NEW BOUNDS FOR HAHN AND KRAWTCHOUK POLYNOMIALS

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ABSTRACT

For the Hahn and Krawtchouk polynomials orthogonal on the set $\{0, \ldots, N\}$ new identities for the sum of squares are derived which generalize the trigonometric identity for the Chebyshev polynomials of the first and second kind. These results are applied in order to obtain conditions (on the degree of the polynomials) such that the polynomials are bounded (on the interval $[0, N]$) by their values at the points 0 and N. As special cases we obtain a discrete analogue of the trigonometric identity and bounds for the discrete Chebyshev polynomials of the first and second kind.

1. Introduction. The Hahn polynomials may be defined in terms of a hypergeometric series

$$Q_n(x, \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, & n+\alpha+\beta+1, & -x; & 1 \\ \alpha+1, & -N & & \end{matrix}\right)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha+\beta+1)_k (-x)_k}{k!\,(\alpha+1)_k (-N)_k} \qquad (n = 0, \ldots, N)$$

where $\alpha, \beta > -1$, $(a)_0 = 1, (a)_k = a(a+1)\ldots(a+k-1)$. These polynomials are limiting cases of some general systems of orthogonal polynomials (see Hahn (1949)) and satisfy for

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$n, m = 0, \ldots, N$ the orthogonality relation

$$\sum_{x=0}^{N} \rho(x,\alpha,\beta,N) Q_m(x,\alpha,\beta,N) Q_n(x,\alpha,\beta,N) \;=\; \frac{\delta_{nm}}{\pi_n(\alpha,\beta,N)} \tag{1.1}$$

where

$$\rho(x,\alpha,\beta,N) \;=\; \frac{\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x}}{\binom{N+\alpha+\beta+1}{N}} \tag{1.2}$$

and

$$\pi_n(\alpha,\beta,N) \;=\; \frac{(-1)^n(-N)_n(\alpha+1)_n(\alpha+\beta+1)_n}{n!\;(N+\alpha+\beta+2)_n(\beta+1)_n}\;\frac{2n+\alpha+\beta+1}{\alpha+\beta+1}. \tag{1.3}$$

For some properties and applications of the Hahn polynomials we refer the reader to the work of Karlin McGregor (1961, 1962), Gasper (1974, 1975) and Wilson (1970). The polynomials $Q_n(x,\alpha,\beta,N)$ can be seen as the discrete analogue of the Jacobi polynomials and most of the "classical" orthogonal polynomials can be obtained as limits from the Hahn polynomials when the parameters tend to infinity (see Gasper (1975)).

As an example we consider the Krawtchouk polynomials which can be defined as the limit ($q = 1-p, p \in (0,1)$)

$$\begin{aligned} k_n(x,p,N) \;&=\; \lim_{t\to\infty} Q_n(x,pt,qt,N) \;=\; {}_2F_1(-n,-x,-N;1/p) \\ &=\; \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!\;(-N)_k}\left(\frac{1}{p}\right)^k \end{aligned} \tag{1.4}$$

and are orthogonal with respect to the jump function

$$\binom{N}{x} p^x (1-p)^{N-x} \qquad\qquad x = 0, \ldots, N \tag{1.5}$$

(see Krawtchouk (1929)).

In this paper we will discuss some new properties of the orthogonal polynomials with respect to the measures (1.2) and (1.5). After presenting some preliminary results in Section 2 we present new identities for squares of Krawtchouk and Hahn polynomials in Section 3 and 4 which generalize the trigonometric identity for the Chebyshev polynomials of the first and second kind. We will apply these results in order to obtain conditions (on the degree of the polynomials) such that the polynomials $Q_n(x,\alpha,\beta,N)$ and $k_n(x,p,N)$ are bounded on the interval $[0,N]$ by their values at the points 0 and N. For the Hahn polynomials these bounds extend and improve results of Zaremba (1975) while for the Krawtchouk polynomials it is shown that

$$|k_n(x,p,N)| \leq \max\{|k_n(0,p,N)|,|k_n(N,p,N)|\} = \max\{1,(\frac{q}{p})^n\} \qquad x\in[0,N]$$

whenever the degree of the polynomial satisfies $n \leq \frac{N}{2}+1$. Similar results are also given for the dual Hahn polynomials and the Hahn- Eberlein polynomials.

2. Preliminaries. In this section we will briefly discuss some general aspects of orthogonal polynomials which will be needed in the following sections. The notation used here is that of Karlin and Shapely (1953) and Karlin and Studden (1966). Let ξ denote a probability measure on the interval $[0,N]$ with moments

$$c_j = \int_0^N x^j d\xi(x) \qquad (j=0,\ldots,N)$$

and let $P_\ell(x), Q_\ell(x), R_\ell(x), S_\ell(x)$ denote the orthonormal polynomials with respect to the measures $d\xi(x), x(N-x)d\xi(x), xd\xi(x)$ and $(N-x)d\xi(x)$, respectively. The leading coefficients of these polynomials can be expressed by ratios of the determinants

$$(2.1) \qquad \begin{cases} \underline{D}_{2\ell}(\xi) = |(c_{i+j})_{i,j=0}^{\ell}| & \overline{D}_{2\ell} = |(Nc_{i+j-1}-c_{i+j})_{i,j=1}^{\ell}| \\ \underline{D}_{2\ell+1}(\xi) = |(c_{i+j+1})_{i,j=0}^{\ell}| & \overline{D}_{2\ell+1} = |(Nc_{i+j}-c_{i+j+1})_{i,j=0}^{\ell}| \end{cases}$$

(see e.g. Karlin and Studden (1966 p. 109)). For a point $(c_1,\ldots,c_\ell)$ in the interior of the moment space

$$\mathcal{M}_\ell = \{(c_1,\ldots,c_\ell)|c_j = \int_0^N x^j d\xi(x) \text{ for some probability measure on } [0,N] \ (j=1,\ldots,l)\}$$

let $(c_1,\ldots,c_{\ell-1},c_\ell^-)$ and $(c_1,\ldots,c_{\ell-1},c_\ell^+)$ denote the boundary points of $\mathcal{M}_\ell$ corresponding to the lower and upper principal representation associated with the point $(c_1,\ldots,c_{\ell-1})\in$ *int* $(\mathcal{M}_{\ell-1})$ (see Karlin and Studden (1966), p.55). It is well known (see e.g. Karlin and Studden (1966), p. 55) that the quantities c_ℓ^+ and c_ℓ^- can be expressed in terms of the determinants (2.1), that is

$$(2.2) \qquad c_\ell^+ = c_\ell + \frac{\overline{D}_\ell(\xi)}{\overline{D}_{\ell-2}(\xi)}, \qquad c_\ell^- = c_\ell - \frac{\underline{D}_\ell(\xi)}{\underline{D}_{\ell-2}(\xi)} \qquad \ell \geq 1$$

where we define $\underline{D}_{-1}(\xi) = \underline{D}_0(\xi) = \overline{D}_{-1}(\xi) = \overline{D}_0(\xi) = 1$ (note that the ratios in (2.2) are well defined because $(c_1, \ldots, c_{\ell-1}) \in \; int \; (\mathcal{M}_{\ell-1})$).

In the following we will make use of the determinants defined in (2.1) where the moment of highest order is replaced by $c_{2\ell}^+$ ($c_{2\ell+1}^+$) in the determinants $\underline{D}_{2\ell}(\xi)$ ($\underline{D}_{2\ell+1}(\xi)$) and by $c_{2\ell}^-$ ($c_{2\ell+1}^-$) in the determinants $\overline{D}_{2\ell}(\xi)$ ($\overline{D}_{2\ell+1}(\xi)$). The corresponding modified determinants are denoted by $\underline{D}_{2\ell}^+(\xi), \underline{D}_{2\ell+1}^+(\xi), \overline{D}_{2\ell}^-(\xi)$ and $\overline{D}_{2\ell+1}^-(\xi)$, respectively. Using the representation (2.2) it is then easy to see that

$$
(2.3) \qquad \begin{cases} \underline{D}_j^+(\xi) &= \underline{D}_j(\xi) + \dfrac{\underline{D}_{j-2}(\xi)}{\overline{D}_{j-2}(\xi)} \overline{D}_j(\xi) \quad j = 2\ell, \; 2\ell+1 \\ \overline{D}_j^-(\xi) &= \overline{D}_j(\xi) + \dfrac{\overline{D}_{j-2}(\xi)}{\underline{D}_{j-2}(\xi)} \underline{D}_j(\xi) \quad j = 2\ell, \; 2\ell+1. \end{cases}
$$

In a recent paper Dette (1994) established new identities for the orthonormal polynomials $P_\ell(x), Q_\ell(x), R_\ell(x)$ and $S_\ell(x)$ with respect to the measures $d\xi(x)$, $x(N-x)d\xi(x)$, $xd\xi(x)$ and $(N-x)d\xi(x)$ respectively. For example, it is shown that for any arbitrary probability measure ξ on the interval $[0, N]$ the corresponding orthonormal polynomials satisfy the identity

$$
\sum_{\ell=1}^{n-1} \frac{\underline{D}_{2\ell-1}(\xi)}{\overline{D}_{2\ell-1}(\xi)} \left[\frac{\overline{D}_{2\ell-2}(\xi)}{\underline{D}_{2\ell-2}(\xi)} - \frac{\overline{D}_{2\ell}(\xi)}{\underline{D}_{2\ell}(\xi)} \right] P_\ell^2(x) \; + \; \frac{\underline{D}_{2n-1}(\xi) \, \overline{D}_{2n-2}(\xi) \, \underline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi) \, \underline{D}_{2n-2}(\xi) \, \underline{D}_{2n}^+(\xi)} P_n^2(x)
$$

$$
(2.4) \qquad + \; (N-x) \sum_{\ell=0}^{n-1} \frac{\overline{D}_{2\ell}(\xi)}{\underline{D}_{2\ell}(\xi)} \left[\frac{\underline{D}_{2\ell-1}(\xi)}{\overline{D}_{2\ell-1}(\xi)} - \frac{\underline{D}_{2\ell+1}(\xi)}{\overline{D}_{2\ell+1}(\xi)} \right] S_\ell^2(x)
$$

$$
= \; 1 \; - \; x(N-x) \frac{\underline{D}_{2n-1}(\xi) \overline{D}_{2n}(\xi)}{\overline{D}_{2n-1}(\xi) \underline{D}_{2n}^+(\xi)} Q_{n-1}^2(x)
$$

(note that the identities were originally stated on the interval $[-1, 1]$ but can easily be transferred to arbitrary intervals). If $N = 1$ and

$$
d\xi(x) = \frac{1}{\pi \sqrt{x(1-x)}}
$$

is the arcsin distribution, then it is straightforward to show that $\overline{D}_{2\ell}(\xi) = \underline{D}_{2\ell}(\xi) = (\frac{1}{2})^{\ell(2\ell+1)}, \overline{D}_{2\ell+1}(\xi) = \underline{D}_{2\ell+1}(\xi) = (\frac{1}{2})^{(\ell+1)(2\ell+1)}$ (see e.g. Karlin and Studden (1966), p.122). The polynomials $P_\ell(x)$ and $Q_\ell(x)$ are proportional to the Chebyshev polynomials of the first and second kind (on the interval $[0,1]$) and the identity (2.4) reduces to the trigonometric identity. In this sense (2.4) can be seen as an extension of the trigonometric identity for arbitrary orthogonal polynomials on compact intervals. For the Jacobi polynomials identities of the form (2.4) have been estiblished in Dette (1994). In order to derive similar results for the Hahn and Krawtchouk polynomials we need explicit expressions for the determinants of the moment matrices corresponding to the jump functions in (1.2) and (1.5) which will be derived in the following sections.

3. Identities and bounds for Hahn polynomials. It follows from (1.1) and (1.3) that the jump function in (1.2) defines a (discrete) probability measure ξ_ρ on the set $\{0,\dots,N\}$ and that the orthonormal polynomials with respect to the measure $d\xi_\rho(x)$ are given by $\sqrt{\pi_n(\alpha,\beta,N)}Q_n(x,\alpha,\beta,N)$ $(n=0,\dots,N)$. Using the elementary properties of the Gamma function and (1.1) we obtain

$$\sum_{x=0}^{N} Q_m(x-1,\alpha+1,\beta+1,N-2)Q_n(x-1,\alpha+1,\beta+1,N-2)x(N-x)\rho(x,\alpha,\beta,N)$$

$$= \sum_{x=0}^{N-2} Q_m(x,\alpha+1,\beta+1,N-2)Q_n(x,\alpha+1,\beta+1,N-2)$$

$$\times \rho(x,\alpha+1,\beta+1,N-2)\frac{N(N-1)(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)}$$

$$= \frac{N(N-1)(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)} \cdot \frac{\delta_{m,n}}{\pi_n(\alpha+1,\beta+1,N-2)}$$

which shows that the polynomials

$$\text{(3.1)} \quad \sqrt{\frac{(\alpha+\beta+2)(\alpha+\beta+3)}{N(N-1)(\alpha+1)(\beta+1)}\pi_n(\alpha+1,\beta+1,N-2)}\; Q_n(x-1,\alpha+1,\beta+1,N-2)$$

$(n=0,\dots,N-2)$ are orthonormal with respect to the measure $x(N-x)d\xi_\rho(x)$. Similarly it can be shown that the orthonormal polynomials with respect to the measures $xd\xi_\rho(x)$ and $(N-x)d\xi_\rho(x)$ are given by

$$\text{(3.2)} \qquad \sqrt{\frac{(\alpha+\beta+2)}{(\alpha+1)N}\pi_n(\alpha+1,\beta,N-1)}\; Q_n(x-1,\alpha+1,\beta,N-1)$$

$(n = 0, \dots, N-1)$ and by

$$\sqrt{\frac{(\alpha+\beta+2)}{(\beta+1)N}\pi_n(\alpha,\beta+1,N-1)}\; Q_n(x,\alpha,\beta+1,N-1) \tag{3.3}$$

$(n = 0, \dots, N-1)$, respectively.

Theorem 3.1. For $\ell = 0, \dots, N$ define $h_\ell(x,\alpha,\beta,N) = \frac{(\alpha+1)_\ell}{(\beta+1)_\ell} Q_\ell(x,\alpha,\beta,N)$, then the Hahn polynomials satisfy the following identities

a) For $n = 0, \dots, N-1$:

$$\sum_{\ell=1}^{n-1} \frac{2\ell+\alpha+\beta+1}{N}\{(\alpha+\beta+1)(2\ell-N)+2\ell^2\}\left\{\frac{(\alpha+\beta+2)_{\ell-1}}{(N+\alpha+\beta+2)_\ell}\binom{N}{\ell}h_\ell(x,\alpha,\beta,N)\right\}^2$$

$$+\ \left\{\binom{N-1}{n-1}\frac{(\alpha+\beta+2)_{n-1}}{(N+\alpha+\beta+2)_{n-1}}h_n(x,\alpha,\beta,N)\right\}^2$$

$$+\ (\beta-\alpha)(1-\frac{x}{N})\sum_{\ell=0}^{n-1}\frac{2\ell+\alpha+\beta+2}{(\beta+1)^2}\left\{\binom{N-1}{\ell}\frac{(\alpha+\beta+2)_\ell}{(\alpha+\beta+N+2)_\ell}h_\ell(x,\alpha,\beta+1,N-1)\right\}^2$$

$$=\ 1\ -\ \frac{x}{N}(1-\frac{x}{N})\left\{\frac{(\alpha+\beta+2)_n}{(\alpha+\beta+N+2)_{n-1}}\binom{N-2}{n-1}\frac{h_{n-1}(x-1,\alpha+1,\beta+1,N-2)}{\beta+1}\right\}^2$$

b) For $n = 0, \dots, N-1$:

$$\sum_{\ell=1}^{n}\frac{2\ell+\alpha+\beta+1}{N}\{(\alpha+\beta+1)(2\ell-N)+2\ell^2\}\left\{\binom{N}{\ell}\frac{(\alpha+\beta+2)_{\ell-1}}{(\alpha+\beta+N+2)_\ell}Q_\ell(x,\alpha,\beta,N)\right\}^2$$

$$+\ \frac{x}{N}\left\{\binom{N-1}{n}\frac{(\alpha+\beta+2)_n}{(\alpha+\beta+N+2)_n}\frac{\alpha+n+1}{\alpha+1}Q_n(x-1,\alpha+1,\beta,N-1)\right\}^2$$

$$+\ (\alpha-\beta)\frac{x}{N}\sum_{\ell=0}^{n-1}(2\ell+\alpha+\beta+2)\left\{\binom{N-1}{\ell}\frac{(\alpha+\beta+2)_\ell}{(\alpha+\beta+N+2)_\ell}\frac{Q_\ell(x-1,\alpha+1,\beta,N-1)}{\alpha+1}\right\}^2$$

$$=\ 1\ -\ (1-\frac{x}{N})\left\{\binom{N-1}{n}\frac{(\alpha+\beta+2)_n}{(\alpha+\beta+N+2)_n}Q_n(x,\alpha,\beta+1,N-1)\right\}^2$$

c) For $n = 0, \ldots, N-2$:

$$x(1-\frac{x}{N})\sum_{\ell=0}^{n-1}\{(\alpha+\beta+\ell+2)(N-2\ell-2)-(\ell+1)N\}(2\ell+\alpha+\beta+3)$$

$$\times\left\{\frac{h_\ell(x-1,\alpha+1,\beta+1,N-2)}{(\beta+1)(N-1)}\right\}^2$$

$$+\ \frac{x}{N}(1-\frac{x}{N})\left\{\frac{(\alpha+\beta+2+n)(N-n-1)}{(\beta+1)(N-1)}h_n(x-1,\alpha+1,\beta+1,N-2)\right\}^2$$

$$+\ (\beta-\alpha)(1-\frac{x}{N})\sum_{\ell=0}^{n}(2\ell+\alpha+\beta+2)\left\{\frac{h_\ell(x,\alpha,\beta+1,N-1)}{\beta+1}\right\}^2 \ =\ 1\ -\ \{h_{n+1}(x,\alpha,\beta,N)\}^2$$

d) For $n = 0, \ldots, N-1$:

$$\sum_{\ell=1}^{n}\frac{2\ell+\alpha+\beta+1}{N}\{(\alpha+\beta+1)(2\ell-N)+2\ell^2\}\left\{\frac{(\alpha+\beta+2)_{\ell-1}}{(\alpha+\beta+N+2)_\ell}\binom{N}{\ell}h_\ell(x,\alpha,\beta,N)\right\}^2$$

$$+\ (\beta-\alpha)(1-\frac{x}{N})\sum_{\ell=0}^{n-1}(2\ell+\alpha+\beta+2)\left\{\binom{N-1}{\ell}\frac{(\alpha+\beta+2)_\ell}{(\alpha+\beta+N+2)_\ell}\frac{h_\ell(x,\alpha,\beta+1,N-1)}{\beta+1}\right\}^2$$

$$+\ (1-\frac{x}{N})\left\{\binom{N-1}{n}\frac{\beta+n+1}{\beta+1}\frac{(\alpha+\beta+2)_n}{(\alpha+\beta+N+2)_n}h_n(x,\alpha,\beta+1,N-1)\right\}^2$$

$$=\ 1\ -\ \frac{x}{N}\left\{\binom{N-1}{n}\frac{(\alpha+\beta+2)_n}{(\alpha+\beta+N+2)_n}h_n(x-1,\alpha+1,\beta,N-1)\right\}^2$$

Proof. We will only give a proof of the identity (a) using the general result in (2.4). All other cases are treated similarly where the identity (2.4) has to be replaced by the corresponding results in Dette (1994). Observing (2.4), (3.1), (3.2) and (3.3) we have to find the determinants $\underline{D}_{2\ell}(\xi_\rho), \overline{D}_{2\ell}(\xi_\rho), \underline{D}_{2\ell-1}(\xi_\rho), \overline{D}_{2\ell-1}(\xi_\rho)$ where ξ_ρ is the probability measure corresponding to the jump function (1.2). But these determinants can easily be calculated from the leading coefficients of the orthonormal polynomials with respect to the measures $d\xi_\rho(x), xd\xi_\rho(x), (N-x)d\xi_\rho(x), x(N-x)d\xi_\rho(x)$ (see e.g. Karlin and Studden (1966), p. 110). For example, the orthonormal polynomial with respect to the measure

$x(N-x)d\xi_\rho(x)$ is the Hahn polynomial given in (3.1) and the leading coefficient is obtained from the definition of the Hahn polynomials in terms of the hypergeometric series (see Section 1). Thus we have for the leading coefficient of the polynomial in (3.1)

$$\sqrt{\frac{(\alpha+\beta+2)(\alpha+\beta+3)}{N(N-1)(\alpha+1)(\beta+1)}\pi_n(\alpha+1,\beta+1,N-2)} \cdot \frac{(n+\alpha+\beta+3)_n}{(\alpha+2)_n(-N+2)_n} = (-1)^n \cdot \sqrt{\frac{\overline{D}_{2n}(\xi_\rho)}{\overline{D}_{2n+2}(\xi_\rho)}}$$

or equivalently (using (1.3))

$$\frac{\overline{D}_{2n+2}(\xi_\rho)}{\overline{D}_{2n}(\xi_\rho)} = \frac{n!\,(\alpha+1)_{n+1}(\beta+1)_{n+1}(N+\alpha+\beta+2)_n(N-n-1)_{n+2}}{(\alpha+\beta+2)_{n+1}(n+\alpha+\beta+3)_n(n+\alpha+\beta+3)_{n+1}}.$$

Similarly we obtain for the ratio of $\underline{D}_{2n}(\xi_\rho)$ and $\underline{D}_{2n-2}(\xi_\rho)$

$$\frac{\underline{D}_{2n}(\xi_\rho)}{\underline{D}_{2n-2}(\xi_\rho)} = \frac{n!\,(\alpha+1)_n(\beta+1)_n(\alpha+\beta+N+2)_n(N-n+1)_n}{(\alpha+\beta+n+1)_{n+1}(\alpha+\beta+n+1)_n(\alpha+\beta+2)_{n-1}}$$

and a straightforward computation yields

$$\text{(3.4)} \qquad \frac{\overline{D}_{2n}(\xi_\rho)}{\underline{D}_{2n}(\xi_\rho)} = \frac{(N-n)(\alpha+\beta+n+1)}{n(N+\alpha+\beta+n+1)}\frac{\overline{D}_{2n-2}(\xi)}{\underline{D}_{2n-2}(\xi)} = \frac{(N-n)_n(\alpha+\beta+2)_n}{n!(N+\alpha+\beta+2)_n}.$$

In the same way we find

$$\text{(3.5)} \qquad \frac{\underline{D}_{2n-1}(\xi_\rho)}{\overline{D}_{2n-1}(\xi_\rho)} = \frac{(\alpha+1)_n}{(\beta+1)_n}, \qquad \frac{\underline{D}_{2n}(\xi_\rho)}{\underline{D}^+_{2n}(\xi_\rho)} = \frac{n}{N}\frac{\alpha+\beta+N+n+1}{\alpha+\beta+2n+1}$$

and

$$\text{(3.6)} \qquad \frac{\overline{D}_{2n}(\xi_\rho)}{\underline{D}^+_{2n}(\xi_\rho)} = \frac{(\alpha+\beta+2)_n(N-n)_n}{(n-1)!\,(N+\alpha+\beta+2)_{n-1}(\alpha+\beta+2n+1)N}$$

where we have used the representation (2.3) and (3.4). The orthonormal polynomials with respect to the measures $(N-x)d\xi_\rho(x)$ and $x(N-x)d\xi_\rho(x)$ are given by (3.3) and (3.1) and the assertion (a) of Theorem 3.1 follows now from (2.4), (3.4), (3.5), (3.6) and straightforward but tedious algebra. □

The Jacobi polynomials $P_\ell^{(\alpha,\beta)}(x)$ orthogonal with respect to the (continuous) measure $(1-x)^\alpha(1+x)^\beta dx$ and with leading coefficient $2^{-\ell}\binom{2\ell+\alpha+\beta}{\ell}$ can be obtained as limits from the Hahn polynomials

$$\text{(3.7)} \qquad P_n^{(\alpha,\beta)}(x) = \lim_{N\to\infty}\binom{n+\alpha}{\alpha}Q_n(N\frac{1-x}{2},\alpha,\beta,N)$$

and replacing x by $-x$ it is straightforward to show that for the limit (3.7) Theorem 3.1 gives the corresponding formulas for the Jacobi polynomials in Dette (1994). For these polynomials it is well known that $|P_n^{(\alpha,\beta)}(x)|$ is bounded by $\max\{|P_n^{(\alpha,\beta)}(-1)|, |P_n^{(\alpha,\beta)}(1)|\}$ $(n \in I\!N)$ if $\max\{\alpha, \beta\} > -\frac{1}{2}$. An upper bound but not necessarily sharp bound for arbitrary parameters is given by Erdélyi, Magnus and Nevai (1992). For the Hahn polynomials the situation is more complicated. Zaremba (1975) showed that

$$|Q_n(x, \alpha, \beta, N)| \;\leq\; 1 \tag{3.8}$$

for $x = 0, \ldots, N$ provided that $\alpha \geq \beta > -1$, $n(n+1) \leq N$ and

$$\alpha^2 + \beta^2 - \alpha\beta + \alpha + \beta \;\geq\; 0. \tag{3.9}$$

In the following theorem we will give an alternative bound for these polynomials, where the restriction on the degree of the polynomials satisfying (3.8) depends on the parameters of the weight function (1.2) and the inequality holds for all $x \in [0, N]$.

Theorem 3.2. Let $\alpha + \beta > -1$ and

$$n(\alpha, \beta, N) \;:=\; -\frac{1}{2}\{(\alpha + \beta - 1) \;-\; \sqrt{(\alpha + \beta + 1)(\alpha + \beta + 2N + 1)}\}, \tag{3.10}$$

then the nth Hahn polynomial satisfies for all $x \in [0, N]$ and all $n \leq n(\alpha, \beta, N)$ the inequality

$$|Q_n(x, \alpha, \beta, N)| \;\leq\; \max\left\{1, \frac{(\beta+1)_n}{(\alpha+1)_n}\right\} \;=\; \max\left\{|Q_n(0, \alpha, \beta, N)|\;,\; |Q_n(N, \alpha, \beta, N)|\right\}.$$

Proof: The second identity follows from Karlin McGregor (1961), equation (1.14). Let $\beta \geq \alpha$ and $\alpha + \beta > -1$, by (3.10) all terms on the left hand side of the identity in Theorem 3.1(c) are positive which yields (here we use the case $n - 1$ in 3.1c))

$$|Q_n(x, \alpha, \beta, N)| \;\leq\; \frac{(\beta+1)_n}{(\alpha+1)_n}$$

for all $x \in [0, N]$. If $\alpha \geq \beta$ we use the symmetry relation

$$Q_n(x, \alpha, \beta, N) \;=\; (-1)^n \frac{(\beta+1)_n}{(\alpha+1)_n} Q_n(N - x, \beta, \alpha, N)$$

(see e.g. Nikifarov, Suslov and Uvarov (1991), equation (2.4.18), or Karlin and McGregor (1961), equation (1.15), but note that both references use a different notation) and obtain from the first part of the proof

$$|Q_n(x,\alpha,\beta,N)| = \left|\frac{(\beta+1)_n}{(\alpha+1)_n} Q_n(N-x,\beta,\alpha,N)\right| \leq 1$$

for all $x \in [0,N]$. This completes the proof of the theorem. □

Remark 3.3. Zaremba (1975) proved (3.8) for $\alpha \geq \beta > -1$ satisfying (3.9),$n(n+1) \leq N$ but only for the integers $x = 0,\ldots,N$ while Theorem 3.2 gives the sup-norm of the Hahn polynomials for all $\alpha+\beta > -1$. By restricting on the set $\{0,1,\ldots,N\}$ and $\alpha \geq \beta > -1$ Zaremba's bound on the degree of the polynomials (such that (3.8) is satisfied) is comparable with (3.10). If $\alpha = \beta = 0$ we obtain from Zaremba (1975) that (3.8) holds for all $n \leq (-1+\sqrt{4N+1})/2$ while Theorem 3.2 establishes the (for $N \geq 13$ weaker) bound $(1+\sqrt{2N+1})/2$. This can be explained by the fact that Zaremba's approach is directly related to the discrete Legendre polynomials $Q_n(x,0,0,N)$ (and to the integers $\{0,\ldots,N\}$) and the general case is obtained using a projection formula and results of Askey and Gasper (1971) (for this step the condition (3.9) is used). However, in most cases Theorem 3.2 will provide a better bound on the degree of the Hahn polynomials such that (3.8) is satisfied. Furthermore the condition (3.9) is not needed for establishing these bounds. For example, if $\alpha+\beta \geq 1$ and $N \geq 3$, then it is easy to see that $(-1+\sqrt{4N+1})/2 \leq n(\alpha,\beta,N)$ and consequently Theorem 3.2 gives a better bound on the degree of the polynomials, compared to the results of Zaremba (1975). Moreover if $n \leq n(\alpha,\beta,N)$, (3.8) is satisfied for all $x \in [0,N]$. As further example consider the case $\alpha = \beta > -\frac{1}{2}$ and $\beta(\beta+2) < 0$, then (3.9) is not satisfied and Zaremba's results can not be applied. However, we obtain readily from Theorem 3.2 that (3.8) holds for all $x \in [0,N]$ whenever $n \leq \{-(2\beta-1)+\sqrt{(2\beta+1)(2\beta+1+2N)}\}/2$.

Zaremba (1975) considered also the example

$$Q_n(2,-\frac{1}{2},-\frac{1}{2},n^2) = -\frac{5}{3} \qquad (n \geq 2) \tag{3.11}$$

in order to show that the condition (3.9) can not be relaxed. In this case Theorem 3.2 is not applicable and (3.11) indicates that the Hahn polynomials $Q_n(x,\alpha,\beta,N)$ may not be bounded by their absolute values at the points 0 and N if $\alpha+\beta \leq -1$. Nevertheless, the following result provides a bound for these polynomials without a restriction on their degree.

Theorem 3.4. Let $\alpha+\beta \le -1$ and $n \in \{0,\dots,N-1\}$ then the Hahn polynomials $Q_n(x,\alpha,\beta,N)$ satisfy for all $x \in [0,N]$ the inequality

$$(3.12)\quad |Q_n(x,\alpha,\beta,N)| \;\le\; \max\left\{ 1\,,\; \frac{(\beta+1)_n}{(\alpha+1)_n} \right\} \cdot \frac{(\alpha+\beta+2+N)_{n-1}}{(\alpha+\beta+2)_{n-1}} \frac{(n-1)!}{(N-n+1)_{n-1}}.$$

Proof: Let $\beta \ge \alpha$, then by the assumptions all terms in the sums of Theorem 3.1(a) are positive. Consequently we have

$$\left| \binom{N-1}{n-1} \frac{(\alpha+\beta+2)_{n-1}}{(\alpha+\beta+N+2)_{n-1}} h_n(x,\alpha,\beta,N) \right| \;\le\; 1$$

which is equivalent to (3.12) for $\beta \ge \alpha$. The case $\alpha \le \beta$ is similar as in the proof of Theorem 3.2 and therefore omitted. □

Remark 3.5. Note that in general the bound (3.12) can not be improved. This follows readily from (3.11) for $N=4, n=2$ $(\alpha=\beta=-\frac{1}{2})$ because in this case the right hand side of (3.12) is also given by $\frac{5}{3}$.

For $\alpha=\beta=-\frac{1}{2}$ we obtain the discrete analogue of the Chebyshev polynomials which are of particular interest and considered in the following corollary. This result gives a "discrete" version of the trigonometric identity (part (a)).

Corollary 3.6. Let $T_n(x,N) = Q_n(x,-\frac{1}{2},-\frac{1}{2},N)$ and $U_n(x,N) = Q_n(x,\frac{1}{2},\frac{1}{2},N)$ denote the discrete Chebyshev polynomials of the first and second kind, respectively, then we have the following for all $x \in [0,N]$.

a) For $n = 0,\dots,N-1$:

$$-x(x-\frac{x}{N})\sum_{\ell=0}^{n-1}(\ell+1)\left\{\frac{4(\ell+1)}{N-1}U_l(x-1,N-2)\right\}^2$$
$$+\; T_{n+1}^2(x,N) \;+\; \frac{x}{N}(1-\frac{x}{N})\left\{\frac{2(n+1)(N-n-1)}{N-1}U_n(x-1,N-2)\right\}^2 \;=\; 1\,.$$

b) For $n = 0, \dots, N-1$:

$$|T_n(x,N)| \leq \prod_{j=1}^{n-1}\left(1+\frac{n}{N-n+j}\right) .$$

c) For $0 \leq n \leq \sqrt{N+1}$:

$$|U_n(x,N)| \leq 1 .$$

Remark 3.7. Observing that the Jacobi polynomials can be obtained as the limit (3.7) from the Hahn polynomials and using formula (4.17) in Szegö (1975) it is easy to see that the first part of Corollary 3.5 yields ($N \to \infty$ $x = \frac{N}{2}(1-z)$) the trigonometric identity $(1-z^2)U_n^2(z) + T_{n+1}^2(z) = 1$ for the Chebyshev polynomial of the first and second kind while part (b) and (c) establish the bounds $|T_n(z)| \leq 1, |U_n(z)| \leq n+1$ ($z \in [-1,1]$) for these polynomials (note that $\lim_{N\to\infty} U_n(\frac{N}{2}(1-z),N) = \frac{U_n(z)}{n+1}$).

We will conclude this section with a brief discussion of related results for the Hahn- Eberlein and the dual Hahn polynomials. The Hahn- Eberlein polynomials are obtained from the Hahn polynomials $Q_n(x,\alpha,\beta,N)$ for $\alpha < -N, \beta < -N$ (see e.g. Rahman (1978) or Eberlein (1964)). For this choice the mass function in (1.2) still defines a probability measure on $\{0,\dots,N\}$ and consequently the orthogonal polynomials $Q_n(x,\alpha,\beta,N)$ with respect to this measure are well defined and called Hahn- Eberlein polynomials. These polynomials have some applications in coding theory (see e.g. Sloane (1975)). Obviously, the identities of Theorem 3.1 can be extended to the region $\alpha < -N, \beta < -N$ and as a consequence we obtain the following bound for the Hahn- Eberlein polynomials.

Theorem 3.8. Let $\alpha < -N, \beta < -N, \alpha+\beta < -2N-1$ and

$$\tilde{n}(\alpha,\beta,N) = -\frac{1}{2}\{(\alpha+\beta-1) + \sqrt{(\alpha+\beta+1)(\alpha+\beta+1+2N)}\}.$$

The Hahn- Eberlein polynomials $Q_n(x,\alpha,\beta,N)$ satisfy for all $x \in [0,N]$ and $n \leq \tilde{n}(\alpha,\beta,N)$ the inequality

$$|Q_n(x,\alpha,\beta,N)| \leq \max\left\{1, \frac{(\beta+1)_n}{(\alpha+1)_n}\right\} = \max\{|Q_n(0,\alpha,\beta,N)|, |Q_n(N,\alpha,\beta,N)|\} .$$

The dual Hahn polynomials $R_k(x,\alpha,\beta,N)$ $(\alpha,\beta > -1)$ are related to the Hahn polynomials by the equation

$$R_k(x(x+\alpha+\beta+1)) = Q_x(k,\alpha,\beta,N)$$

$(k,x = 0,\ldots,N)$ and are orthogonal on the interval $[0,N(N+\alpha+\beta+1)]$. For a detailed discription of these polynoimals including the recurrence relation and the orthogonality relation we refer the reader to the work of Karlin and McGregor (1961). By a similar analysis as in the proof of Theorem 3.2 we obtain the following bound for these polynomials.

Theorem 3.9. Let $\alpha,\beta > -1$, $N+1+\beta-\alpha \geq 0$ and

$$n^*(\alpha,\beta,N) = \frac{1}{2}\min\{N+2,\ N+1+\beta-\alpha\}.$$

If $n \leq n^*(\alpha,\beta,N)$ then the dual Hahn polynomial $R_n(x,\alpha,\beta,N)$ satisfies for all $x \in [0,N(N+\alpha+\beta+1)]$ the inequality

$$|R_n(x,\alpha,\beta,N)| \leq \frac{(N+1+\beta-n)_n}{(\alpha+1)_n} = |R_n(N(N+\alpha+\beta+1),\alpha,\beta,N)| \ .$$

Proof: Let ξ_D denote the measure which puts masses

$$\xi_D(\lambda_x) = \pi_x(\alpha,\beta,N)\rho(0)$$

at the points $\lambda_x = x(x+\alpha+\beta+1)$ $(x=0,\ldots,N)$ where $\pi_x(\alpha,\beta,N)$ and $\rho(x) = \rho(x,\alpha,\beta,N)$ are defined in (1.3) and (1.2), respectively. By the results of Karlin and McGregor (1961) (equation (1.20)) it follows that ξ_D defines a probability measure on the interval $[0,N(N+\alpha+\beta+1)]$ and that the orthonormal polynomials with respect to $d\xi_D(x)$ are given by

$$P_l(x) = \sqrt{\frac{\rho(l)}{\rho(0)}}R_l(x,\alpha,\beta,N) \qquad (l=0,\ldots,N). \tag{3.13}$$

According to Theorem 3.1 in Dette (1994) it follows that the orthonormal polynomials $P_l(x)$, $Q_l(x)$ and $S_l(x)$ with respect to the measures $d\xi_D(x)$, $x[N(N+\alpha+\beta+1)-x]d\xi_D(x)$

and $[N(N+\alpha+\beta+1)-x]d\xi_D(x)$ satisfy the identity

$$
\begin{aligned}
(3.14)\qquad & x[N(N+\alpha+\beta+1)-x]\sum_{\ell=0}^{n-1}\frac{\underline{D}_{2\ell+1}(\xi_D)}{\overline{D}_{2\ell+1}(\xi_D)}\left[\frac{\underline{D}_{2\ell}(\xi_D)}{\overline{D}_{2\ell}(\xi_D)}-\frac{\underline{D}_{2\ell+2}(\xi_D)}{\overline{D}_{2\ell+2}(\xi_D)}\right]Q_\ell^2(x) \\
&+ x[N(N+\alpha+\beta+1)-x]\frac{\underline{D}_{2n}(\xi_D)\ \underline{D}_{2n+1}(\xi_D)\ \overline{D}_{2n+2}(\xi_D)}{\overline{D}_{2n}(\xi_D)\ \overline{D}_{2n+1}(\xi_D)\ \overline{\underline{D}}_{2n+2}(\xi_D)}Q_n^2(x) \\
&+ [N(N+\alpha+\beta+1)-x]\sum_{\ell=0}^{n}\frac{\underline{D}_{2\ell}(\xi_D)}{\overline{D}_{2\ell}(\xi_D)}\left[\frac{\underline{D}_{2\ell-1}(\xi_D)}{\overline{D}_{2\ell-1}(\xi_D)}-\frac{\underline{D}_{2\ell+1}(\xi_D)}{\overline{D}_{2\ell+1}(\xi_D)}\right]S_\ell^2(x) \\
&= 1-\frac{\underline{D}_{2n+1}(\xi_D)\underline{D}_{2n+2}(\xi_D)}{\overline{D}_{2n+1}(\xi_D)\overline{\underline{D}}_{2n+2}(\xi_D)}P_{n+1}^2(x)
\end{aligned}
$$

$(n=0,\dots,N-1)$. By a similar reasoning as in the proof of Theorem 3.1 we obtain for the ratios of the determinants in (3.14)

$$\frac{\underline{D}_{2l}(\xi_D)}{\overline{D}_{2l}(\xi_D)}-\frac{\underline{D}_{2l+2}(\xi_D)}{\overline{D}_{2l+2}(\xi_D)}=\frac{l!}{(N-l-1)_{l+1}}(N-2l-2)\qquad (l=0,\dots,n-1),$$

$$\frac{\underline{D}_{2l-1}(\xi_D)}{\overline{D}_{2l-1}(\xi_D)}-\frac{\underline{D}_{2l+1}(\xi_D)}{\overline{D}_{2l+1}(\xi_D)}=\frac{(\alpha+1)_l}{(N+\beta-l)_{l+1}}(N-1+\beta-\alpha-2l)\qquad (l=0,\dots,n)$$

and

$$
\begin{aligned}
\frac{\underline{D}_{2n+1}(\xi_D)\underline{D}_{2n+2}(\xi_D)}{\overline{D}_{2n+1}(\xi_D)\overline{\underline{D}}_{2n+2}(\xi_D)}P_{n+1}^2(x) &= \frac{(\alpha+1)_{n+1}(n+1)!}{(N+\beta-n)_{n+1}(N-n)_{n+1}}\frac{\rho(n+1)}{\rho(0)}R_{n+1}^2(x,\alpha,\beta,N) \\
&= \left(\frac{(\alpha+1)_{n+1}}{(N+\beta-n)_{n+1}}R_{n+1}(x,\alpha,\beta,N)\right)^2
\end{aligned}
$$

where we have used (3.13) and (1.2) in the last identity. By the assumptions of the theorem all terms on the left hand side in (3.14) are positive and the assertion follows from

$$R_n(N(N+\alpha+\beta+1)) = Q_N(n,\alpha,\beta,N) = (-1)^n\frac{(N+1+\beta-n)_n}{(\alpha+1)_n}$$

which can easily be proved by an induction argument. □

4. Krawtchouk polynomials. In this Section we will apply the results of Section 2 and 3 in order to obtain similar results for the Krawtchouk polynomials. We will mainly use the representation (1.4) of $k_n(x,p,N)$ as the limit of the Hahn polynomials $Q_n(x,\alpha,\beta,N)$

when $\alpha = pt, \beta = qt$ and $t \to \infty$. By this relation the following results are immediate consequences of Theorem 3.1 and 3.2.

Theorem 4.1. For $\ell = 0, \ldots, N$ define $\tilde{k}_\ell(x,p,N) = \binom{N}{\ell}(\frac{p}{q})^\ell k_\ell(x,p,N)$. The Krawtchouk polynomials satisfy the following identities.

a) For $n = 0, \ldots, N-1$:

$$\sum_{\ell=1}^{n-1}(\frac{2\ell}{N} - 1)\{\tilde{k}_\ell(x,p,N)\}^2 + \{\frac{n}{N}\tilde{k}_n(x,p,N)\}^2 + (1-\frac{x}{N})\frac{q-p}{q^2}\sum_{\ell=0}^{n-1}\{\tilde{k}_\ell(x,p,N-1)\}^2$$
$$= 1 - \frac{x}{N}(1-\frac{x}{N})\left\{\frac{k_{n-1}(x-1,p,N-2)}{q}\right\}^2 .$$

b) For $n = 0, \ldots, N-1$:

$$\sum_{\ell=1}^{n}(\frac{2\ell}{N} - 1)\left\{\binom{N}{\ell}k_\ell(x,p,N)\right\}^2 + \frac{x}{N}\left\{\binom{N-1}{n}k_n(x-1,p,N-1)\right\}^2$$

$$+ \frac{p-q}{p^2}\frac{x}{N}\sum_{\ell=0}^{n-1}\left\{\binom{N-1}{\ell}k_\ell^2(x-1,p,N-1)\right\}^2$$

$$= 1 - (1-\frac{x}{N})\left\{\binom{N-1}{n}k_n(x,p,N-1)\right\}^2 .$$

c) For $n = 0, \ldots, N-2$:

$$x(1-\frac{x}{N})\sum_{\ell=0}^{n-1}(N-2\ell-2)\left\{\frac{p^\ell}{q^{\ell+1}}\frac{k_\ell(x-1,p,N-2)}{(N-1)}\right\}^2$$

$$+ \frac{q-p}{q^2}(1-\frac{x}{N})\sum_{\ell=0}^{n}\left\{\frac{p^\ell}{q^\ell}k_\ell(x,p,N-1)\right\}^2$$

$$+ \frac{x}{N}(1-\frac{x}{N})\left\{\frac{(N-n-1)p^n k_n(x-1,p,N-2)}{q^{n+1}(N-1)}\right\}^2 = 1 - \left\{\frac{p^{n+1}}{q^{n+1}}k_{n+1}(x,p,N)\right\}^2$$

d) For $n = 0, \ldots, N-1$:

$$\sum_{\ell=1}^{n}(\frac{2\ell}{N}-1)\{\tilde{k}_\ell(x,p,N)\}^2 + \frac{q-p}{q^2}(1-\frac{x}{N})\sum_{\ell=0}^{n-1}\{\tilde{k}_\ell(x,p,N-1)\}^2$$

$$+ \ (1-\frac{x}{N})\{\tilde{k}_n(x,p,N-1)\}^2 \ = \ 1 \ - \ \frac{x}{N}\{\tilde{k}_n(x-1,p,N-1)\}^2 \ .$$

<u>**Theorem 4.2.**</u> Let $n \le \frac{N}{2}+1$, then the nth Krawtchouk polynomial $k_n(x,p,N)$ satisfy for all $x \in [0,N]$ the inequality

$$|k_n(x,p,N)| \ \le \ \max\left\{1,(\frac{q}{p})^n\right\} \ = \ \max\{|k_n(0,p,N)| \ , \ |k_n(N,p,N)|\} \ .$$

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