteristic functions, Jordan decomposition, regular variation, absolutely continuous.

**1. Introduction.** In this article, we study the solvability of the convolution equation $X \stackrel{d}{=} Z + Y$, where $X$ and $Z$ have given distributions, say $F$ and $G$. Usually $F$ and $G$ will be chosen as familiar distributions common in statistical or probabilistic modeling and applications. Sometimes, we will seek a solution $Y$ in a restricted class of random variables (equivalently distributions), for instance the class of Lebesgue absolutely continuous random variables. The typical result will be either a necessary or a necessary and sufficient condition for the appropriate convolution equation to have a solution. We will demonstrate that in a broad class of examples, unsolvability of appropriate convolution equations directly leads to the consequence that location and scale mixtures of a fixed type of law are mutually incompatible. Specifically, for instance, it will be shown that if $F$ is any distribution on the line with a regularly varying characteristic function, then no scale mixture $\int F(\frac{x}{\sigma})d\mu(\sigma)$ may be represented as $\int F(x-\theta)d\nu(\theta)$ with an absolutely continuous $\nu$. The Double Exponential law $F$ with a Lebesgue density $f(x) = \frac{1}{2}e^{-|x|}$ is one example. In addition to these, we will also derive in some cases properties of the convolution $X = Z + Y$ when $Z$ has a given distribution $F$ and $Y$ is of some appropriate specified type. For example, $F$ may be taken as a symmetric stable law and $Y$ as symmetric and absolutely continuous. The article will also present some positive examples, i.e., we give examples of (common) random variables $X$ and $Z$ such that $X \stackrel{d}{=} Z + Y$ is in fact solvable.

Convolutions have a long and rich history in probability and statistics. Besides the inherent mathematical interest in them, convolution models relate to measurement error models in statistics and to the marginal distribution of the data $X$ in Bayesian statistics. In fact, every result in this article can be restated as a result in Bayes theory when $X \sim F(x-\theta)$ and $\theta$ is an unknown location parameter. There is a truly vast literature in probability and statistics on convolutions and their properties. Directly relevant to this article are the works of Gastwirth, Rubin and Wolff (1967), Widder (1971), Pollard (1953) and an earlier article of the present author, DasGupta (1992). For a general basic exposition, one can see Feller (1973), and Champeney (1987), Lukacs (1970) for the theory of Fourier transforms, a major tool in the study of convolutions. Section 2 treats the case when $X, Z$ are both normal scale mixtures, i.e., $X \sim F = \int N(0,\sigma^2)d\mu(\sigma^2)$ and $Z \sim G = \int N(0,\sigma^2)d\Gamma(\sigma^2)$; section 3 treats the case when $Z$ has a symmetric stable distribution and section 4 considers the general case of $Z \sim F$ when $F$ has a characteristic function of regular variation.

**2. Normal Scale Mixtures.** Let $X \sim F = \int_{[0,\infty)} N(0,\sigma^2)d\mu(\sigma^2)$ and $Z \sim G = \int_{[0,\infty)} N(0,\sigma^2)d\Gamma(\sigma^2)$. We consider solvability of the convolution equation $X = Z + Y$. If $\Gamma$ is degenerate, say at $\sigma^2 = 1$, then the question being asked is what is the intersection of normal location and scale mixtures. See DasGupta (1992). We prove the following theorem.

**Theorem 2.1.** Let $X \sim F = \int_{[0,\infty)} N(0,\sigma^2)d\mu(\sigma^2)$ and $Z \sim G = \int_{[0,\infty)} N(0,\sigma^2)d\Gamma(\sigma^2)$. Let $\nu = \mu - \Gamma$ and let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of $\nu$. Then the convolution equation $X \stackrel{d}{=} Z + Y$ is unsolvable if there exists $t > 0$ such that $\nu^+[0,t) > 0$ and supp $(\nu^-) \cap [0,t)$ is empty.

**Proof:** Suppose in fact there exists $Y$ such that the convolution equation $X \stackrel{d}{=} Z + Y$ holds. Let $\psi_i(s), i = 1,2,3$ denote the characteristic functions of $X, Z, Y$ respecitvely. Then, clearly,

$$\psi_3(s) = \frac{\psi_1(s)}{\psi_2(s)} = \frac{\int e^{-\frac{s^2}{2}u}d\mu(u)}{\int e^{-\frac{s^2}{2}u}d\Gamma(u)}. \tag{2.1}$$

Clearly, then, if $\exists w > 0$ such that

$$\frac{\int e^{-wu}d\mu(u)}{\int e^{-wu}d\Gamma(u)} > 1, \tag{2.2}$$

then we would have reached a contradiction. However,

$$\begin{aligned}
(2.2) &\iff \exists w > 0 \ni \int e^{-wu}d\mu(u) > \int e^{-wu}d\Gamma(u) \\
&\iff \exists w > 0 \ni \int e^{-wu}d\nu(u) > 0 \\
&\iff \exists w > 0 \ni \int e^{-wu}d\nu^+(u) > \int e^{-wu}d\nu^-(u) \\
&\iff \exists w > 0 \ni \frac{\int e^{-wu}d\nu^+(u)}{\int e^{-wu}d\nu^-(u)} > 1. \qquad (2.3)
\end{aligned}$$

By hypothesis, however, $\exists\ \delta > 0 \ni \nu^+[0,t-\delta) > 0$ and $\nu^-[0,t) = 0$.

$$\begin{aligned}
\therefore \text{ for } w > 0, &\int e^{-wu}d\nu^+(u) \\
&\geq \int_{[0,t-\delta)} e^{-wu}\, d\nu^+(u) \\
&\geq e^{-w(t-\delta)}\nu^+[0,t-\delta], \qquad (2.4) \\
\text{whereas } &\int e^{-wu}d\nu^-(u) \\
&= \int_{[t,\infty)} e^{-wu}\, d\nu^-(u) \\
&\leq e^{-wt}\nu^-[t,\infty) \qquad (2.5)
\end{aligned}$$

From (2.4) and (2.5), then,

$$\begin{aligned}
&\frac{\int e^{-wu}d\nu^+(u)}{\int e^{-wu}d\nu^-(u)} \\
&\geq \frac{e^{-w(t-\delta)}\nu^+[0,t-\delta)}{e^{-wt}\nu^-[t,\infty)} \longrightarrow \infty \text{ as } w \longrightarrow \infty,
\end{aligned}$$

since $\nu^+[0, t-\delta) > 0$ and $\nu^-[t, \infty) < \infty$. The theorem is therefore proved.

**Corollary 2.2.** Let $\Gamma$ be the measure degenerate at 1. Then $X \stackrel{d}{=} Z + Y$ has no solution if $\mu[0,1) > 0$.

**Proof:** This follows on noting that the Jordan decomposition of $\nu = \mu - \Gamma$ in this case is

$$\begin{aligned} \nu^+(E) &= \mu(E) \text{ if } E \not\ni 1 \\ &= \mu(E - \{1\}) \text{ if } E \ni 1 \end{aligned} \tag{2.6}$$

$$\begin{aligned} \text{and } \nu^-(E) &= 0 \text{ if } E \not\ni 1 \\ &= 1 - \mu\{1\} \text{ if } E \ni 1 \end{aligned} \tag{2.7}$$

It then follows $\nu^-[0,1) = 0$ and $\nu^+[0,1) > 0$ and Theorem 2.1 applies. This corollary says that any normal scale mixture $\int N(0, \sigma^2) d\mu(\sigma^2)$ such that the mixing distribution $\mu$ assigns positive measure to $[0,1)$ cannot be a normal location mixture. This therefore implies that in particular for each $X$ as follows, $X$ cannot be written as $X \stackrel{d}{=} N(0,1) + Y$:

$X =$ any $t$ random variable

$X =$ any Double Exponential random variable

$X =$ any Logistic random variable

$X =$ any Hyperbolic cosine random variable.

Some of these can be established by other more direct methods; see DasGupta (1992) where it is proved the converse of this Corollary is also true, i.e., $\mu[0,1) = 0$ is sufficient for solvability of $X \stackrel{d}{=} Z + Y$.

**Corollary 2.3.** Suppose $\mu, \Gamma$ are each (Lebesgue) absolutely continuous with densities $f, g$ respectively. Then $X \stackrel{d}{=} Z + Y$ does not have a solution if $f > g$ in a neighborhood of zero.

**Proof:** Again, Theorem 2.1 applies because

$$\text{supp } (\nu^+) = \{u\colon\ f(u) > g(u)\}$$
$$\text{supp } (\nu^-) = \{u\colon\ f(u) < g(u)\}$$

Corollary 2.3 leads to many examples. We will only give one.

**Example 1.** Let

$$\begin{aligned} f(\sigma^2) &= \frac{\beta^2}{2} e^{-\frac{\beta^2}{2}\sigma^2} \\ \text{and } g(\sigma^2) &= \frac{\gamma^\alpha}{\Gamma(\alpha)} e^{-\frac{\gamma}{\sigma^2}} \left(\frac{1}{\sigma^2}\right)^{(\alpha+1)}. \end{aligned} \tag{2.8}$$

Thus $X$ has the Double Exponential $(0,\beta)$ distribution and $Y$ has the $t$-distribution with scale parameter $2\gamma$ and degrees of freedom $2\alpha$.

Since clearly $\frac{f(u)}{g(u)} \to \infty$ as $u \to 0$ and $f,g$ are continuous, it follows that the equation Double Exponential $(0,\beta)\stackrel{d}{=}t(0,2\gamma,2\alpha)+Y$ has no solution for any $\alpha,\beta,\gamma>0$.

It is somewhat disappointing but unfortunately it is the case that if $X,Z$ are each normal scale mixtures, then typically (but evidently not always) $X$ cannot be written as $X\stackrel{d}{=}Z+Y$. The following example is an example to the contrary.

**Example 2.** Let $X$ have the Standard Cauchy distribution and let $Z_3, Z_5$ respectively have the $t$-distribution with location prameter 0, scale parameter 1, and 3 and 5 degrees of freedom. We will show that $X\stackrel{d}{=}Z_i+Y$ has a solution for each $i=3,5$.

The characteristic function of every random variable under consideration is even and so it is enough to consider only positive arguments. For $t>0$, the characteristic functions of $X, Z_3$ and $Z_5$ are respectively

$$\begin{aligned} \psi_1(t) &= e^{-t} \\ \psi_3(t) &= e^{-t}(1+t) \\ \text{and } \psi_5(t) &= \frac{1}{3}e^{-t}(3+3t+t^2) \end{aligned} \tag{2.9}$$

$$\begin{aligned} \therefore \frac{\psi_1(t)}{\psi_3(t)} &= \frac{1}{1+t} \\ \text{and } \frac{\psi_1(t)}{\psi_5(t)} &= \frac{3}{3+3t+t^2} \end{aligned} \tag{2.10}$$

Clearly, each of $\frac{\psi_1(t)}{\psi_3(t)}$ and $\frac{\psi_1(t)}{\psi_5(t)}$ are decreasing for $t>0$ and easy calculus also shows that for $t>0$, they are convex. It follows that they are both Polya functions and hence characteristic functions (see Feller (1973)). The example is complete. In fact, by virtue of the Polya property, it also follows that in each case, the corresponding random variable $Y$ is absolutely continuous (see Lukacs (1970)).

**3. Symmetric Stable Laws.** In Section 2, it was proved that a given normal scale mixture $F=\int N(0,\sigma^2)d\mu(\sigma^2)$ is also a Gaussian convolution if and only if $\mu[0,1)=0$. We will first give a theorem which establishes the same phenomenon for any symmetric stable law.

**Theorem 3.1.** Let $Z\sim F$ be a symmetric stable random variable of given exponent $\alpha, 0<\alpha\le 2$. Let $X\sim \int_{[0,\infty)} F(\frac{x}{\sigma})d\mu(\sigma)$. Then $X\stackrel{d}{=}Z+Y$ if and only if $\mu[0,1)=0$, in which case $Y\sim\int F(\frac{x}{\eta})d\Gamma(\eta)$ where $\Gamma$ is the distribution of $\eta=(\sigma^\alpha-1)^{\frac{1}{\alpha}}$ when $\sigma\sim\mu$.

**Proof:** The proof of Theorem 2.1 can be repeated for the only if part. However, the following is a simple proof covering both parts.

Using the notation of Theorem 2.1,

$$\psi_1(s) = \int e^{-|s|^\alpha \sigma^\alpha} d\mu(\sigma) = e^{-|s|^\alpha} \cdot \psi_3(s)$$

$$\Longleftrightarrow \psi_3(s) = \int e^{-|s|^\alpha(\sigma^\alpha - 1)} d\mu(\sigma)$$

$$\geq \int\limits_{[0,1)} e^{-|s|^\alpha(\sigma^\alpha - 1)} d\mu(\sigma) \tag{3.1}$$

From (3.1), by Fatou's lemma, $\liminf_{s\to\infty} \psi_3(s) = \infty$ if $\mu[0,1) > 0$. This establishes the 'only if' part and also says that

$$\psi_3(s) = \int\limits_{[1,\infty)} e^{-|s|^\alpha(\sigma^\alpha - 1)} d\mu(\sigma) \tag{3.2}$$

and hence the remaining assertions of the Theorem also follow.

The next result goes one step further and characterizes the situation when $X$ can be represented as $X \stackrel{d}{=} Z + Y$ for an absolutley continuous random variable $Y$.

**Theorem 3.2.** Let $Z$ and $X$ be as in Theorem 3.1. Then the convolution equation $X \stackrel{d}{=} Z + Y$ holds for some absolutely continuous random variable $Y$ if and only if $\mu[0,1] = 0$.

**Proof:** Clearly, we need to show that

$$\psi_3(s) = \int\limits_{[0,\infty)} e^{-|s|^\alpha(\sigma^\alpha - 1)} d\mu(\sigma)$$

is the characteristic function of an absolutely continuous distribution if and only $\mu[0,1] = 0$. In view of Theorem 3.1, then, it is enough to show that

$$\psi_{3(s)} = \int\limits_{[0,\infty)} e^{-|s|^\alpha \eta^\alpha} d\Gamma(\eta)$$

is the characteristic function of an absolutely continuous distribution if and only if $\Gamma\{0\} = 0$. However, $\psi_3(s)$ is the characteristic function of an absolutely continuous distribution if and only if

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \psi_3^2(s) ds = 0$$

(see Lukacs (1970))

$$\Longleftrightarrow \lim_{n\to\infty} \frac{1}{n} \int_0^n \iint\limits_{[0,\infty)\times[0,\infty)} e^{-|s|^\alpha(u^\alpha + v^\alpha)} d\Gamma(u) d\Gamma(v) ds = 0$$

$$\Longleftrightarrow \Gamma^2\{0\} + \lim_{n\to\infty} \frac{1}{n} \int_0^n \iint\limits_{(0,\infty)\times(0,\infty)} e^{-|s|^\alpha(u^\alpha + v^\alpha)} d\Gamma(u) d\Gamma(v) ds = 0 \tag{3.3}$$

In (3.3), the limit of the second term exists and equals 0 by change of variable to $t = ns$ and then an application of the Dominated convergence theorem, and therefore the theorem is proved.

We will finally prove a property common to all smooth densities that arise as convolutions of any symmetric stable variable $Z$ with any other symmetric variable $Y$. This property comes out as an amusing consequence of a result in Integral transforms, obtained by Tagamlitzki (1946). We state this result for ease of reference.

**Lemma 3.3 (Tagamlitzki).** Let $f(x)$ be any measurable function on the line such that it is infinitely differentiable and satisfies for every $k \geq 0$ and $x \geq 0$, $|f^{(k)}(x)| \leq \alpha e^{-x}$ for some $\alpha > 0$. Then $f(x) = Ae^{-x}$ for $x > 0$, for some constant $A$.

**Theorem 3.4.** Let $Z$ have a symmetric stable distribution $F$. Then there does not exist any symmetric random variable $Y$ such that the density function $f$ of the convolution $X = Z + Y$ satisfies $|f^{(k)}(x)| \leq \alpha e^{-x}$ for all $k, x \geq 0$.

**Proof:** A verbal but rigorous explanation of the proof is the easiest to understand. By Lemma 3.3, if such a random variable $Y$ exists, then the convolution density $f$ will have to be the standard Double Exponential density $\frac{1}{2}e^{-|x|}$, which has the characteristic function $\frac{1}{1+s^2}$. However, if the exponent of the stable law of $Z$ is $\alpha$, then the characteristic function of the convolution $X = Z + Y$ has to be at least as small as $e^{-|s|^\alpha}$, and therefore $f$ cannot be the standard Double Exponential density, a contradiction.

## 4. Distributions with a Characteristic Function of Regular Variation.

The results of sections 2 and 3 give evidence that location and scale mixtures of the same type of law are frequently incompatible in the sense discussed in these sections. In this section, we will describe a broad class of problems in which this incompatibility becomes extreme. In the following, we make it precise. We remind the reader that Feller (1973) and Resnick (1987) give excellent expositions to functions of regular variation.

**Theorem 4.1.** Let $Z \sim F$ such that the characteristic function $\psi(s)$ of $F$ is of regular variation, and has no roots. Let $X \sim \int_{[0,\infty)} F(\frac{x}{\sigma}) d\mu(\sigma)$, where $\mu$ is an arbitrary probability measure on $[0, \infty)$, not degenerate at 0. Then there cannot exist any absolutely continuous random variable $Y$ such that the convolution equation $X \stackrel{d}{=} Z + Y$ holds.

**Proof:** Suppose such an absolutely continuous $Y$ exists and suppose its characteristic function is $\psi_2(s)$. Then,

$$\psi(s) \cdot \psi_2(s) = \int_{[0,\infty)} \psi(s\sigma) d\mu(\sigma)$$

$$\Longrightarrow \psi_2(s) = \int\limits_{[0,\infty)} \frac{\psi(s\sigma)}{\psi(s)} d\mu(\sigma). \tag{4.1}$$

Suppose $\psi$ is of regular variation of order $\rho$; therefore, for every $\sigma > 0$,

$$\lim_{s\to\infty} \frac{\psi(s\sigma)}{\psi(s)} = \sigma^\rho \tag{4.2}$$

Combining (4.1) and (4.2) with Fatou's Lemma, one has

$$\liminf_{s\to\infty} \psi_2(s) \geq \liminf_{s\to\infty} \int\limits_{(0,\infty)} \frac{\psi(s\sigma)}{\psi(s)} d\mu(\sigma) \geq \int\limits_{(0,\infty)} \sigma^\rho d\mu(\sigma) > 0,$$

and hence by the Riemann-Lebesgue Lemma, $\psi_2$ cannot be the characteristic function of an absolutely continuous distribution. The proof is complete.

**Example 3.** The standard Double Exponential distribution with density $\frac{1}{2}e^{-|x|}$ has the characteristic function $\frac{1}{1+s^2}$, which is of regular variation of order $-2$. Therefore no (nontrivial) Double Exponential scale mixture can be an absolutely continuous Double Exponential location mixture. The case when $X$ has the scaled Double Exponential $(0,\beta)$ distribution with $\beta > 1$ is of special interest. In this case, with $Z$ as the standard Double Exponential, $X \stackrel{d}{=} Z + Y$ is solvable, with $Y \sim (1 - \frac{1}{\beta^2})$ Double Exponential $(0,\beta) + \frac{1}{\beta^2}\delta\{0\}$, where $\delta\{0\}$ denotes point mass at 0. This example should be considered as known, however.

**Discussion on Theorem 4.1.** The applicability of Theorem 4.1 depends on the availability of characteristic functions of regular variation. There are many. For instance, take any absolutely continuous distribution $H$ with characteristic function $f$, say. Now consider the distribution $F$ with characteristic function $e^{-\lambda(1-f(s))}$; then this is slowly varying and hence of regular variation of order 0. Polya functions frequently are of regular variation; for instance, $f(s) = \frac{1}{1+s}$ is of regular variation of order $-1$. We state these as a corollary.

**Corollary 4.2.** Let the characteristic function $\psi(s)$ of $F$ be any of the following types:

- **i** $\psi(s) = e^{-\lambda(1-f(s))}$, where $\lambda > 0$ and $f(s)$ is the characteristic function of an absolutely continuous random variable;
- **ii** $\psi(s)$ is the Polya function

$$\psi(s) = \int\limits_0^\infty \left(1 - \frac{|s|}{\sigma}\right) I_{|s|\leq\sigma} d\mu(\sigma),$$

where the probability measure $\mu$ has a Lebesgue density $h$ of regular variation.

Then $F$ satisfies the hypothesis and hence the conclusion of Theorem 4.1.

**Proof of Corollary 4.2:** Both results are easy to prove.

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