INTEGRABLE EXPANSIONS FOR POSTERIOR DISTRIBUTIONS FOR A TWO PARAMETER EXPONENTIAL FAMILY

by

Dongchu Sun
University of Missouri-Columbia

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Department of Statistics
Purdue University

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Dongchu Sun

University of Missouri–Columbia

Abstract

Asymptotic Expansions of posterior distributions are derived for a two dimensional exponential family, which includes Normal, Gamma, Inverse Gamma and Inverse Gaussian distributions. Reparameterization allows us to use a data dependent transformation, convert the likelihood function to the two dimensional standard normal density, and apply a version of Stein's Identity to assess the posterior distributions. An application to repeated likelihood ratio tests are discussed briefly.

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Key words and Phrases: posterior distributions, parameter transformations, Stein's Identity, martingale convergence theorem, stopping time, Jeffrey's prior, repeated likelihood ratio tests.

1 Introduction

There are many design problems where it is necessary to compute the Bayesian risks in order to see the effect of the design parameters. Those Bayesian risks usually depend on asymptotic expansions of posterior distributions which are integrated with respect to the marginal distribution of data. Examples can be found from Woodroofe (1986, 1989).

The question of integrable expansions is of independent interest, as discussed in Woodroofe (1989), and Bickel and Ghosh (1990), and can be dated from Laplace (1847). In the last two decades, integrable expansions for posterior distributions became one of the most widely studied problems in both statistical theory and application. Johnson (1970) is among the first few authors who investigate pointwise posterior expansions rigorously. More recent papers about general posterior expansions can be found from Ghosh, Sinha and Joshi (1983), Bickel, Goetze and Van Zwet (1985), and Bickel and Ghosh (1990) and their references. The main technic they used is Taylor's expansion under various regularity conditions.

Another approach, which was introduced by Woodroofe (1992) for one-parametric exponential family by using a data dependent transformation which allows one to convert the likelihood function to exact normality and to avoid Taylor's expansions. The data dependent transformation has also been used in Woodroofe (1986, 1989), Woodroofe and Hardwick (1990), and Keener and Woodroofe (1992) for various Bayesian design problems.

Of course, it is important to consider multiparameter problems. However, in some multiparameter exponential families, parameters are depending on each other. A typical example is Multinomial distribution. See Brown (1986). Such dependence makes the problem more difficult.

In this paper, we consider a certain two parameter exponential family of distributions, which contains Normal, Inverse Gaussian, Gamma and Inverse Gamma distributions. This family was discussed first by Bar-Lev and Reiser (1982) for constructing UMPU tests, and recently, by Bose and Boukai (1990) for finding sequential point estimators for one of the parameters with minimum risk. We take advantage of the special structure of this class of distributions, by using reparameterization so that the new parameters vary independently. A data dependent transformation is applied to convert the likelihood function to the two dimensional standard normal density. A version of Stein's Identity plays an important role to the posterior distributions so that the remainder terms can be written as conditional expectations, which will be treated by the martingale convergence theory.

The symmetric functions of the above data transformation are of special interest to us. We will show that the generalized log-likelihood ratio statistic is a special case, which has been greatly attracted to reachers. See Woodroofe (1982), Lalley (1988) and Hu (1988). It turns out that the second term for a high order expansion is always zero. Therefore the first term is a quite accurate estimator for the desired quantities.

This paper is divided as follows. Section 2 gives the basic materials we need later. We first review, for subsequent use, some of the properties of the two parameter exponential family mentioned above in Section 2.1, The data dependent transformation is then introduced in Section 2.2. Some useful inequalities are given in Section 2.3 and Stein's Identity is stated in Section 2.4.

The main results are given in Section 3. Integrable expansions with the fixed sample size are illustrated in Section 3.1. The second and high order expansions after sequential experimentation are established in Section 3.2 and 3.3, respectively. Expansions for symmetric functions and their rescaling are discussed in Section 3.4.

Some applications of our results are stated in section 4. In particular, we apply our result to approximate sampling distribution of sequential log-likelihood ratio test statistic for the normal distribution with unknown mean and covariance.

2 Preliminaries

2.1 The Model

Consider a two parameter exponential family of densities on the Borel sets of $\mathbb{R}$, say,

$$p_{\boldsymbol{\beta}}(x) = \exp\{\beta_1 U_1(x) + \beta_2 U_2(x) - \psi(\boldsymbol{\beta})\}, \quad \boldsymbol{\beta} = (\beta_1, \beta_2), \tag{1}$$

with respect to Lebesgue measure. Let $\mathcal{N}$ denote the corresponding natural parameter space . It is well known that for any $\boldsymbol{\beta} \in \mathcal{N}^o$ (the interior of $\mathcal{N}$), the random vector $\boldsymbol{U} = (U_1(X), U_2(X))$ has moments of all orders. In particular,

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\beta}}(\boldsymbol{U}) &= \nabla\psi(\boldsymbol{\beta}) = (\partial\psi(\boldsymbol{\beta})/\partial\beta_1, \partial\psi(\boldsymbol{\beta})/\partial\beta_2) \equiv (\mu_1, \mu_2), \\ Var_{\boldsymbol{\beta}}(\boldsymbol{U}) &= \nabla^2\psi(\boldsymbol{\beta}) = \left(\partial^2\psi(\boldsymbol{\beta})/\partial\beta_i\partial\beta_j\right)_{i,j=1,2}, \end{aligned}$$

are the mean vector and covariance matrix of $\boldsymbol{U}$. Throughout this paper, ∇ is used to denote the gradient.

Let $X_1, X_2, \cdots$ be random variables which are i.i.d. with common density (1) for $\boldsymbol{\beta} \in \mathcal{N}$. Bar-Lev and Reiser (1982) first introduced a subfamily of (1) for constructing UMPU tests. This subfamily admits a single ancillary statistic for β_2 in the presence of β_1, (i.e., its distribution depends only on β_1) and characterized by the following two assumptions:

Assumption A: The parameter β_2 can be represented as: $\beta_2 = -\beta_1 G_2'(\mu_2)$, where $G_2'(\mu_2) = dG_2(\mu_2)/d\mu_2$, for some function G_2.

Assumption B: $U_2(x)$ is $1-1$ function on the support of (1).

One can introduce a parameterization of the exponential family (1) by means of the mapping $(\beta_1, \beta_2) \longrightarrow (\beta_1, \mu_2)$, where $\mu_2 = \mathbb{E}_{\boldsymbol{\beta}}(\boldsymbol{U}_2(X))$. From Theorem 8.4 of Barnddoff-Nielsen (1978), this mapping is a homeomorphism and $(\beta_1, \mu_2) \in \Theta_1 \times \Theta_2 \equiv \boldsymbol{\Theta}$ (i.e., components β_1 and μ_2 vary independently.) For simplicity, we will denote $\boldsymbol{\theta} \equiv (\theta_1, \theta_2) = (\beta_1, \mu_2)$.

Denote

$$T_{nj} = \sum_{i=1}^{n} U_j(X_i), \qquad \overline{T}_{nj} = \frac{1}{n} T_{nj}, \quad j = 1, 2, \tag{2}$$

$$Y_n = T_{n1} - nG_2(\overline{T}_{n2}), \quad \overline{Y}_n = \frac{1}{n} Y_n. \tag{3}$$

The following facts can be found from Bar-Lov and Reiser (1982).

Facts: Under the Assumptions A and B,

1. G_2 is infinitely differentiable and G_2' is not identically constant;

2. the variance of U_2 is given by $\frac{-1}{\theta_1 G_2''(\theta_2)}$, (> 0);

3. the functions $\psi(\boldsymbol{\beta})$ and $\mu_1(\boldsymbol{\beta})$, when expressed by (θ_1, θ_2), have the following form:

$$\begin{cases} \psi(\theta_1, \theta_2) = -\theta_1[\theta_2 G_2'(\theta_2) - G_2(\theta_2)] + G_1(\theta_1), \\ \mu_1 = G_2(\theta_2) + G_1'(\theta_1), \end{cases}$$

   where $G_1(\theta_1)$ is an infinitely differentiable function on Θ_1 for which $G_1''(\theta_1) > 0$, for all $\theta_1 \in \Theta_1$.

4. The distribution of Y_n belongs to the one parameter exponential family with natural parameter θ_1 and density of the form

$$p_{Y_n}(y_n, \theta_1) = q(y_n) \exp\{\theta_1 y_n - H_n(\theta_1)\}, \quad \theta_1 \in \Theta_1,$$

   where

$$H_n(\theta_1) = nG_1(\theta_1) - G_1(n\theta_1).$$

One of immediate consequences of Fact 2 is that either $\Theta_1 \subset \mathbb{R}^-$ or $\Theta_1 \subset \mathbb{R}^+$. In this paper, we follow Assumptions A and B. Without loss of generality, we will assume that $\Theta_1 \subset \mathbb{R}^-$. Then we have

Fact 5. $G_2''(\theta_2) > 0$ for all $\theta_2 \in \Theta_2$ so that G_2 is strictly convex.

Fact 6. Both G_1' and G_1'' are positive. Therefore G_1 is strictly convex and monotonically increasing.

The Normal, Inverse Gaussian, Gamma, Inverse Gamma distributions are special cases in (1) satisfying Assumptions A and B. See Table 1. Any monotone functions of these random variables satisfy these two Assumptions, too. For details, see Bar-Lov and Reiser (1982).

For the parameter $\boldsymbol{\theta}$, (1) becomes

$$p_{\boldsymbol{\theta}}(x) = \exp\{\theta_1 U_1(x) - \theta_1 G_2'(\theta_2) U_2(x) - \psi(\boldsymbol{\theta})\}, \tag{4}$$

where $\psi(\boldsymbol{\theta})$ is given by Fact 3. It is easy to verify that the Kullback-Leibler number is

$$\begin{aligned} I(\boldsymbol{\omega};\boldsymbol{\theta}) &= \psi(\theta_1,\theta_2) - \psi(\omega_1,\omega_2) - (G_2(\omega_2) + G_1'(\omega_1),\ \omega_2)\begin{pmatrix} \theta_1 - \omega_1 \\ \omega_1 G_2'(\omega_2) - \theta_1 G_2'(\theta_2) \end{pmatrix} \\ &= I_1(\omega_1,\theta_1) - \theta_1 I_2(\omega_2,\theta_2), \quad \boldsymbol{\omega},\boldsymbol{\theta} \in \boldsymbol{\Theta}, \end{aligned} \tag{5}$$

where

$$I_1(\omega_1,\theta_1) = G_1(\theta_1) - G_1(\omega_1) - G_1'(\omega_1)(\theta_1 - \omega_1), \quad \omega_1,\theta_1 \in \Theta_1, \tag{6}$$

$$I_2(\omega_2,\theta_2) = G_2(\omega_2) - G_2(\theta_2) - G_2'(\theta_2)(\omega_2 - \theta_2), \quad \omega_2,\theta_2 \in \Theta_2. \tag{7}$$

Note that $I_j(\omega_j,\theta_j) \geq 0$ and the equality holds iff $\omega_j = \theta_j$. Therefore $I(\boldsymbol{\omega},\boldsymbol{\theta}) \geq 0$ and the equality holds iff $\boldsymbol{\omega} = \boldsymbol{\theta}$.

Derivatives of I_1 and I_2 are needed in the subsequential sections. For $\omega_1,\theta_1 \in \Theta_1$, $I_{1,10}(\omega_1,\theta_1) = -G_1''(\omega_1)(\theta_1 - \omega_1)$, $I_{1,11}(\omega_1,\theta_1) = -G_1''(\omega_1)$, $I_{1,01}(\omega_1,\theta_1) = G_1'(\theta_1) - G_1'(\omega_1)$, and $I_{1,02}(\omega_1,\theta_1) = G_1''(\theta_1)$, where $I_{1,jk}(\omega_1,\theta_1) = \partial^{j+k} I_1(\omega_1,\theta_1)/\partial\omega_1^j\partial\theta_1^k$ for $j,k = 0,1,\cdots$. Similarly, $I_{2,10}(\omega_2,\theta_2) = G_2'(\omega_2) - G_2'(\theta_2)$, $I_{2,20}(\omega_2,\theta_2) = G_2''(\omega_2)$, $I_{2,11}(\omega_2,\theta_2) = -G_2''(\theta_2)$, $I_{2,01}(\omega_2,\theta_2) = G_2''(\theta_2)(\theta_2 - \omega_2)$, $I_{2,02}(\omega_2,\theta_2) = G_2'''(\theta_2)(\theta_2 - \omega_2) + G_2''(\theta_2)$ and $I_{2,03}(\omega_2,\theta_2) = G_2^{\text{iv}}(\theta_2)(\theta_2 - \omega_2) + 2G_2'''(\theta_2)$.

2.2 Data Dependent Transformation

The log-likelihood function of $\boldsymbol{\theta} = (\theta_1,\theta_2)$ given $X_1,\cdots,X_n$ is

$$\begin{aligned} L_n(\boldsymbol{\theta}) &= \theta_1 T_{n1} - \theta_1 G_2'(\theta_2) T_{n2} - n\psi(\boldsymbol{\theta}) \\ &= \theta_1 Y_n - \theta_1[G_2'(\theta_2)T_{n2} - nG_2(\overline{T}_{n2})] + n\theta_1[\theta_2 G_2'(\theta_2) - G_2(\theta_2)] - nG_1(\theta_1), \end{aligned}$$

for $\boldsymbol{\theta} \in \boldsymbol{\Theta}, n \geq 1$, where T_{nj} are given by (2), and let $\hat{\boldsymbol{\theta}}_n = (\hat{\theta}_{n1}, \hat{\theta}_{n2})$ denote the maximum likelihood estimators of $\boldsymbol{\theta}$. It is easy to show that under the above assumptions, $\hat{\theta}_{n2} = \overline{T}_{n2}$ and that $\hat{\theta}_{n1}$ satisfies the equation:

$$G_1'(\hat{\theta}_{n1}) = \overline{Y}_n. \tag{8}$$

Then one can verify

$$L_n(\boldsymbol{\theta}) - L_n(\hat{\boldsymbol{\theta}}_n) = -nI(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) = -nI_1(\hat{\theta}_{n1}, \theta_1) + n\theta_1 I_2(\hat{\theta}_{n2}, \theta_2). \tag{9}$$

The data dependent transformation mentioned above is

$$\boldsymbol{Z} \equiv \boldsymbol{Z}_n \equiv \begin{pmatrix} Z_{n1} \\ Z_{n2} \end{pmatrix} = \begin{pmatrix} \sqrt{2nI_1(\hat{\theta}_{n1}, \theta_1)}\ \mathrm{sign}(\theta_1 - \hat{\theta}_{n1}) \\ \sqrt{-2n\theta_1 I_2(\hat{\theta}_{n2}, \theta_2)}\ \mathrm{sign}(\theta_2 - \hat{\theta}_{n2}) \end{pmatrix}. \tag{10}$$

Note that $Z_{n1}(Z_{n2})$ is increasing in $\theta_1(\theta_2)$, since $\partial Z_{n1}/\partial\theta_1 = \sqrt{n}|I_{1,01}|/\sqrt{2I_1} > 0$ ($\partial Z_{n2}/\partial\theta_2 = \sqrt{-n\theta_1}|I_{2,01}|/\sqrt{2I_2} > 0$) and the likelihood function is exactly normal in $\boldsymbol{Z}_n$. Also,

$$\det\left(\frac{\partial(Z_{n1}, Z_{n2})}{\partial(\theta_1, \theta_2)}\right) = \det\begin{pmatrix} \partial Z_{n1}/\partial\theta_1 & 0 \\ * & \partial Z_{n2}/\partial\theta_2 \end{pmatrix} > 0.$$

Thus, $\boldsymbol{\theta} \to \boldsymbol{Z}$ is one-to-one and onto.

Now let us consider a Bayesian model in which the prior density of $\boldsymbol{\theta} = (\theta_1, \theta_2)$ is ξ and $X_1, X_2, \cdots$ are conditionally i.i.d. with common density $p_{\boldsymbol{\theta}}(x)$ given $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. We will denote the probability and expectation in the Bayesian model by $\mathbb{P}^\xi$ and $\mathbb{E}^\xi$; denote the conditional probability and expectation given $\boldsymbol{\theta}$ by $\mathbb{P}_{\boldsymbol{\theta}}$ and $\mathbb{E}_{\boldsymbol{\theta}}$, respectively; and denote the conditional expectation given $X_1, X_2, \cdots, X_n$ by $\mathbb{E}_n^\xi$. For simplicity, $\boldsymbol{\theta}$ will be treated as either a random variable or its observation. Define

$$J_k(\omega_k, \theta_k) = \frac{\sqrt{2I_k(\omega_k, \theta_k)}}{|I_{k,01}(\omega_k, \theta_k)|}, \quad k = 1, 2. \tag{11}$$

Then the conditional density of $\boldsymbol{Z}_n$ given $X_1, X_2, \cdots, X_n$ is

$$\zeta_n(\boldsymbol{z}) \propto \xi(\boldsymbol{\theta}) J(\hat{\theta}_{n1}, \hat{\theta}_{n2}; \theta_1, \theta_2) \exp\left\{-\frac{1}{2}\boldsymbol{z}\boldsymbol{z}^\tau\right\} I_{R_n}(\boldsymbol{z}), \tag{12}$$

where

$$J(\omega_1, \omega_2; \theta_1, \theta_2) = \frac{1}{\sqrt{-\theta_1}} J_1(\omega_1, \theta_1) J_2(\omega_2, \theta_2), \quad \omega_k, \theta_k \in \Theta_k, \quad k = 1, 2. \tag{13}$$

The partial derivatives of J_k are also needed: for $\omega_k, \theta_k \in \Theta_k$ and $\omega_k \neq \theta_k$, $k = 1, 2$,

$$J_{k,01}(\omega_k, \theta_k) = \frac{1}{\sqrt{2I_k}}\Big\{1 - I_{k,02}J_k^2\Big\}\mathrm{sign}(\theta_k - \omega_k); \tag{14}$$

$$J_{k,02}(\omega_k, \theta_k) = -\Big\{\frac{|I_{k,01}|}{2I_k}J_{k,01} + \frac{1}{\sqrt{2I_k}}[I_{k,03}J_k^2 + 2I_{k,02}J_kJ_{k,01}]\Big\}\mathrm{sign}(\theta_k - \omega_k). \tag{15}$$

The value of J_k and its partial derivatives on the diagonal may be obtained from L'Hospital's rule as $J_k(\omega_k, \omega_k) = \frac{1}{\sqrt{G_k''(\omega_k)}}$, $k = 1, 2$, $J_{1,01}(\omega_1, \omega_1) = -\frac{G_1'''(\omega_1)}{3G_1''(\omega_1)^{3/2}}$, $J_{1,02}(\omega_1, \omega_1) = \frac{11G_1'''(\omega_1)^2}{36G_1''(\omega_1)^{5/2}} - \frac{G_1^{\mathrm{iv}}(\omega_1)}{4G_1''(\omega_1)^{3/2}}$, $J_{2,01}(\omega_2, \omega_2) = -\frac{2G_2'''(\omega_2)}{3G_2''(\omega_2)^{3/2}}$ and $J_{2,02}(\omega_2, \omega_2) = \frac{11G_2'''(\omega_2)^2}{9G_2''(\omega_2)^{5/2}} - \frac{3G_2^{\mathrm{iv}}(\omega_2)}{4G_2''(\omega_2)^{3/2}}$.

2.3 Inequalities

Define

$$A_n = \{\overline{Y}_n \in G'(\Theta_1)\} \quad \text{and} \quad B_n = \{\overline{T}_{n2} \in \Theta_2\}. \tag{16}$$

Lemma 2.1 For all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $m = 2, 3 \cdots$,

$$\mathbb{P}_{\boldsymbol{\theta}}(A_n \text{ and } \hat{\theta}_{n1} \geq \omega_1, \exists\, n \geq m) \leq \exp\{-mI_1(\omega_1, \theta_1)\}, \quad \text{if } \omega_1 > \theta_1,$$

$$\mathbb{P}_{\boldsymbol{\theta}}(A_n \text{ and } \hat{\theta}_{n1} \leq \omega_1, \exists\, n \geq m) \leq \exp\{-mI_1(\omega_1, \theta_1) + G_1(m\theta_1) - G_1(m\omega_1)\}, \quad \text{if } \omega_1 < \theta_1.$$

Proof. See Appendix. □

Lemma 2.2 For all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $m = 2, 3, \cdots$,

$$\mathbb{P}_{\boldsymbol{\theta}}(B_n \text{ and } \hat{\theta}_{n2} \geq \omega_2, \exists\, n \geq m) \leq \exp\{m\theta_1 I_2(\omega_2, \theta_2)\}, \quad \text{if } \omega_2 > \theta_2,$$

$$\mathbb{P}_{\boldsymbol{\theta}}(B_n \text{ and } \hat{\theta}_{n2} \leq \omega_2, \exists\, n \geq m) \leq \exp\{m\theta_1 I_2(\omega_2, \theta_2)\}, \quad \text{if } \omega_2 < \theta_2.$$

Proof. See Appendix. □

For further analysis, we need an additional assumption on the function G_1 for the rest of the paper.

Assumption C: $\sup_{x \leq \theta_1} |x| G_1'(x) = \gamma(\theta_1) < \infty.$

Note that Assumption C holds for the normal, inverse Gamma, Gamma and inverse Gamma distributions. As it is shown by Woodroofe (1977) the left tail behavior of the underlying distribution function is crucial in various risk's assessments and posterior distributions associated with sequential design. Basically, Assumption C controls the left tail behavior of the underlying distribution function and a similar condition has been used by Bose and Boukai (1989).

Lemma 2.3 For any $x > 0$, $m = 2, 3, \cdots, \boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$\mathbb{P}_{\boldsymbol{\theta}}(\sup_{n \geq m} I(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) I_{A_n B_n} \geq x) \leq [e^{m\gamma(\theta_1)} + 4] \exp\{-\frac{(m-1)x}{2}\}.$$

Proof. For any fixed $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \boldsymbol{\Theta}$,

$$\begin{aligned} & \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n \geq m} I(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) I_{A_n B_n} \geq x) \\ = \; & \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n \geq m} [I_1(\hat{\theta}_{n1}, \theta_1) - \theta_1 I_2(\hat{\theta}_{n2}, \theta_2)] I_{A_n B_n} \geq x) \\ \leq \; & \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n \geq m} I_1(\hat{\theta}_{n1}, \theta_1) I_{A_n} \geq \frac{x}{2}) + \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n \geq m} I_2(\hat{\theta}_{n2}, \theta_2) I_{B_n} \geq \frac{x}{-2\theta_1}). \end{aligned} \tag{17}$$

If G_1 is bounded, the assertion holds. Otherwise,

$$\begin{aligned} & -I_1(\omega_1, \theta_1) + G_1(m\theta_1) - G_1(m\omega_1) \\ = \; & -[G_1(\theta_1) - G_1(m\theta_1)] + G_1(\omega_1) - G_1(m\omega_1) + G_1'(\omega_1)(\theta_1 - \omega_1) \\ = \; & (m-1)\theta_1 G_1'(\theta_1^*) - (m-1)\omega_1 G_1'(\omega_1^*) + \theta_1 G_1'(\omega_1) - \omega_1 G_1'(\omega_1), \end{aligned} \tag{18}$$

for some $\theta_1^* \in [m\theta_1, \theta_1]$ and $\omega_1^* \in [m\omega_1, \omega_1]$. Since G_1' is positive and strictly increasing, $(m-1)\theta_1 G_1'(\theta_1^*) < 0$, $G_1'(\omega_1)\theta_1 < 0$, and $0 < -(m-1)\omega_1 G_1'(\omega_1^*) < -(m-1)\omega_1 G_1'(\omega_1)$. Thus The right hand side of (18) is bounded by $-m\omega_1 G_1'(\omega_1) \leq m\gamma(\theta_1)$ for all $\omega_1 < \theta_1$, by assumption C, Therefore, the right hand side of (17) is bounded by

$$e^{m\gamma(\theta_1)} \exp\{-\frac{(m-1)x}{2}\} + 3\exp\{-\frac{mx}{2}\}.$$

This completes the proof from Lemmas 4.1 and 4.2 and the monotonicity of I_j. □

Lemma 2.4 For any $x \geq 1$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $n = 1, 2, \cdots$

$$\mathbb{P}_{\boldsymbol{\theta}}(\max_{k \leq n} \|\boldsymbol{Z}_k\| I_{A_k B_k} \geq x) \leq [e^{m\gamma(\theta_1)} + 4](1 + \log_2 n) \exp\{-\frac{x^2}{4}\},$$

where $\log_2$ denote logarithm to the base two.

Proof. For each $k \geq 1$, there is a unique $m \geq 1$ for which $2^{m-1} \leq k < 2^m$. Moreover, if $A_k B_k$ occurs and $\|\boldsymbol{Z}_k\| > x \geq 1$ for some $k \in [2^{m-1}, 2^m]$, then $I(\hat{\boldsymbol{\theta}}_k, \boldsymbol{\theta}) > x^2/2^{m+1}$ for some $k \geq 2^{m-1}$. Let M_n be the least integer which exceeds $\log_2 n$. Then for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $n \geq 1$,

$$\begin{aligned} \mathbb{P}_{\boldsymbol{\theta}}(\max_{k \leq n} \|\boldsymbol{Z}_k\| I_{A_k B_k} \geq x) & \leq \sum_{m=1}^{M_n} \mathbb{P}_{\boldsymbol{\theta}}\Big\{A_k B_k \text{ and } \|\boldsymbol{Z}_k\| \geq x, \exists\, k \in [2^{m-1}, 2^m]\Big\} \\ & \leq \sum_{m=1}^{M_n} \mathbb{P}_{\boldsymbol{\theta}}\Big\{A_k B_k \text{ and } \|\boldsymbol{Z}_k\| \geq x, \exists\, k \geq 2^{m-1}\Big\} \\ & \leq [e^{m\gamma(\theta_1)} + 4] M_n \exp\{-\frac{x^2}{4}\}. \end{aligned}$$

The last inequality follows from Lemma 2.3 and the fact that $G_1(\cdot)$ is monotonically increasing on Θ_1. □

For a fixed ξ, let $\mathcal{W}^\xi$ denote the class of all functions $W : \boldsymbol{\Theta} \times \boldsymbol{\Theta} \to \mathbb{R}$ for which

$$\int \sup_{n \geq m} |W(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta})| I_{A_n \cap B_n} d\mathbb{P}^\xi < \infty, \tag{19}$$

for some $m = 2, 3, \cdots$. Observe that $\mathcal{W}^\xi$ is a linear space which contains all constant functions.

Lemma 2.5 Assume that ξ is a density on $\boldsymbol{\Theta}$ with a compact support, and w is a real function on $\boldsymbol{\Theta}$ such that $|w|\xi$ is integrable. For any fixed m_0, define

$$W(\boldsymbol{\omega}; \boldsymbol{\theta}) = w(\boldsymbol{\theta}) \exp\{m_0 I(\boldsymbol{\omega}, \boldsymbol{\theta})\}, \quad \boldsymbol{\omega}, \boldsymbol{\theta} \in \boldsymbol{\Theta}.$$

Then $W \in \mathcal{W}^\xi$.

Proof. The conclusion is true if $m_0 \leq 0$. For $m_0 > 0$, choose an integer $m \geq 4(m_0 + 1)$,

$$\begin{aligned}
\int \sup_{n \geq m} W(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta})\xi(\boldsymbol{\theta}) d\mathbb{P}^\xi &\leq \int \sup_{n \geq m} \exp\Big\{m_0 I(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta})\Big\} |w(\boldsymbol{\theta})| \xi(\boldsymbol{\theta}) d\mathbb{P}^\xi \\
&\leq \int_{\boldsymbol{\Theta}} \int_0^\infty \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n \geq m} \exp\Big\{m_0 I(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta})\Big\} > s) ds\, |w(\boldsymbol{\theta})| \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} \\
&\leq \int_{\boldsymbol{\Theta}} \Big\{ \int_{10}^\infty \mathbb{P}_{\boldsymbol{\theta}} \Big[\sup_{n \geq m} I(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) > \frac{\log(s)}{m_0} \Big] ds + 10 \Big\} |w(\boldsymbol{\theta})| \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} \\
&\leq \int_{\boldsymbol{\Theta}} \Big\{ \Big[e^{m\gamma(\theta_1)} + 4 \Big] \int_{10}^\infty \exp\Big[-\frac{m \log(s)}{2m_0} \Big] ds + 10 \Big\} |w(\boldsymbol{\theta})| \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} \\
&\leq \int_{\boldsymbol{\Theta}} \Big\{ \frac{e^{m\gamma(\theta_1)} + 4}{10} + 10 \Big\} |w(\boldsymbol{\theta})| \xi(\boldsymbol{\theta}) d\boldsymbol{\theta}.
\end{aligned}$$

Since $\gamma(\theta_1)$ is bounded on the compact support of $\xi(\cdot)$, there is a constant $C > 0$ such that

$$\int \sup_{n \geq m} W(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) \xi(\boldsymbol{\theta}) d\mathbb{P}^\xi \;\leq\; C \int_{\boldsymbol{\Theta}} |w(\boldsymbol{\theta})| \xi(\boldsymbol{\theta}) d\boldsymbol{\theta},$$

which is finite by assumptions. □

2.4 Stein's Identity

Let $\mathcal{H}$ denote the collection of measurable functions $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of polynomial growth; let $\mathcal{H}_p = \{h \in \mathcal{H} : sup_{\boldsymbol{z} \in \mathbb{R}^2} |h(\boldsymbol{z})|/(1 + |z_1|^p + |z_2|^p) \leq 1\}$, $\tilde{\mathcal{H}}_p = \{h : h/c \in \mathcal{H}_p$, for some $c > 0\}$. Thus $\mathcal{H} = \cup_{p \geq 0} \tilde{\mathcal{H}}_p$. Let $\|\boldsymbol{z}\|$ be the Euclidean norm of a vector $\boldsymbol{z}$. If $|h(\boldsymbol{z})| \leq c(1 + \|\boldsymbol{z}\|^p)$ for some $0 < c < \infty$, then $h \in \tilde{\mathcal{H}}_p$.

let Φ_1 and Φ denote one and two dimensional standard normal distributions, respectively; let ϕ_1 and ϕ denote one and two dimensional standard normal densities, respectively; and let

$$\begin{aligned}
\Phi h &= \int_{\mathbb{R}^2} h(z)\phi(z)dz, \\
\Phi_1^h(z_1) &= \int_{-\infty}^{\infty} h(z_1, z_2)\phi_1(z_2)dz_2, \\
V^h(z) &= (V_1^h(z), V_2^h(z)) = \begin{pmatrix} [\phi_1(z_1)]^{-1}\int_{z_1}^{\infty}\{\Phi_1^h(y_1) - \Phi h\}\phi_1(y_1)dy_1 \\ [\phi_1(z_2)]^{-1}\int_{z_2}^{\infty}\{h(z_1, y_2) - \Phi_1^h(z_1)\}\phi_1(y_2)dy_2 \end{pmatrix}^{\tau}.
\end{aligned}$$

For example, for $z = (z_1, z_2) \in \mathbb{R}^2$ if $h(z) = z_1$, then $V^h(z) = (1, 0)$, if $h(z) = z_2$, then $V^h(z) = (0, 1)$, and if $h(z) = z_1^2 + z_2^2$, then $V^h(z) = z$. Note that V_1^h, as a function on $\mathbb{R}^2$, is a constant in its last variable. The transformation from h to V^h is a linear operator from $\mathcal{H}_p$ into $\mathcal{H}_p \times \mathcal{H}_p$ and satisfies

$$\Phi V^h = \int_{\mathbb{R}^2} zh(z)\Phi(dz), \tag{20}$$

$$\Phi(V \circ V^h) = \frac{1}{2}\int_{\mathbb{R}^2} \begin{pmatrix} z_1^2 - 1 & 0 \\ 2z_1z_2 & z_2^2 - 1 \end{pmatrix} h(z)\Phi(dz). \tag{21}$$

Here and in the following, $V \circ V$ is the composition of V with itself.

Lemma 2.6 Given any nonnegative integer p, there is a constant C_p so that

$$\|V^h(z)\| \le C_p(1 + |z_1|^p + |z_2|^p), \quad \forall\, z \in \mathbb{R}^2,$$

for all $h \in \mathcal{H}_p$.

Proof. The assertion for $p = 0$ is proved by Stein (1986). For $p \geq 1$, the proof is similar and simpler. □

The following result is similar to one by Stein (1987) or Woodroofe (1989, 1992), in which Γ denotes a finite signed measure of the form $d\Gamma = fd\Phi$, where f is an integrable function with respect to Φ on $\mathbb{R}^2$; and $\Gamma h = \int_{\mathbb{R}^2} hd\Gamma$, when $h \in \mathcal{H}$ and the integral exits.

Stein's Identity Let p be a nonnegative integer. Assume that $d\Gamma = fd\Phi$, where f is absolutely continuous on every compact subset of $\mathbb{R}^2$. Denote that $\bigtriangledown f = (f_{10}(z), f_{01}(z))$, where $f_{j,k}(z) = \frac{\partial^{j+k}}{\partial z_1^j \partial z_2^k} f(z)$. If

$$\int_{\mathbb{R}^2} (|z_1|^p + |z_2|^p)[|f_{10}(z)| + |f_{10}(z)|]\Phi(dz) < \infty, \tag{22}$$

then

$$\Gamma h - \Gamma 1 \cdot \Phi h = \int_{\mathbb{R}^2} \boldsymbol{V}^h [\nabla f]^\tau \Phi(d\boldsymbol{z}), \ \ \forall\, h \in \tilde{\mathcal{H}}_p.$$

Proof. See Appendix. □

3 Asymptotic Expansions

3.1 Basic Consequences

If $A_n \cap B_n$ occurs, then the posterior distribution, Γ_n say, of $\boldsymbol{Z}_n$ given $X_1, \cdots, X_n$ is of the form $d\Gamma_n = f_n d\Phi$, where

$$f_n(\boldsymbol{z}) = c_n \xi(\boldsymbol{\theta}) J(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) I_{S_n}(\boldsymbol{z}), \qquad \boldsymbol{z} \in \mathbb{R}^2, \tag{23}$$

where $0 < c_n = c_n(X_1, \cdots, X_n) < \infty$, S_n denotes the range of $\boldsymbol{Z}_n$, and $\boldsymbol{z}$ and $\boldsymbol{\theta}$ are related by (10).

Let Ξ_0 denote the class of all absolutely continuous densities with compact support in $\boldsymbol{\Theta}$; Let AC denote the collections of all absolutely continuous functions on $\boldsymbol{\Theta}$; for $\alpha > 1$, define two subclasses of Ξ_0 by

$$\Xi_1^\alpha = \Big\{ \xi \in \Xi_0 : \int_{\boldsymbol{\Theta}} \Big\{ \Big|\frac{\xi_{10}}{\xi}\Big|^\alpha + \Big|\frac{\xi_{01}}{\xi}\Big|^\alpha \Big\} \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} < \infty \Big\}, \tag{24}$$

$$\Xi_2^\alpha = \Big\{ \xi \in \Xi_1 : \xi_{10}, \xi_{01} \in AC, \int_{\boldsymbol{\Theta}} \Big\{ \Big|\frac{\xi_{20}}{\xi}\Big|^\alpha + \Big|\frac{\xi_{11}}{\xi}\Big|^\alpha + \Big|\frac{\xi_{02}}{\xi}\Big|^\alpha \Big\} \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} < \infty \Big\}, \tag{25}$$

where $\xi_{jk}(\boldsymbol{\theta}) = \frac{\partial^{j+k}}{\partial \theta_1^j \partial \theta_1^k} \xi(\boldsymbol{\theta})$. Let $\tilde{\xi}(\boldsymbol{\theta}) = \xi(\boldsymbol{\theta})/\sqrt{-\theta_1}$, then $\frac{\tilde{\xi}_{0,k}}{\tilde{\xi}} = \frac{\xi_{0,k}}{\xi}$, for $k = 0, 1, 2, \cdots$. If $\xi \in \Xi_1$, then $\int_{\boldsymbol{\Theta}} (|\tilde{\xi}_{10}/\xi|^\alpha + |\tilde{\xi}_{01}/\xi|^\alpha) \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} < \infty$, and if $\xi \in \Xi_2$, then $\int_{\boldsymbol{\Theta}} (|\tilde{\xi}_{20}/\xi|^\alpha + |\tilde{\xi}_{11}/\xi|^\alpha + |\tilde{\xi}_{02}/\xi|^\alpha) \xi(\boldsymbol{\theta}) d\boldsymbol{\theta} < \infty$. For $\boldsymbol{\omega}, \boldsymbol{\theta} \in \boldsymbol{\Theta}$, define

$$\boldsymbol{K}_1^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) = \begin{pmatrix} K_{1,1}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) \\ K_{1,2}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) \end{pmatrix}^\tau \quad \text{and} \quad \boldsymbol{K}_2^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) = \begin{pmatrix} K_{2,11}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) & K_{2,12}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) \\ K_{2,12}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) & K_{2,22}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) \end{pmatrix},$$

where

$$\begin{aligned} K_{1,1}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) &= \frac{\tilde{\xi}_{10}}{\tilde{\xi}} J_1 + J_{1,01}, \\ K_{1,2}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) &= \frac{\tilde{\xi}_{01}}{\tilde{\xi}} J_2 + J_{2,01}, \\ K_{2,11}^\xi(\boldsymbol{\omega}, \boldsymbol{\theta}) &= \frac{\tilde{\xi}_{20}}{\tilde{\xi}} J_1^2 + \frac{3\tilde{\xi}_{10}}{\tilde{\xi}} J_1 J_{1,01} + J_1 J_{1,02} + J_{1,01}^2, \end{aligned}$$

$$K^{\xi}_{2,12}(\boldsymbol{\omega},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{11}}{\tilde{\xi}}J_1J_2 + \frac{\tilde{\xi}_{10}}{\tilde{\xi}}J_1J_{2,01} + \frac{\tilde{\xi}_{01}}{\tilde{\xi}}J_{1,01}J_2 + J_{1,01}J_{2,01},$$

$$K^{\xi}_{2,22}(\boldsymbol{\omega},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{02}}{\tilde{\xi}}J_2^2 + \frac{3\tilde{\xi}_{01}}{\tilde{\xi}}J_2J_{2,01} + J_2J_{2,02} + J_{2,01}^2.$$

In particular,

$$K^{\xi}_{1,1}(\boldsymbol{\theta},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{10}}{\tilde{\xi}}\frac{1}{\sqrt{G_1''(\theta_1)}} - \frac{G_1'''(\theta_1)}{3[G_1''(\theta_1)]^{3/2}},$$

$$K^{\xi}_{1,2}(\boldsymbol{\theta},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{01}}{\tilde{\xi}}\frac{1}{\sqrt{G_2''(\theta_2)}} - \frac{2G_2'''(\theta_2)}{3[G_2''(\theta_2)]^{3/2}},$$

$$K^{\xi}_{2,11}(\boldsymbol{\theta},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{20}}{\tilde{\xi}}\frac{1}{G_1''} - \frac{\tilde{\xi}_{10}}{\tilde{\xi}}\frac{G_1'''}{[G_1'']^2} + \frac{5[G_1''']^2}{12[G_1'']^3} - \frac{G_1^{\text{iv}}}{4[G_1'']^2},$$

$$K^{\xi}_{2,12}(\boldsymbol{\theta},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{11}}{\tilde{\xi}}\frac{1}{\sqrt{G_1''G_2''}} - \frac{\tilde{\xi}_{10}}{\tilde{\xi}}\frac{2G_2'''}{3\sqrt{G_1''[G_2'']^3}} - \frac{\tilde{\xi}_{01}}{\tilde{\xi}}\frac{G_1'''}{3\sqrt{[G_1'']^3G_2''}} + \frac{2G_1'''G_2'''}{9[G_1''G_2'']^{3/2}},$$

$$K^{\xi}_{2,22}(\boldsymbol{\theta},\boldsymbol{\theta}) = \frac{\tilde{\xi}_{02}}{\tilde{\xi}}\frac{1}{G_2''} - \frac{\tilde{\xi}_{01}}{\tilde{\xi}}\frac{2G_2'''}{[G_2'']^2} + \frac{5[G_2''']^2}{3[G_2'']^3} - \frac{3G_2^{\text{iv}}}{4[G_2'']^2}.$$

Lemma 3.1 Suppose that $\xi \in \Xi_1^{\alpha}$ for some $\alpha > 1$, and A_nB_n occurs, then f_n is absolutely continuous with

$$\nabla f_n(\boldsymbol{z}) = \frac{1}{\sqrt{n}}f_n(\boldsymbol{z})\boldsymbol{K}^{\xi}_1(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta}), \quad \boldsymbol{z} \in \mathbb{R}^2, \tag{26}$$

and if $\xi \in \Xi_2^{\alpha}$ for some $\alpha > 1$, and A_nB_n occurs, then f_n is twice continuously differentiable for which,

$$\nabla^2 f_n(\boldsymbol{z}) = \frac{1}{n}f_n(\boldsymbol{z})\boldsymbol{K}^{\xi}_2(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta}), \quad \boldsymbol{z} \in \mathbb{R}^2. \tag{27}$$

Proof. It follows from assumptions immediately. □

Lemma 3.2 Let $\alpha > 1$. If $\xi \in \Xi_1^{\alpha}$, then $|K^{\xi}_{1,1}|^{\alpha} + |K^{\xi}_{1,2}|^{\alpha} \in \mathcal{W}_{\xi}$; if $\xi \in \Xi_2^{\alpha}$, then $|K^{\xi}_{1,11}|^{\alpha} + |K^{\xi}_{1,12}|^{\alpha} + |K^{\xi}_{1,22}|^{\alpha} \in \mathcal{W}_{\xi}$.

Proof. See Appendix. □

Theorem 3.1 Suppose that $\xi \in \Xi_1^{\alpha}$ for some $\alpha > 1$. There is $m \geq 2$ such that for all $h \in \mathcal{H}$,

$$\mathbb{E}^{\xi}_n\{h(\boldsymbol{Z}_n)\} = \Phi h + \frac{1}{\sqrt{n}}\mathbb{E}^{\xi}_n\Big\{\boldsymbol{K}^{\xi}_1(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})[\boldsymbol{V}^h(\boldsymbol{Z}_n)]^{\tau}\Big\}, \tag{28}$$

a.s. $(\mathbb{P}^{\xi})$ on A_nB_n for all $n \geq m$. Suppose that $\xi \in \Xi_2^{\alpha}$ for some $\alpha > 1$. Then there is $m \geq 2$ such that for all $h \in \mathcal{H}$,

$$\mathbb{E}^{\xi}_n\{h(\boldsymbol{Z}_n)\} = \Phi h + \frac{1}{\sqrt{n}}(\Phi\boldsymbol{V}^h)\mathbb{E}^{\xi}_n\{\boldsymbol{K}^{\xi}_1(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})\}^{\tau} + \frac{1}{n}tr\Big\{\mathbb{E}^{\xi}_n\Big\{\boldsymbol{K}^{\xi}_2(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})[\boldsymbol{V}\circ\boldsymbol{V}^h(\boldsymbol{Z}_n)]\Big\}\Big\}, \tag{29}$$

a.s. $(\mathbb{P}^{\xi})$ on A_nB_n for all $n \geq m$, for some $m \geq 2$.

Proof. If $\xi \in \Xi_1^\alpha$, $|K_{1,1}^\xi|^\alpha + |K_{1,1}^\xi|^\alpha \in \mathcal{W}^\xi$ from Lemma 3.2. Let $m \geq 2$ be defined by (19). If $n \geq m$ and $A_n B_n$ occurs, then for all $0 \leq p < \infty$,

$$\sqrt{n} \int_{\mathbb{R}^2} (|z_1|^p + |z_2|^p) \Big| \frac{\partial}{\partial z_1} f_n(\boldsymbol{z}) \Big| \Phi(d\boldsymbol{z}) = \mathbb{E}_n^\xi \{ (|Z_{n1}|^p + |Z_{n2}|^p) |K_{1,1}^\xi(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta})| \},$$

which is finite w.p. 1 for all $0 \leq p < \infty$ by Hölder's Inequality. Similarly, it is finite w.p. 1 for all $0 \leq p < \infty$ if $\frac{\partial}{\partial z_1} f_n(\boldsymbol{z})$ is replaced by $\frac{\partial}{\partial z_2} f_n(\boldsymbol{z})$. Now the first assertion is established by Stein's Identity. For the second, it is sufficient to verify that for any $0 \leq p < \infty$,

$$\sqrt{n} \int_{\mathbb{R}^2} (|z_1|^p + |z_2|^p) \Big| \frac{\partial^2}{\partial z_i \partial z_j} f_n(\boldsymbol{z}) \Big| \Phi(d\boldsymbol{z}) I_{A_n B_n} < \infty, \quad a.e.,$$

which is valid from Hölder's Inequality and the second assertion of Lemma 3.2. □

For further discussion, we need the concept of nearly dominated. Let $Y_1, Y_2, \cdots$ denote a sequence of random variables adaptive to a family of σ–algebras, $\mathcal{F}_1, \mathcal{F}_2, \cdots$, on a probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$; Let $\mathcal{T}$ denote the collection of all finite stopping time t with respect to $\mathcal{F}_1, \mathcal{F}_2, \cdots$. $Y_1, Y_2, \cdots$ is said to be nearly dominated iff $\{Y_t : t \in \mathcal{T}\}$ is uniformly integrable. Observe that then $\sup_{t \in \mathcal{T}} \mathbb{E}(Y_t) < \infty$ and $\sup_{t \in \mathcal{T}} |Y_t| < \infty$, w.p.1. If $W_1, W_2, \cdots$ are random variables for which $\mathbb{E}\Big\{\sup_{t \in \mathcal{T}} |W_t|\Big\} < \infty$, then $Y_n = \mathbb{E}(W_n | \mathcal{F}_n), n \geq 1$ is nearly dominated.

Also if $1 < \alpha, \beta < \infty$ are conjugate values and if $|W_n|^\alpha, n \geq 1$ and $|Y_n|^\beta, n \geq 1$ are both nearly dominated, then so is $\{W_n Y_n\}_{n \geq 1}$. In the following, we will use $\mathcal{F}_n = \sigma\{X_1, X_2, \cdots, X_n\}, n = 1, 2, \cdots$ and $\mathbb{P} = \mathbb{P}^\xi$.

Lemma 3.3 Let $1 \leq p < \infty$; if $\xi \in \Xi_1^{p+1}$, then

$$\operatorname*{ess\,sup}_{h \in \mathcal{H}_p} \sqrt{n} |\mathbb{E}_n^\xi [h(\boldsymbol{Z}_n)] - \Phi h| I_{A_n B_n}, \quad n \geq m,$$

are nearly dominated for some $m \geq 2$.

Proof. Let m be in (19) for $W = |K_{1,1}^\xi|^{p+1} + |K_{1,2}^\xi|^{p+1}$. If $h \in \mathcal{H}_p$, then for $n \geq m$,

$$\begin{aligned} & \sqrt{n} |\mathbb{E}_n^\xi [h(\boldsymbol{Z}_n)] - \Phi h| \\ = \; & |\sum_{j=1}^2 \mathbb{E}_n^\xi \{ K_{1,j}^\xi (\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}) V_j^h(\boldsymbol{Z}_n) \}| \\ \leq \; & \sum_{j=1}^2 \Big\{ \mathbb{E}_n^\xi [|K_{1,j}^\xi (\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta})|^{p+1}] \Big\}^{\frac{1}{p+1}} \Big\{ \mathbb{E}_n^\xi [|V_j^h(\boldsymbol{Z}_n)|^{\frac{p+1}{p}}] \Big\}^{\frac{p}{p+1}}, a.e. \text{ on } A_n B_n, \end{aligned}$$

by Theorem 3.1 and Hölder's Inequality. Let $C_j(p)$ denote positive constants depending only on p. From Lemma 2.6,

$$|V_j^h(\boldsymbol{z})|^{\frac{p+1}{p}} \leq C_1(p) \{1 + |z_1|^{p+1} + |z_2|^{p+1}\}, \quad \forall \; h \in \mathcal{H}_p.$$

Thus

$$\begin{aligned}
&\sup_{n\geq m}\sqrt{n}|\mathbb{E}_n^\xi[h(\boldsymbol{Z}_n)]-\Phi h|I_{A_nB_n}\\
\leq\ & C_2(p)\sum_{j=1}^{2}\Big\{\mathbb{E}_n^\xi[|K_{1,j}^\xi(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})|^{p+1}]\Big\}^{\frac{1}{p+1}}\Big\{\mathbb{E}_n^\xi[1+|Z_{n1}|^{p+1}+|Z_{n2}|^{p+1}]\Big\}^{\frac{p}{p+1}}I_{A_nB_n}, a.e.
\end{aligned}$$

Let $g_p(\boldsymbol{z})=1+|z_1|^p$ for $p\geq 1$. Then $V_1^{g_{p+1}}(\boldsymbol{z})\leq C_3(p)g_p(\boldsymbol{z})\leq C_4(p)g_{p+1}(\boldsymbol{z})$ and $V_2^{g_{p+1}}(\boldsymbol{z})=0$. Therefore

$$\sqrt{n}|\mathbb{E}_n^\xi[g_{p+1}(\boldsymbol{Z}_n)]-\Phi g_{p+1}|\leq C_5(p)\Big\{\mathbb{E}_n^\xi[|K_{1,1}^\xi(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})|^{p+1}]\Big\}^{\frac{1}{p+1}}\Big\{\mathbb{E}_n^\xi[|g_{p+1}(\boldsymbol{Z}_n)|^{\frac{p+1}{p}}]\Big\}^{\frac{p}{p+1}}.$$

Using an inequality: if $0<b,c<\infty, 1\leq p<\infty$ and $0\leq x\leq b+cx^{p/(p+1)}$, then $x\leq pb+c^{p+1}$. Now, let $b=\Phi g_{p+1}$, $c=\frac{C_5(p)}{\sqrt{n}}\Big\{\mathbb{E}_n^\xi[|K_{1,1}^\xi(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})|^{p+1}]\Big\}^{\frac{1}{p+1}}$, $x=\mathbb{E}_n^\xi[g_{p+1}(\boldsymbol{Z}_n)]^{\frac{p+1}{p}}$, then

$$\mathbb{E}_n^\xi[|g_{p+1}(\boldsymbol{Z}_n)]\leq pb+c^{p+1}=p\Phi g_{p+1}+\Big(\frac{C_5(p)}{\sqrt{n}}\Big)^{p+1}\mathbb{E}_n^\xi[|K_{1,1}^\xi(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})|^{p+1}]\Big\}. \tag{30}$$

Similarly, let $\tilde{g}_p(z)=1+|z_2|^p$. Then

$$\mathbb{E}_n^\xi[|\tilde{g}_{p+1}(Z_n)]\leq p\Phi\tilde{g}_{p+1}+\Big(\frac{C_6(p)}{\sqrt{n}}\Big)^{p+1}\mathbb{E}_n^\xi[|K_{1,2}^\xi(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})|^{p+1}]\Big\}. \tag{31}$$

Now the assertion follows from (30)–(31). □

Corollary 3.1 If $1\leq p<\infty$ and $|K_{1,j}^\xi|^{p+1}\in\mathcal{G}^\xi$, $j=1,2$, then

$$\operatorname*{ess\,sup}_{h\in\mathcal{H}_p}\sqrt{n}|\mathbb{E}_n^\xi[h(\boldsymbol{Z}_n)]|I_{A_nB_n},\quad n\geq m,$$

are nearly dominated for some $m\geq 2$.

3.2 The Second Order Expansions

Let $t=t_a, a\geq 1$ denote an increasing family of stopping times, with respect to $\mathcal{F}_n=\sigma(X_1,\cdots,X_n)$. We will assume that

$$\mathbb{P}_{\boldsymbol{\theta}}\{A_tB_t\}\ \ =\ \ 1,\ \forall\ a\geq 1,\boldsymbol{\theta}\in\boldsymbol{\Theta}. \tag{32}$$

Let ξ be a member in Ξ_1^{p+1} ($p\geq 1$). Observe that $\mathbb{E}^\xi|h(\boldsymbol{Z}_t)|^p<\infty$ for any For $h\in\mathcal{H}_p$ and $a\geq 1$, by Corollary 3.1. For the next theorem, let

$$R_a(\xi,h)=\sqrt{a}\Big\{\mathbb{E}_t^\xi[h(\boldsymbol{Z}_t)]-\Phi h-\frac{1}{\sqrt{a}}(\Phi\boldsymbol{V}^h)\mathbb{E}_t^\xi[\rho(\boldsymbol{\theta})\boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})]^\tau\Big\}. \tag{33}$$

Theorem 3.2 Let $\xi \in \Xi_1^{p+1}$ for some $p \geq 1$, and let $t = t_a$ are stopping time satisfying

$$\frac{a}{t_a} \to \rho^2(\boldsymbol{\theta}), \text{ in } \mathbb{P}^\xi - \text{probability}, \quad \text{as } a \to \infty, \tag{34}$$

and

$$\lim_{a\to\infty} a^q \mathbb{P}^\xi\{t_a \leq \eta a\} = 0, \ \exists\ \eta > 0,\ q > \frac{1}{2}. \tag{35}$$

Then

$$\lim_{a\to\infty} \mathbb{E}^\xi\Big\{\operatorname*{ess\ sup}_{h\in\mathcal{H}_p}\Big|R_a(\xi,h)\Big|\Big\} = 0.$$

Proof. Denote $C_1, C_2, \cdots$ positive constants, and $C_1(p), C_2(p), \cdots$ positive constants depending only on p. Then,

$$\begin{aligned}\int_{\{t\leq\eta a\}} \operatorname*{ess\ sup}_{h\in\mathcal{H}_p}|R_a(\xi,h)|d\mathbb{P}^\xi &\leq C_1(p)\sqrt{a}\int_{\{t\leq\eta a\}}\Big(1+|Z_{t1}|^p+|Z_{t2}|^p\Big)d\mathbb{P}^\xi \\ &+ C_1(p)\int_{\{t\leq\eta a\}}\sum_{j=1}^2 \rho(\boldsymbol{\theta})|K_{1,j}^\xi(\boldsymbol{\theta},\boldsymbol{\theta})|d\mathbb{P}^\xi.\end{aligned}$$

The second integral $\to 0$ as $a \to \infty$ by assumptions (32)–(35). For the first, let η and q be in (35).

$$\begin{aligned}\sqrt{a}\int_{\{t\leq\eta a\}}\Big(1+|Z_{t1}|^p+|Z_{t2}|^p\Big)d\mathbb{P}^\xi &\leq 2^p\sqrt{a}\int_{\{t\leq\eta a\}}\Big(1+\|\boldsymbol{Z}_t\|^p\Big)d\mathbb{P}^\xi \\ &\leq C_2\sqrt{a}\Big[\mathbb{P}^\xi(t\leq\eta a)\Big]^{\frac{1}{\alpha}}\mathbb{E}^\xi\Big\{\max_{k\leq\eta a}\Big(1+\|\boldsymbol{Z}_k\|^p\Big)^\beta\Big\}^{\frac{1}{\beta}}. \end{aligned}\tag{36}$$

Since G_1 is bounded on the compact support of ξ, it follows from Lemma 2.4 that

$$\mathbb{P}^\xi(\max_{k\geq n}\|\boldsymbol{Z}_k\|I_{A_kB_k} \geq x) \leq C_3(1+\log_2 n)\exp\{-\frac{x^2}{4}\},\ x \geq 1,.$$

The right most factor is then of order $(\log a)^p$, and (36) is of order $\sqrt{a}a^{-q/\alpha}(\log a)^p = o(1)$. For the integral over $\{t > \eta a\}$, we may write

$$R_a(\xi,h) = \sum_{j=1}^4 R_{a,j}(\xi,h),$$

where

$$\begin{aligned}R_{a,1}(\xi,h) &= \sqrt{\frac{a}{t}}\mathbb{E}_t^\xi\Big\{\Big[\boldsymbol{K}_1^\xi(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) - \boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\Big[\boldsymbol{V}^h(\boldsymbol{Z}_t)\Big]^\tau\Big\}, \\ R_{a,2}(\xi,h) &= \sqrt{\frac{a}{t}}\mathbb{E}_t^\xi\Big\{\Big[\boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta}) - \mathbb{E}_t^\xi[\boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})]\Big]\Big[\boldsymbol{V}^h(\boldsymbol{Z}_t)\Big]^\tau\Big\}, \\ R_{a,3}(\xi,h) &= \sqrt{\frac{a}{t}}\mathbb{E}_t^\xi\Big\{\mathbb{E}_t^\xi\Big[\boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\mathbb{E}_t^\xi\Big[\boldsymbol{V}^h(\boldsymbol{Z}_t) - \Phi\boldsymbol{V}^h\Big]^\tau\Big\}, \\ R_{a,4}(\xi,h) &= \Phi\boldsymbol{V}^h\mathbb{E}_t^\xi\Big\{\Big[\sqrt{\frac{a}{t}} - \rho(\boldsymbol{\theta})\Big]\Big[\boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]^\tau\Big\}.\end{aligned}$$

It is convenient to write $R^{\#}_{a,j} = \operatorname*{ess\,sup}_{h\in\tilde{\mathcal{H}}_p} R_{a,j}(\xi,h)I_{\{t>\eta a\}}$. By Hölder's Inequality and Lemma 2.6,

$$\mathbb{E}^{\xi}(R^{\#}_{a,1}) = \frac{C_4(p)}{\sqrt{\eta}}\sum_{j=1}^{2}\left\{\mathbb{E}^{\xi}\left|K^{\xi}_{1,j}(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) - K^{\xi}_{1,j}(\boldsymbol{\theta},\boldsymbol{\theta})\right|^{p+1}\right\}^{\frac{1}{p+1}}\left\{\mathbb{E}^{\xi}\left[(1+|Z_{t1}|^p+|Z_{t2}|^p)^{\frac{p+1}{p}}\right]\right\}^{\frac{p}{p+1}}.$$

By Corollary 3.1, the right factor in the summand is bounded. The first fact in the summand is not more than $\left\{\mathbb{E}^{\xi}\sup_{n\geq\eta n}\left|K^{\xi}_{1,j}(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta}) - K^{\xi}_{1,j}(\boldsymbol{\theta},\boldsymbol{\theta})\right|^{p+1}\right\}^{\frac{1}{p+1}}$, which approaches zero as $a\to\infty$, by the dominated and convergence theorems. Proving $\lim_{a\to\infty}\mathbb{E}^{\xi}(R^{\#}_{a,2})\to 0$ is similar. By Lemma 3.3 and Hölder Inequality, $R^{\#}_{a,3}$, $a\geq 1$, is uniformly integrable. It is clear that $R^{\#}_{a,3}\to 0$, w.p.1. as $a\geq 1$. Therefore, $\lim_{a\to\infty}\mathbb{E}^{\xi}(R^{\#}_{a,3})\to 0$. It follows from (34) and (35) that $|\sqrt{a/t}-\rho(\boldsymbol{\theta})|I_{\{t>\eta a\}}\leq |\sqrt{a/t}-\rho(\boldsymbol{\theta})|\to 0$ in probability as $a\to\infty$ and that $|\sqrt{a/t}-\rho(\boldsymbol{\theta})|I_{\{t>\eta a\}}\leq 1/\sqrt{\eta}+\rho(\boldsymbol{\theta})$, which is essentially bounded. Therefore, $\mathbb{E}^{\xi}(R^{\#}_{a,4})\leq C_5(p)\sum_{j=1}^{2}\mathbb{E}^{\xi}(|K_{1,j}(\boldsymbol{\theta},\boldsymbol{\theta})||\sqrt{a/t}-\rho(\boldsymbol{\theta})|I_{\{t>\eta a\}})\to 0$ as $a\to\infty$, by dominated convergence theorem. □

For the corollaries below, suppose that $t_a, a\geq 1$, satisfying (34), and for every compact $\boldsymbol{\Theta}_0\subset\boldsymbol{\Theta}$, there is an $\eta=\eta(\boldsymbol{\Theta}_0)$, for which

$$\lim_{a\to\infty} a^q\int_{\boldsymbol{\Theta}_0}\mathbb{P}_{\boldsymbol{\theta}}(t_a\leq\eta)d\boldsymbol{\theta} = 0,\quad \text{for some } q>\frac{1}{2}. \tag{37}$$

Condition (37) implies (35) for every ξ with compact support. Assume further that ρ is absolutely continuous on all compact subsets of $\boldsymbol{\Theta}$. Let

$$\begin{aligned}
\kappa_{1,1}(\boldsymbol{\theta}) &= \frac{G_1'''(\theta_1)}{6[G_1''(\theta_1)]^{3/2}} - \frac{1}{\sqrt{G_1''(\theta_1)}}\frac{\frac{\partial}{\partial\theta_1}[\rho(\boldsymbol{\theta})\sqrt{-\theta_1}]}{\rho(\boldsymbol{\theta})\sqrt{-\theta_1}},\\
\kappa_{1,2}(\boldsymbol{\theta}) &= -\frac{G_2'''(\theta_2)}{6[G_2''(\theta_2)]^{3/2}} - \frac{1}{\sqrt{G_2''(\theta_2)}}\frac{\frac{\partial}{\partial\theta_2}\rho(\boldsymbol{\theta})}{\rho(\boldsymbol{\theta})},\\
\tilde{\kappa}^{\xi}_{1,j} &= \int_{\boldsymbol{\Theta}}\kappa_{1,j}(\boldsymbol{\theta})\rho(\boldsymbol{\theta})\xi(\boldsymbol{\theta})d\boldsymbol{\theta},\quad j=1,2.
\end{aligned}$$

Corollary 3.2 If $p\geq 1$ and $\xi\in\Xi_0^{p+1}$, then

$$\lim_{a\to\infty}\mathbb{E}^{\xi}\left\{\operatorname*{ess\,sup}_{h\in\mathcal{H}_p}\sqrt{a}\left|\mathbb{E}^{\xi}_t[h(\boldsymbol{Z}_t)] - \Phi h - \frac{1}{\sqrt{a}}(\Phi\boldsymbol{V}^h)(\tilde{\kappa}^{\xi}_{1,1},\tilde{\kappa}^{\xi}_{1,2})^{\tau}\right|\right\} = 0, \tag{38}$$

and

$$\lim_{a\to\infty}\sup_{h\in\mathcal{H}_p}\sqrt{a}\left|\mathbb{E}^{\xi}[h(\boldsymbol{Z}_t)] - \Phi h - \frac{1}{\sqrt{a}}(\Phi\boldsymbol{V}^h)(\tilde{\kappa}^{\xi}_{1,1},\tilde{\kappa}^{\xi}_{1,2})^{\tau}\right| = 0. \tag{39}$$

Proof. We only prove (39). The proof for the other is similar. In view of the definition of R_a, the term on the left hand side of (39) may be written as the absolute value of the sum of

$$I_a = \mathbb{E}^{\xi}\{R_a(\xi,h)\}\quad\text{and}\quad II = \Phi\mathrm{V}^h\{\mathbb{E}^{\xi}[\rho(\boldsymbol{\theta})\boldsymbol{K}^{\xi}_1(\boldsymbol{\theta},\boldsymbol{\theta})] - (\tilde{\kappa}^{\xi}_{1,1},\tilde{\kappa}^{\xi}_{1,2})\}^{\tau},$$

for all $\xi \in \Xi_1^{p+1}$, and $h \in \mathcal{H}_p, (a \geq 1)$. It is clear from Theorem 1 that $I_a \to 0$ as $a \to \infty$ uniformly in $h \in \mathcal{H}_p$. However,

$$\mathbb{E}^\xi\{\rho(\boldsymbol{\theta})K_{1,1}^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\} = \int_{\boldsymbol{\Theta}} \rho(\boldsymbol{\theta})\Big\{\frac{\tilde{\xi}_{10}}{\tilde{\xi}}\frac{1}{\sqrt{G_1''(\theta_1)}} - \frac{G_1'''(\theta_1)}{3[G_1''(\theta_1)]^{3/2}}\Big\}\xi(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

By the assumptions on ξ,

$$\begin{aligned}\int_{\boldsymbol{\Theta}_1} \frac{\rho(\boldsymbol{\theta})\tilde{\xi}_{10}(\boldsymbol{\theta})}{\sqrt{G_1''(\theta_1)\tilde{\xi}(\boldsymbol{\theta})}}\xi(\boldsymbol{\theta})d\theta_1 &= \int_{\boldsymbol{\Theta}_1} -\frac{\partial}{\partial\theta_1}\Big[\frac{\rho(\boldsymbol{\theta})\sqrt{-\theta_1}}{\sqrt{G_1''(\theta_1)}}\Big]\frac{1}{\sqrt{-\theta_1}}\xi(\boldsymbol{\theta})d\theta_1 \\ &= \int_{\boldsymbol{\Theta}_1}\Big\{\frac{G_1'''(\theta_1)}{2[G_1''(\theta_1)]^{2/3}} - \frac{1}{\sqrt{G_1''(\theta_1)}}\frac{\frac{\partial}{\partial\theta_1}[\rho(\boldsymbol{\theta})\sqrt{-\theta_1}]}{\rho(\boldsymbol{\theta})\sqrt{-\theta_1}}\Big\}\rho(\boldsymbol{\theta})\xi(\boldsymbol{\theta})d\boldsymbol{\theta}_1.\end{aligned}$$

Thus, $\mathbb{E}^\xi\{\rho(\boldsymbol{\theta})K_{1,1}^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\} = \int_{\boldsymbol{\Theta}} \kappa_{1,1}(\boldsymbol{\theta})\rho(\boldsymbol{\theta})\xi(\boldsymbol{\theta})d\boldsymbol{\theta} = \tilde{\kappa}_{1,1}^\xi$. Similarly, $\mathbb{E}^\xi\{\rho(\boldsymbol{\theta})K_{1,2}^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\} = \tilde{\kappa}_{1,2}^\xi$. It turns out that $II = 0$. This completes the proof. □

3.3 Higher Order Expansions

In this subsection, $t = t_a, a \geq 1$ denote stopping times for which (32), (34) and (37) hold for some $q > 1$; and

$$S_a(\xi,h) = a\Big(\mathbb{E}_t^\xi[h(\boldsymbol{Z}_t)] - \Phi h - \frac{1}{\sqrt{t}}(\Phi W^h)\mathbb{E}_t^\xi[\boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})]^\tau - \frac{1}{a}tr\Big\{\Phi(\boldsymbol{V}\circ\boldsymbol{V}^h)\mathbb{E}_t^\xi\Big[\rho^2(\boldsymbol{\theta})\boldsymbol{K}_2^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\Big\}\Big),$$

for $h \in \mathcal{H}_p$ and $\xi \in \Xi_2^{p+1}, (p \geq 2)$.

Theorem 3.3 As $a \to \infty$,

$$\lim_{a\to\infty} \mathbb{E}^\xi\Big\{\operatorname*{ess\,sup}_{h\in\mathcal{H}_p}\Big|S_a(\xi,h)\Big|\Big\} = 0.$$

Proof. Since $q > 1$, on the analogy of the proof of Theorem 3.2,

$$\int_{\{t\leq\eta a\}} \operatorname*{ess\,sup}_{h\in\mathcal{H}_p}|S_a(\xi,h)|d\mathbb{P}^\xi \to 0, \quad \text{as } a \to \infty.$$

To estimate the integral over $\{t > \eta a\}$, it is convenient to decompose

$$\begin{aligned}S_a(\xi,h) &= \frac{a}{t}tr\ \mathbb{E}_t^\xi\Big\{\Big[\boldsymbol{K}_2^\xi(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) - \boldsymbol{K}_2^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\Big[\boldsymbol{V}\circ\boldsymbol{V}^h(\boldsymbol{Z}_t)\Big]\Big\} \\ &+ \frac{a}{t}tr\ \mathbb{E}_t^\xi\Big\{\Big[\boldsymbol{K}_2^\xi(\boldsymbol{\theta},\boldsymbol{\theta}) - \mathbb{E}_t^\xi[\boldsymbol{K}_2^\xi(\boldsymbol{\theta},\boldsymbol{\theta})]\Big]\Big[\boldsymbol{V}\circ\boldsymbol{V}^h(\boldsymbol{Z}_t)\Big]\Big\} \\ &+ \frac{a}{t}tr\ \mathbb{E}_t^\xi\Big[\boldsymbol{K}_2^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\mathbb{E}_t^\xi\Big[\boldsymbol{V}\circ\boldsymbol{V}^h(\boldsymbol{Z}_t) - \Phi\boldsymbol{V}^h\Big] \\ &+ tr\ \Big[\Phi(\boldsymbol{V}\circ\boldsymbol{V}^h)\Big]\mathbb{E}_t^\xi\Big\{\Big[\sqrt{\frac{a}{t}} - \rho(\boldsymbol{\theta})\Big]\Big[\boldsymbol{K}_2^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\Big\} \\ &+ \sqrt{a}\sqrt{\frac{a}{t}}\Big[\Phi\boldsymbol{V}^h\Big]\mathbb{E}_t^\xi\Big[\boldsymbol{K}_1^\xi(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) - \boldsymbol{K}_1^\xi(\boldsymbol{\theta},\boldsymbol{\theta})\Big]^\tau \\ &\overset{say}{=} \sum_{j=1}^5 S_{a,j}.\end{aligned}$$

The analyses of $S_{a,1}$—$S_{a,4}$ are similar to these of $R_{a,1}$—$R_{a,4}$ in Theorem 3.2. For $S_{a,5}$, it is sufficient to prove

$$I_j \overset{say}{=} \sqrt{a}\int_{\{t>\eta a\}}\Big|\mathbb{E}_t^{\xi}\Big[\boldsymbol{K}_{1,j}^{\xi}(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) - \boldsymbol{K}_{1,j}^{\xi}(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\Big|d\mathbb{P}^{\xi} \to 0, \quad \text{as } a\to\infty, \quad j=1,2. \tag{40}$$

Let $\boldsymbol{\Theta}_{\xi}$ denote the compact support of ξ; let $\boldsymbol{\Theta}_0 = [\theta_{1L},\theta_{1U}]\times[\theta_{2L},\theta_{2U}]$ denote a compact rectangle, for which $\boldsymbol{\Theta}_{\xi}\subset\boldsymbol{\Theta}_0^0\subset\boldsymbol{\Theta}_0\subset\boldsymbol{\Theta}$. define $D_1=\{t>\eta a, \hat{\theta}_{t1}\notin[\theta_{1L},\theta_{1U}]\}$ and $D_2=\{t>\eta a, \hat{\theta}_{t1}\in[\theta_{1L},\theta_{1U}]\}$. First,

$$\begin{aligned}
&\sqrt{a}\int_{D_1}\sqrt{\frac{a}{t}}\Big|\mathbb{E}_t^{\xi}\Big[K_{1,1}^{\xi}(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) - K_{1,1}^{\xi}(\boldsymbol{\theta},\boldsymbol{\theta})\Big]\Big|d\mathbb{P}^{\xi}\\
\le\ & 2\sqrt{\frac{a}{\eta}}\mathbb{E}^{\xi}\Big\{\sup_{n\ge\eta a}\Big|K_{1,1}^{\xi}(\hat{\boldsymbol{\theta}}_n,\boldsymbol{\theta})\Big|^{p+1}\Big\}^{\frac{1}{p+1}}\times\mathbb{P}^{\xi}\Big(\hat{\theta}_{n1}\notin[\theta_{1L},\theta_{1U}], \exists\ n\ge\eta a\Big)^{\frac{p}{p+1}},
\end{aligned}$$

which approaches zero as $a\to\infty$ by Lemmas 3.2 and 2.3. Since $K_{1,1}^{\xi}(\boldsymbol{\omega},\boldsymbol{\theta})$ does not depend on ω_1, we may write $K_{1,1}^{\xi}(\hat{\boldsymbol{\theta}}_t,\boldsymbol{\theta}) = K_{1,1}^{\xi}(\hat{\theta}_{t1},\boldsymbol{\theta})$ and $K_{1,1}^{\xi}(\boldsymbol{\theta},\boldsymbol{\theta}) = K_{1,1}^{\xi}(\theta_1,\boldsymbol{\theta})$. Let

$$\sqrt{n}\{K_{1,1}^{\xi}(\hat{\theta}_{n1},\boldsymbol{\theta}) - K_{1,1}^{\xi}(\theta_1,\boldsymbol{\theta})\} = I_n + II_n,$$

where

$$\begin{aligned}
I_n &= \sqrt{n}\{K_{1,1}^{\xi}(\hat{\theta}_{n1},\boldsymbol{\theta}) - K_{1,1}^{\xi}(\theta_1,\boldsymbol{\theta})\} - \frac{Z_{n1}}{\sqrt{G_1''(\hat{\theta}_{n1})}}\frac{\partial}{\partial\omega_1}K_{1,1}^{\xi}(\omega_1,\boldsymbol{\theta})\Big|_{\omega_1=\hat{\theta}_{n1}},\\
II_n &= \frac{Z_{n1}}{\sqrt{G_1''(\hat{\theta}_{n1})}}\frac{\partial}{\partial\omega_1}K_{1,1}^{\xi}(\omega_1,\boldsymbol{\theta})\Big|_{\omega_1=\hat{\theta}_{n1}}.
\end{aligned}$$

Let $C_j (j\ge 1)$ be constants. By the Mean Value Theorem, we can see $I_t\to 0$ as $a\to\infty$, and

$$|I_t|\le C_1|Z_{t1}|, w.p.1. \quad \text{on } D_2 \ \text{ for all } a\ge 1.$$

So

$$\int_{\mathrm{D}_2}|\mathbb{E}_t^{\xi}(I_t)|d\mathbb{P}^{\xi}\le\int_{\mathrm{D}_2}|I_t|d\mathbb{P}^{\xi}\to 0 \ \text{ as } a\to\infty.$$

From Theorem 3.1, and the continuity of $K_{1,1}(\omega,\theta)$ and $G_1''(\theta_1)$,

$$\begin{aligned}
\int_{\mathrm{D}_2}|\mathbb{E}_t^{\xi}(II_t)|d\mathbb{P}^{\xi} &= \int_{\mathrm{D}_2}\Big|\frac{\partial}{\partial\omega_1}K_{1,1}^{\xi}(\omega_1,\boldsymbol{\theta})\Big|_{\omega_1=\hat{\theta}_{t1}}\frac{\mathbb{E}_t^{\xi}(Z_{n1})}{\sqrt{G_1''(\hat{\theta}_{n1})}}\Big|d\mathbb{P}^{\xi}\\
&\le C_2\int_{\mathrm{D}_2}\Big|\mathbb{E}_t^{\xi}(Z_{n1})\Big|d\mathbb{P}^{\xi}\\
&= C_2\int_{\mathrm{D}_2}\frac{1}{\sqrt{t}}\Big|K_{1,1}^{\xi}(\hat{\theta}_{t1},\boldsymbol{\theta})\Big|d\mathbb{P}^{\xi}\\
&\le C_3\int_{\mathrm{D}_2}\frac{1}{\sqrt{t}}d\mathbb{P}^{\xi}\to 0,
\end{aligned}$$

as $a \to \infty$. Therefore, (40) holds for $j = 1$. The proof for $j = 2$ is similar. □

For the Corollary below, assume that $\rho(\cdot)$ is twice absolutely continuous; and let

$$\begin{aligned}
\kappa_{2,11}(\boldsymbol{\theta}) &= \frac{\frac{\partial^2}{\partial\theta_1^2}[\rho^2\sqrt{-\theta_1}]}{G_1''\rho^2\sqrt{-\theta_1}} - \frac{G_1'''}{G_1''^2}\frac{\frac{\partial}{\partial\theta_1}[\rho^2\sqrt{-\theta_1}]}{\rho^2\sqrt{-\theta_1}} + \frac{5G_1'''^2}{12G_1''^3} - \frac{G_1^{\text{iv}}}{4G_1''^2}, \\
\kappa_{2,12}(\boldsymbol{\theta}) &= \frac{\frac{\partial^2}{\partial\theta_1\partial\theta_2}[\rho^2\sqrt{-\theta_1}]}{\sqrt{G_1''G_2''}\,\rho^2\sqrt{-\theta_1}} + \frac{G_2'''\frac{\partial}{\partial\theta_1}[\rho^2\sqrt{-\theta_1}]}{6\sqrt{G_1''G_2''^3}\rho^2\sqrt{-\theta_1}} - \frac{G_1'''\frac{\partial}{\partial\theta_2}[\rho^2\sqrt{-\theta_1}]}{6\sqrt{G_1''^3G_2''}\rho^2\sqrt{-\theta_1}} - \frac{19G_1'''G_2'''}{36[G_1''G_2'']^{3/2}}, \\
\kappa_{2,22}(\boldsymbol{\theta}) &= \frac{\frac{\partial^2}{\partial\theta_2^2}\rho^2}{G_2''\rho^2} - \frac{G_2'''^2}{3G_2''^3} + \frac{G_2^{\text{iv}}}{4G_2''^2}, \\
\tilde{\kappa}_{2,ij}^{\xi} &= \int_{\boldsymbol{\Theta}} \kappa_{2,ij}(\boldsymbol{\theta})\rho^2(\boldsymbol{\theta})\xi(\boldsymbol{\theta})d\boldsymbol{\theta}.
\end{aligned}$$

Corollary 3.3 If $p \geq 2$ and $\xi \in \Xi_2^{p+1}$, and for every compact $\boldsymbol{\Theta}_0 \subset \boldsymbol{\Theta}$,

$$\lim_{a\to\infty} \sqrt{a} \int_{\boldsymbol{\Theta}_0} \left| \mathbb{E}_{\boldsymbol{\theta}}(\sqrt{\frac{a}{t}}) - \rho(\boldsymbol{\theta}) \right| d\boldsymbol{\theta} = 0, \tag{41}$$

then

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}_p} a \left| \mathbb{E}^{\xi}[h(\boldsymbol{Z}_t)] - \Phi h - \frac{1}{\sqrt{a}}\Phi \mathrm{V}^h \begin{pmatrix} \tilde{\kappa}_{1,1}^{\xi} \\ \tilde{\kappa}_{1,2}^{\xi} \end{pmatrix} - \frac{1}{a} tr\left\{ \Phi(\mathrm{V}\circ\mathrm{V}^h) \begin{pmatrix} \tilde{\kappa}_{2,11}^{\xi} & \tilde{\kappa}_{2,12}^{\xi} \\ \tilde{\kappa}_{2,12}^{\xi} & \tilde{\kappa}_{2,22}^{\xi} \end{pmatrix} \right\} \right| = 0. \tag{42}$$

Proof. First as in the proof of Corollary 3.2, $\mathbb{E}_t^{\xi}\left[\rho^2(\boldsymbol{\theta})\boldsymbol{K}_2^{\xi}(\boldsymbol{\theta},\boldsymbol{\theta})\right] = \begin{pmatrix} \tilde{\kappa}_{2,11}^{\xi} & \tilde{\kappa}_{2,12}^{\xi} \\ \tilde{\kappa}_{2,12}^{\xi} & \tilde{\kappa}_{2,22}^{\xi} \end{pmatrix}$. By the definition of R_a, the term on left hand side of (42) is no more than the absolute value of the sum of $I_a = \mathbb{E}^{\xi}\{S_{2,a}(\xi,h)\}$ and $II_a = \sqrt{a}\left\{\mathbb{E}^{\xi}\sqrt{\frac{a}{t}}\mathbb{E}_t^{\xi}\left[\boldsymbol{K}_1^{\xi}(\boldsymbol{\theta},\boldsymbol{\theta})\right]^{\tau} - \Phi\mathrm{V}^h\binom{\tilde{\kappa}_{1,1}^{\xi}}{\tilde{\kappa}_{1,2}^{\xi}}\right\}$. From Theorem 3.3, $\lim_{a\to\infty}\sup_{h\in\mathcal{H}_p}|I_a| = 0$. However, $\lim_{a\to\infty}\sup_{h\in\mathcal{H}_p}|II_a| = 0$ by (41). □

3.4 Asymptotic Expansions by Rescaling.

For $\boldsymbol{z} \in \mathbb{R}^2, \boldsymbol{\theta} \in \boldsymbol{\Theta},\ t = t_a$, and $a \geq 1$, let

$$F_{a,\boldsymbol{\theta}}(\boldsymbol{z}) = \mathbb{P}_{\boldsymbol{\theta}}(\boldsymbol{Z}_t \leq \boldsymbol{z}),$$

and define the following two signed measures by

$$\begin{aligned}
\Phi_{a,\boldsymbol{\theta}}^{(1)}h &= \Phi h + \frac{1}{\sqrt{a}}(\kappa_{1,1}(\boldsymbol{\theta}), \kappa_{1,2}(\boldsymbol{\theta}))(\Phi\boldsymbol{V}^h)^{\tau}, \\
\Phi_{a,\boldsymbol{\theta}}^{(2)}h &= \Phi h + \frac{1}{\sqrt{a}}(\kappa_{1,1}(\boldsymbol{\theta}), \kappa_{1,2}(\boldsymbol{\theta}))(\Phi\boldsymbol{V}^h)^{\tau} + \frac{1}{a}tr\left\{ \begin{pmatrix} \kappa_{2,11}(\boldsymbol{\theta}) & \kappa_{2,12}(\boldsymbol{\theta}) \\ \kappa_{2,12}(\boldsymbol{\theta}) & \kappa_{2,22}(\boldsymbol{\theta}) \end{pmatrix} \Phi(\boldsymbol{V}\circ\boldsymbol{V}^h) \right\}.
\end{aligned}$$

Then $\Phi_{a,\boldsymbol{\theta}}^{(1)}h$ and $\Phi_{a,\boldsymbol{\theta}}^{(2)}h$ give a second order and a high order approximations to $F_{a,\boldsymbol{\theta}}$, respectively.

Corollary 3.4 Under the conditions of Theorem 3.2,

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}_1} \Big|\int_{\boldsymbol{\Theta}} \sqrt{a}[F_{a,\boldsymbol{\theta}}h - \Phi^{(1)}_{a,\boldsymbol{\theta}}h]\xi(\boldsymbol{\theta})d\boldsymbol{\theta}\Big| = 0.$$

Furthermore, under the conditions of Theorem 3.2 and Corollary 3.3,

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}_1} \Big|\int_{\boldsymbol{\Theta}} a[F_{a,\boldsymbol{\theta}}h - \Phi^{(2)}_{a,\boldsymbol{\theta}}h]\xi(\boldsymbol{\theta})d\boldsymbol{\theta}\Big| = 0.$$

Proof. The results here are restatements of Corollary 3.2 and 3.3. □

One of significant features of the transformation (10) is that the correction terms in the asymptotic expression may be described by rescaling. Define

$$\Phi^{(3)}_{a,\boldsymbol{\theta}}(\boldsymbol{z}) \quad = \quad \Phi\Big(\boldsymbol{z} - \frac{\rho(\boldsymbol{\theta})}{\sqrt{a}}[\kappa_{1,1}(\boldsymbol{\theta}), \kappa_{1,2}(\boldsymbol{\theta})]\Big),$$

for $\boldsymbol{z} \in \mathbb{R}^2, \boldsymbol{\theta} \in \boldsymbol{\Theta}$, and $a \geq 1$.

Theorem 3.4 Let $p \geq 1$ be an integer. For all $\xi \in \Xi_2^{p+1}$,

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}_p} \sqrt{a}\Big|\int_{\boldsymbol{\Theta}} \Big[F_{a,\boldsymbol{\theta}}h - \Phi^{(3)}_{a,\boldsymbol{\theta}}h\Big]\xi(\boldsymbol{\theta})d\boldsymbol{\theta}\Big| = 0.$$

Proof. The proof is similar and simpler than the proof of the next theorem. □

In repeated significance tests, the log-likelihood ratio test statistic for testing $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ is versus $\mathrm{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ is

$$\Lambda_n(\boldsymbol{\theta}_0) = L_n(\hat{\boldsymbol{\theta}}_n) - L_n(\boldsymbol{\theta}_0) = nI(\hat{\boldsymbol{\theta}}_n, \boldsymbol{\theta}_0),$$

by (9). Many important quantities are associated with $\Lambda_n(\boldsymbol{\theta})$, which equals to $\frac{1}{2}(Z_{n1}^2 + Z_{n2}^2)$ by (9) and (10). See Woodroofe (1982, Chapter 8). Then any function of $\Lambda_n(\boldsymbol{\theta})$ is symmetric about both Z_{n1} and Z_{n2}. Generally, define a subset of symmetric functions in $\mathcal{H}_p$ by

$$\mathcal{H}_p^* = \{h \in \mathcal{H}_p : \ h(z_1, z_2) = h(-z_1, z_2) = h(z_1, -z_2) = h(-z_1, -z_2), \ \forall\ (z_1, z_2) \in \mathbb{R}^2\}.$$

Note that for any $h \in \mathcal{H}_p^*, \Phi \boldsymbol{V}^h = 0$, Φh is the approximation for both $\mathbb{E}_t^\xi[h(\boldsymbol{Z}_t)]$ and $\mathbb{E}^\xi[h(\boldsymbol{Z}_t)]$ of the second order accuracy for any $\xi \in \Xi_1^{p+1}$, $h \in \mathcal{H}_p^*$, and any stopping times satisfying (34) and (37). Furthermore, it is also possible to have a high order rescaled expansion. Assume that $\kappa_{2,11}(\boldsymbol{\theta}) \geq 0, j = 1, 2$. Define

$$\Phi^{(4)}_{a,\boldsymbol{\theta}}(\boldsymbol{z}) \quad = \quad \Phi\Big(\boldsymbol{z} - \frac{\rho(\boldsymbol{\theta})}{\sqrt{a}}\Big(\sqrt{\kappa_{2,11}(\boldsymbol{\theta})}, \sqrt{\kappa_{2,22}(\boldsymbol{\theta})}\Big)\Big),$$

for $\boldsymbol{z} \in \mathbb{R}^2, \boldsymbol{\theta} \in \boldsymbol{\Theta}$, and $a \geq 1$.

Theorem 3.5 For all $\xi \in \Xi_2^{p+1}$,

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}_p^*} a\Big|\int_{\boldsymbol{\Theta}} \Big[F_{a,\boldsymbol{\theta}}h - \Phi_{a,\boldsymbol{\theta}}^{(4)}h\Big]\xi(\boldsymbol{\theta})d\boldsymbol{\theta}\Big| = 0.$$

Proof. First, we can write

$$a\int_{\boldsymbol{\Theta}} \Big[F_{a,\boldsymbol{\theta}}h - \Phi_{a,\boldsymbol{\theta}}^{(4)}h\Big]\xi(\boldsymbol{\theta})d\boldsymbol{\theta} = S_1 + S_2,$$

where

$$S_1 = a\{\mathbb{E}^{\xi}[h(\boldsymbol{Z}_t)] - \Phi h - \frac{1}{2a}\sum_{j=1}^{2}\tilde{\kappa}_{2,jj}^{\xi}\int_{\mathbb{R}^2}(z_j^2-1)h(\boldsymbol{z})\Phi(d\boldsymbol{z})\},$$

$$S_2 = -\int_{\boldsymbol{\Theta}} a\Big\{\Phi_{a,\boldsymbol{\theta}}^{(4)}h - \Phi h - \frac{1}{2a}\sum_{j=1}^{2}\rho^2(\boldsymbol{\theta})\kappa_{2,jj}(\boldsymbol{\theta})\int_{\mathbb{R}^2}(z_j^2-1)h(\boldsymbol{z})\Phi(d\boldsymbol{z})\Big\}\xi(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

For any $h \in \mathcal{H}_p^*$, $\int_{\mathbb{R}^2}\boldsymbol{z}h(\boldsymbol{z})\Phi(d\boldsymbol{z}) = (0,0)$ and $\int_{\mathbb{R}^2}z_1z_2h(\boldsymbol{z})\Phi(d\boldsymbol{z}) = 0$. It follows from (20) and (21) that

$$\Phi\boldsymbol{V}^h = (0,0) \quad\text{and}\quad \Phi(\boldsymbol{V}\circ\boldsymbol{V}^h) = \frac{1}{2}\int_{\mathbb{R}^2}\begin{pmatrix} z_1^2-1 & 0\\ 0 & z_2^2-1\end{pmatrix}h(\boldsymbol{z})\Phi(d\boldsymbol{z}),\quad h\in\mathcal{H}_p^*.$$

Note that for any $h \in \mathcal{H}_p^*$, the term II_a in the proof of Corollary 3.3 is 0. By the same argument of Corollary 3.3 that $\lim_{a\to\infty}\sup_{h\in\mathcal{H}_p^*}|S_1| = 0$. By Taylor's expansions, the integrand in S_2 is

$$\begin{aligned}
&a\Big|\int_{R^2}\Big\{\phi\Big(\boldsymbol{z} - \frac{\rho(\boldsymbol{\theta})}{\sqrt{a}}[\sqrt{\kappa_{2,11}(\boldsymbol{\theta})},\sqrt{\kappa_{2,22}(\boldsymbol{\theta})}]\Big) - \phi(\boldsymbol{z})\Big\}h(\boldsymbol{z})d\boldsymbol{z}\\
&\qquad\qquad - \frac{\rho^2(\boldsymbol{\theta})}{2a}\sum_{j=1}^{2}\kappa_{2,jj}(\boldsymbol{\theta})\int_{\mathbb{R}^2}(z_j^2-1)h(\boldsymbol{z})\phi(\boldsymbol{z})d\boldsymbol{z}\Big|\\
&\le \frac{1}{\sqrt{a}}\int_{\mathbb{R}^2}\sum_{k=0}^{3}\binom{3}{k}\rho^3(\boldsymbol{\theta})\Big[\kappa_{2,11}(\boldsymbol{\theta})\Big]^{\frac{k}{2}}\Big[\kappa_{2,22}(\boldsymbol{\theta})\Big]^{\frac{3-k}{2}}\Big|\frac{\partial^3}{\partial z_1^k\partial z_2^{3-k}}\phi(\boldsymbol{z}^*)\Big|(1+|z_1|^p+|z_2|^p)d\boldsymbol{z}, \quad (43)
\end{aligned}$$

where $\|\boldsymbol{z}^* - \boldsymbol{z}\| \le \frac{\rho(\boldsymbol{\theta})}{\sqrt{a}}\sqrt{\kappa_{2,11}(\boldsymbol{\theta}) + \kappa_{2,22}(\boldsymbol{\theta})}$. The right hand side of (43) is independent of $h \in \mathcal{H}_p^*$, approaches zero as $a \to \infty$, for each $\boldsymbol{\theta}$, and is bounded by a constant multiple of $1 + Q(\boldsymbol{\theta})$. Here $Q(\theta)$ depends only on $1 + \|\nabla\rho(\boldsymbol{\theta})\| + \|\nabla^2\rho(\boldsymbol{\theta})\|$, which is bounded on the compact support of ξ. So $R_2 \to 0$ uniformly in $h \in \mathcal{H}_p^*$, by the dominated convergence theorem. □

4 An Example

One application of Theorem 3.5 is to approximate the sampling distribution of the log-likelihood ratio test statistics mentioned earlier. Consider the problem, in which $X_1, X_2, \cdots$ are i.i.d.

$N(\mu, \sigma^2)$, where both $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$ are unknown, and

$$t = t_a = \min(b_2 a,\ \inf\Big\{n \geq b_1 a : \sum_{i=1}^{n} X_i^2 - n - n\log(\hat{\sigma}_n^2)\ > 2a\Big\}),$$

where $0 < b_1 < b_2 < \infty$ are two prespecified numbers, $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X}_n)^2$, and $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then

$$2\Lambda_t = \|\boldsymbol{Z}_t\|^2 = \frac{1}{\sigma^2}\sum_{i=1}^{t}(X_i - \mu)^2 + t\log(\sigma^2) - t - t\log(\hat{\sigma}_t^2),$$

on $\sum_{j=1}^{t}(X_j - \overline{X}_t)^2 > 0$. From Table 1, $\boldsymbol{\theta} = (-\frac{1}{2\sigma^2}, \mu)$. Theorem 8.3 of Woodroofe (1982) implies that

$$\frac{a}{t_a} \to \begin{cases} b_2, & \text{if } \rho^2(\boldsymbol{\theta}) < 1/b_2, \\ \rho^2(\boldsymbol{\theta}), & \text{if } 1/b_2 < \rho^2(\boldsymbol{\theta}) < 1/b_1, \\ b_1, & \text{if } \rho^2(\boldsymbol{\theta}) > 1/b_1, \end{cases}$$

in $\mathbb{P}_{\boldsymbol{\theta}}$-Probability, as $a \to \infty$, where

$$\begin{aligned} \rho^2(\boldsymbol{\theta}) = I(\boldsymbol{\theta}, (-0.5, 0)) &= G_1(\theta_1) - G_1(-0.5) - G_1'(-0.5)(\theta_1 + 0.5) - \theta_1\theta_2^2 \\ &= \frac{1}{2}\Big\{\frac{\mu^2+1}{\sigma^2} + \log(\sigma^2) - 1\Big\}. \end{aligned}$$

Let $h_u(\boldsymbol{z}) = I_{(\|\boldsymbol{z}\|^2 \leq 2u)}$, for $\boldsymbol{z} \in \mathbb{R}^2$. Then Theorem 3.5 suggested the approximation

$$\mathbb{P}_{\boldsymbol{\theta}}(\Lambda_t \leq u) \approx \Phi_{a,\boldsymbol{\theta}}^{(4)} h_u \approx \delta(u; a, \boldsymbol{\theta}),$$

where

$$\delta(u; a, \boldsymbol{\theta}) = \Phi h_u + \frac{1}{2a}\sum_{j=1}^{2}\rho^2(\boldsymbol{\theta})\kappa_{2,jj}(\boldsymbol{\theta})\int_{\mathbb{R}^2}(z_j^2 - 1)h(\boldsymbol{z})\Phi(d\boldsymbol{z}).$$

It is easy to verify that $\rho^2(\boldsymbol{\theta})\kappa_{2,11}(\boldsymbol{\theta}) = \frac{\mu^2+1}{\sigma^2} + \frac{11}{6}\rho^2(\boldsymbol{\theta})$, $\rho^2(\boldsymbol{\theta})\kappa_{2,22}(\boldsymbol{\theta}) = -\theta_1 = \frac{1}{2\sigma^2}$. Also, $\Phi h_u = \mathbb{P}(\chi_2^2 \leq 2u) = 1 - e^{-u}$ and $\int_{\mathbb{R}^2}(z_j^2 - 1)h(\boldsymbol{z})\Phi(d\boldsymbol{z}) = -u\ e^{-u}$, for $j = 1, 2$ and $u \geq 0$, where χ_2^2 is a chi-square random variable with 2 degrees of freedom. Therefore,

$$\delta(u; a, \boldsymbol{\theta}) = 1 - e^{-u} - \frac{ue^{-u}}{2a}\Big\{\frac{\mu^2 + 1.5}{\sigma^2} + \frac{11}{6}\rho^2(\boldsymbol{\theta})\Big\}, \quad u \geq 0. \tag{44}$$

Figure 1 shows Monte Carlo estimates of $\mathbb{P}_{\boldsymbol{\theta}}(\Lambda_t \leq u)$ for $a = 8$, $b_1 = 0.5$, $b_2 = 50$, against u in [0,10] for various combinations $(\mu, \sigma^2) = (0.5, 1.25), (0.5, 0.8), (1, 1.25)$ and $(1, 0.8)$, together with directed χ_2^2 approximation $\Phi h_u = 1 - e^{-u}, (u \geq 0)$, and the corrected approximation $\delta(u; a, \theta)$ given by (44). The χ_2^2-approximation seriously overestimates the probabilities. The corrected term in (44) is always negative and the corrected approximations are closer to Monte Carlo estimates in all cases as showed in Figure 1.

4.1 A Remark

The results of this paper can be generalized to some multivariate density functions. For simplicity, we discuss the bivariate case only.

Case 1. Assume that the two components of a bivariate random vector are independent, and each of them follows an one-parameter exponential family.

Case 2. Assume that (X, Y) is a bivariate random vector, whose density function with respect to Lebesgue measure on $\mathbb{R}^2$ has the form

$$p_{(\beta_1,\beta_2)}(x,y) = \exp\{U_1(x,y)\beta_1 + U_1(x,y)\beta_2 - \psi(\beta_1,\beta_2)\}.$$

Note that the Bi-variate Gamma distribution given in Mihram and Hultquist (1967) is a special case in Case 2.

Under analogous assumptions for the bivariate case, the main results developed in this paper would be valid.

Appendix

Proof of Lemma 2.1. If A_n occurs and $\hat{\theta}_{n1} \geq \omega_1 > \theta_1$ for some $n \geq 2$ then $\overline{Y}_n \geq G_1'(\omega_1)$ for the same n. Expanding $G_2(\overline{T}_{n2})$ about μ_2, we get

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} V_i - \xi_n \equiv \overline{V}_n - \xi_n, \tag{45}$$

where

$$V_i = U_1(X_i) - G_2'(\mu_2)U_2(X_i) + [G_2'(\mu_2)\mu_2 - G_2(\mu_2)], \tag{46}$$

$$\xi_n = (\overline{T}_{n2} - \mu_2)^2 G_2''(\tau_n)/2, \tag{47}$$

and τ_n is some intermediate point satisfying $|\tau_n - \mu_2| \leq |\overline{T}_{n2} - \mu_2|$. Clearly, $\xi_n \geq 0$ so that

$$\begin{aligned} \mathbb{P}_{\boldsymbol{\theta}}(A_n \text{ and } \hat{\theta}_{n1} \geq \omega_1, \exists\, n \geq m) &= \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n\geq m} \overline{Y}_n \geq G_1'(\omega_1)) \\ &= \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n\geq m} (\overline{V}_n - \xi_n) \geq G_1'(\omega_1)) \\ &\leq \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n\geq m} \overline{V}_n \geq G_1'(\omega_1)). \end{aligned}$$

Let $b = m(\omega_1 - \theta_1)$. By Assumption A, the moment generating function of V_1 is given by

$$M_{V_1}(s) = \exp\{G_1(s+\theta_1) - G_1(\theta_1)\}, \quad s + \theta_1 \in \Theta_1. \tag{48}$$

Then by the submartingale inequality (to the reverse submartingale $\exp(b\overline{V}_n), n \geq m$),

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\theta}}(\sup_{n\geq m} \overline{V}_n \geq G_1'(\omega_1)) &\leq \exp\{-bG_1'(\omega_1)\}\mathbb{E}_{\boldsymbol{\theta}}\{\exp(b\overline{V}_m)\} \\ &= \exp\Big\{m[G_1(\theta_1 + \frac{b}{m}) - G_1(\theta_1)] - bG_1'(\omega_1)\Big\} \\ &= \exp\{-mI_1(\omega_1, \theta_1)\},\end{aligned}$$

which implies the first assertion of the Lemma. For the second, note that $b = m(\omega_1 - \theta_1) < 0$ and

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\theta}}(A_n \text{ and } \hat{\theta}_{n1} \leq \omega_1, \exists\, n \geq m) &= \mathbb{P}_{\boldsymbol{\theta}}\Big[\inf_{n\geq m} \overline{Y}_n \leq G_1'(\omega_1)\Big] \\ &\leq \mathbb{P}_{\boldsymbol{\theta}}\Big\{\sup_{n\geq m} \exp(b\overline{Y}_n) \geq \exp[b\, G_1'(\omega_1)]\Big\}. \qquad (49)\end{aligned}$$

It follows from Fact 4 that the moment generating function of Y_n is given by

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}}\{\exp(sY_n)\} &= \exp\{H_n(s + \theta_1) - H_n(\theta_1)\} \\ &= \exp\{n[G_1(s + \theta_1) - G_1(\theta_1)] - [G_1(n(s + \theta_1)) - G_1(n\theta_1)]\}, \qquad (50)\end{aligned}$$

for $s + \theta_1 \in \Theta_1$. Since $G_2''(\cdot)$ is positive, $G_2(\cdot)$ is convex and $\{\overline{Y}_n\}_{n\geq m}$ is a reverse supermartingale. Therefore $\{\exp(b\overline{Y}_n)\}_{n\geq m}$ is a reverse submartingale. It follows from the reverse submartingale inequality to the sequence $\{\exp(b\,\overline{Y}_n)\}_{n\geq m}$ that

$$\begin{aligned}&\mathbb{P}_{\boldsymbol{\theta}}\{\sup_{n\geq m} \exp(t\,\overline{Y}_n) \geq \exp[b\, G_1'(\omega_1)]\} \\ \leq\;& \exp\{-bG_1'(\omega_1)\}\mathbb{E}_{\boldsymbol{\theta}}\{\exp(b\overline{Y}_m)\} \\ =\;& \exp\Big\{m[G_1(\theta_1 + \frac{b}{m}) - G_1(\theta_1)] - [G_1(m(\frac{b}{m} + \theta_1)) - G_1(m\theta_1)] - bG_1'(\omega_1)\Big\} \\ =\;& \exp\{-mI_1(\omega_1, \theta_1) - [G_1(m\omega_1) - G_1(m\theta_1)]\}.\end{aligned}$$

Now the second assertion is established. □

Proof of Lemma 2.2. Note that the moment generating function of $U_2(X_1)$ is

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}}\{\exp[sU_2(X_1)]\} &= \exp\{\psi(\theta_1, \tilde{\theta}_2) - \psi(\theta_1, \theta_2)\} \\ &= \exp\{\theta_1[G_2(\tilde{\theta}_2) - G_2(\theta_2) + \theta_2 G_2'(\theta_2) - \tilde{\theta}_2 G_2'(\tilde{\theta}_2)]\}, \qquad (51)\end{aligned}$$

where $\tilde{\theta}_2$ is the unique solution of $G_2'(\tilde{\theta}_2) = G_2'(\theta_2) - s/\theta_1$. For $\omega_2 > \theta_2$, let $b = -m\theta_1[G_2'(\omega_2) - G_2'(\theta_2)]$. Then $b > 0$ from the monotonicity of $G_2'(\cdot)$. It follows from the submartingale inequality (to the reverse submartingale $\{\exp(b\overline{T}_{n2})\}_{n\geq m}$) and (51) that

$$\mathbb{P}_{\boldsymbol{\theta}}(B_n \text{ and } \hat{\theta}_{n2} \geq \omega_2, \exists\, n \geq m) = \mathbb{P}_{\boldsymbol{\theta}}(\sup_{n\geq m} \overline{T}_{n2} \geq \omega_2)$$

$$
\begin{aligned}
&\le \exp\{-b\,\omega_2\}\mathbb{E}_{\boldsymbol{\theta}}\{\exp(\frac{b}{m}T_{2m})\} \\
&= \exp\Big\{\theta_1[G_2(\tilde\theta_2) - G_2(\theta_2) + \theta_2 G_2'(\theta_2) - \tilde\theta_2 G_2'(\tilde\theta_2)] - b\,\omega_2\Big\},
\end{aligned}
$$

where $\tilde\theta_2$ is the unique solution of $G_2'(\tilde\theta_2) = G_2'(\theta_2) - b/m\theta_1$. From the definition of t and the monotonicity of $G_2'(\cdot)$, $\tilde\theta_2 = \omega_2$. The first assertion then is established by simple algebra. The second assertion can be established similarly. □

Proof of Stein's Identity. For any $h \in \mathcal{H}_p$, Lemma 2.6 implies that $V_j^h \in \mathcal{H}_p$, and $(|z_1|^p + |z_2|^p)[|f_{10}(\boldsymbol{z})| + |f_{10}(\boldsymbol{z})|]$ is integrable with respect to Φ from (22). Thus

$$
\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty \Big\{\int_{z_1}^\infty [\Phi_1^h(y_1) - \Phi h] e^{-\frac{1}{2}y_1^2} dy_1\Big\} f_{10}(\boldsymbol{z}) \frac{1}{2\pi} e^{-\frac{1}{2}z_2^2} dz_1 dz_2 \\
= &\int_0^\infty \int_{-\infty}^\infty \int_0^{z_1} f_{10}(\boldsymbol{z}) dz_1 [\Phi_1^h(y_1) - \Phi h] \phi(y_1, z_2) dy_1 dz_2 \\
= &\int_0^\infty \int_{-\infty}^\infty \Big[f(z_1, z_2) - f(0, z_2)\Big]\Big[\Phi_1^h(z_1) - \Phi h\Big] \phi(\boldsymbol{z}) dz_1 dz_2;
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^0 \int_{-\infty}^\infty \Big\{\int_{z_1}^\infty [\Phi_1^h(y_1) - \Phi h] e^{-\frac{1}{2}y_1^2} dy_1\Big\} f_{10}(\boldsymbol{z}) \frac{1}{2\pi} e^{-\frac{1}{2}z_2^2} dz_1 dz_2 \\
= &-\int_{-\infty}^0 \int_{-\infty}^\infty \Big\{\int_{-\infty}^{z_1} [\Phi_1^h(y_1) - \Phi h] e^{-\frac{1}{2}y_1^2} dy_1\Big\} f_{10}(\boldsymbol{z}) \frac{1}{2\pi} e^{-\frac{1}{2}z_2^2} dz_1 dz_2 \\
= &\int_{-\infty}^0 \int_{-\infty}^\infty [f(z_1, z_2) - f(0, z_2)][\Phi_1^h(z_1) - \Phi h] \phi(\boldsymbol{z}) dz_1 dz_2;
\end{aligned}
$$

We then have

$$
\int_{\mathbb{R}^2} V_1^h(\boldsymbol{z}) f_{10}(\boldsymbol{z}) \phi(\boldsymbol{z}) d\boldsymbol{z} = \int_{\mathbb{R}^2} \Big[\Phi_1^h(z_1) - \Phi^h\Big] f(\boldsymbol{z}) \phi_2(\boldsymbol{z}) d\boldsymbol{z}. \tag{52}
$$

Similarly, we get

$$
\int_0^\infty \int_{-\infty}^\infty V_2^h(\boldsymbol{z}) f_{01}(\boldsymbol{z}) \phi(\boldsymbol{z}) dz_1 dz_2 = \int_0^\infty \int_{-\infty}^\infty [f(\boldsymbol{z}) - f(z_1, 0)][h(\boldsymbol{z}) - \Phi_1^h(z_2)] \phi(\boldsymbol{z}) dz_1 dz_2,
$$

and

$$
\int_{-\infty}^0 \int_{-\infty}^\infty V_2^h(\boldsymbol{z}) f_{01}(\boldsymbol{z}) \phi(\boldsymbol{z}) dz_1 dz_2 = \int_{-\infty}^0 \int_{-\infty}^\infty [f(\boldsymbol{z}) - f(z_1, 0)][h(\boldsymbol{z}) - \Phi_1^h(z_1)] \phi(\boldsymbol{z}) dz_1 dz_2,
$$

which implies that

$$
\int_{\mathbb{R}^2} V_2^h(\boldsymbol{z}) f_{01}(\boldsymbol{z}) \phi(\boldsymbol{z}) d\boldsymbol{z} = \int_{\mathbb{R}^2} [h(\boldsymbol{z}) - \Phi_1^h(z_1)] f(\boldsymbol{z}) \phi(\boldsymbol{z}) d\boldsymbol{z}. \tag{53}
$$

The desired result follows from (52) and (53). □

Proof of Lemma 3.2 We will denote $C_1, C_2, \cdots$ positive constants. If $\xi \in \Xi_1^\alpha$, let $\mathbf{\Theta}_\xi$ denote the compact support of ξ, and let $\mathbf{\Theta}_0$ denote another compact set for which $\mathbf{\Theta}_\xi \subset \mathbf{\Theta}_0^o \subset \mathbf{\Theta}_0 \subset \mathbf{\Theta}$, where $\mathbf{\Theta}_0^o$ is the interior of $\mathbf{\Theta}_0$. We will treat J_2 and $J_{2,01}$ as functions of $(\boldsymbol{\omega}, \boldsymbol{\theta})$ which are independent of (ω_1, θ_1). Since G_2 is strictly convex and twice differentiable on $\boldsymbol{\theta}_2$,

$$J_2 \le C_1 \quad \text{and} \quad J_{2,01}2 \le C_1, \quad \text{on } \mathbf{\Theta}_0 \times \mathbf{\Theta}_\xi;$$
$$\frac{1}{\sqrt{I_2}} \le C_2 \quad \text{and} \quad \frac{1}{\sqrt{I_{2,01}}} \le C_2, \quad \text{on } (\mathbf{\Theta} - \mathbf{\Theta}_0) \times \mathbf{\Theta}_\xi.$$

Thus, on $(\mathbf{\Theta} - \mathbf{\Theta}_0) \times \mathbf{\Theta}_\xi$, $J_2 \le \frac{\sqrt{2}I_2}{|I_{2,01}|\sqrt{I_2}} \le 2C_2^2 I_2$ and $|I_{2,02}|J_2^2 = \frac{2|G_2(\theta_2)(\theta_2-\omega_2)+G_2''(\theta_2)|I_2}{[G_2''(\theta_2)(\theta_2-\omega_2)]^2} \le C_3 I_2$, which imply that

$$|J_{2,01}| \le \frac{1}{\sqrt{I_2}} + \frac{|I_{2,02}|J_2^2}{\sqrt{I_2}} \le C_2 + C_2 C_3 I_2, \quad \text{on } (\mathbf{\Theta} - \mathbf{\Theta}_0) \times \mathbf{\Theta}_\xi,$$

Since $\mathbf{\Theta}_\xi$ is bounded,

$$J_2 \le C_4(1 + I_2) \le C_5(1 + I_1 - \theta_1 I_2) \le C_5(1 + I), \text{ and } J_{2,01} \le C_5(1 + I), \text{ on } \mathbf{\Theta} \times \mathbf{\Theta}_\xi,$$

which implies that on $\mathbf{\Theta} \times \mathbf{\Theta}_\xi$,

$$\left|\frac{\xi_{01}}{\xi} J_2\right|^\alpha \le C_6 \left|\frac{\xi_{01}}{\xi} J_2\right|^\alpha \exp\{\alpha I\} \quad \text{and} \quad \left|J_{2,01}\right|^\alpha \le C_6 \exp\{\alpha I\}.$$

It follows from Lemma 2.5 that $|\xi_{01}/\xi J_2|^\alpha \in \mathcal{W}^\xi$ and $|J_{2,01}|^\alpha \in \mathcal{W}^\xi$. Therefore $|K_{1,2}^\xi|^\alpha \in \mathcal{W}^\xi$. The proof of $|K_{1,1}^\xi|^\alpha \in \mathcal{W}^\xi$ is similar. This assure the first assertion. The second assertion can be proved similarly. □

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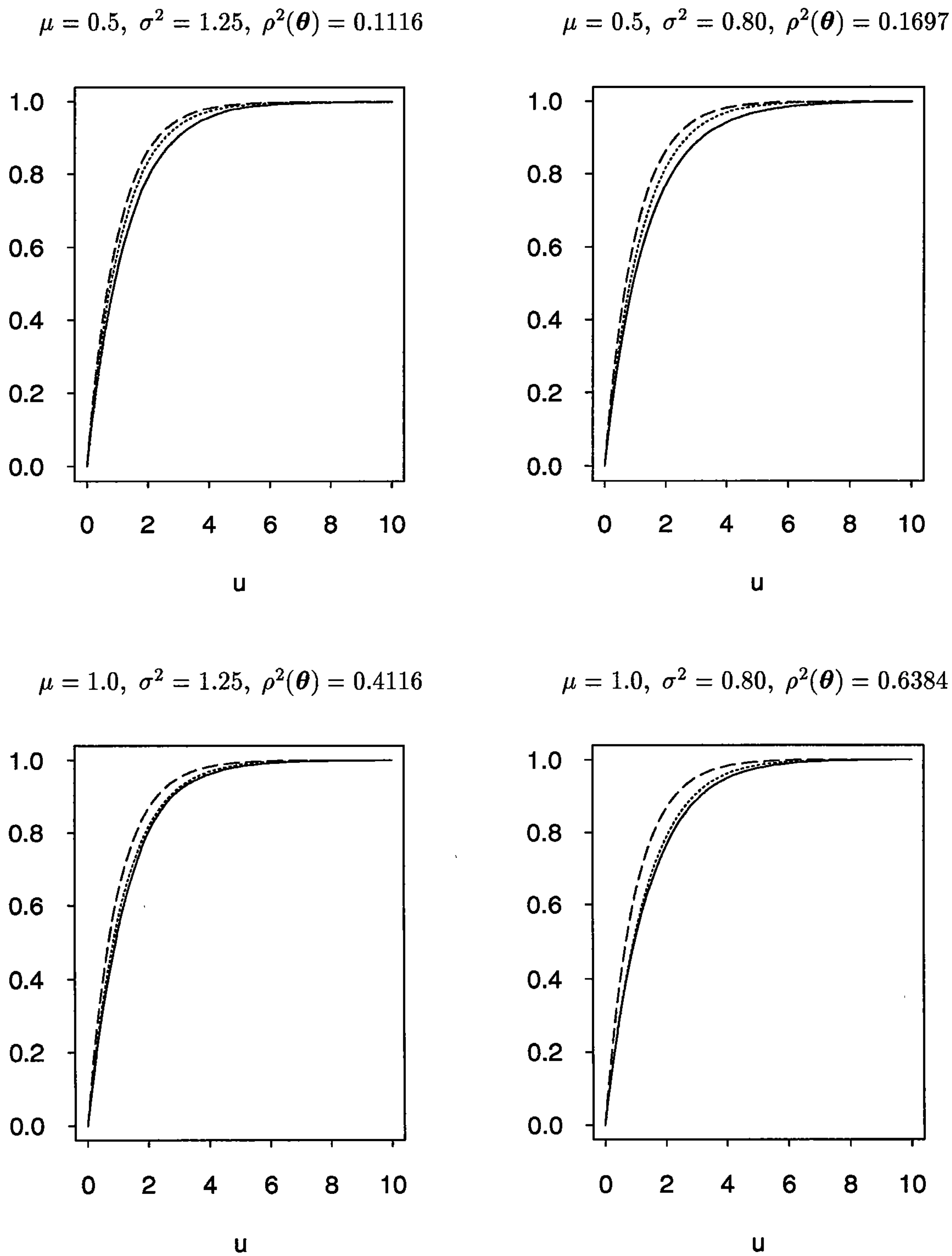
Table 1: Examples of a Certain Two Parameter Exponential Family of Distributions

Name	Normal	Inverse Gaussian	Gamma	Inverse Gamma
density	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\sqrt{\frac{\alpha}{2\pi x^3}}e^{-\frac{\alpha}{2x}-\frac{\mu x}{2}+\sqrt{\alpha\mu}}$	$\frac{\alpha^\alpha x^{\alpha-1}}{\Gamma(\alpha)\mu^\alpha}e^{-\frac{\alpha x}{\mu}}$	$\frac{\alpha^\alpha}{\Gamma(\alpha)\mu^\alpha x^{\alpha+1}}e^{-\frac{\alpha}{\mu x}}$
$U_1(x)$	x^2	$\frac{1}{x}$	$-\log(x)$	$-\log(x)$
$U_2(x)$	x	x	x	$\frac{1}{x}$
$\beta_1 = \theta_1$	$-\frac{1}{2\sigma^2}$	$-\frac{\alpha}{2}$	$-\alpha$	$-\alpha$
β_2	$\frac{\mu}{\sigma^2}$	$-\frac{\mu}{2}$	$-\frac{\alpha}{\mu}$	$-\frac{\alpha}{\mu}$
$\theta_2 = \mu_2$	μ	$\sqrt{\frac{\alpha}{\mu}}$	μ	μ
$\mathcal{N}$	$\mathbb{R}^- \times \mathbb{R}$	$\mathbb{R}^- \times (\mathbb{R}^- \cup \{0\})$	$\mathbb{R}^- \times \mathbb{R}^-$	$\mathbb{R}^- \times \mathbb{R}^-$
Θ	$\mathbb{R}^- \times \mathbb{R}$	$\mathbb{R}^- \times \mathbb{R}^+$	$\mathbb{R}^- \times \mathbb{R}^+$	$\mathbb{R}^- \times \mathbb{R}^+$
$G_1(\theta_1)$	$-\frac{1}{2}\log(-2\theta_1)$	$-\frac{1}{2}\log(-2\theta_1)$	$\theta_1 + \theta_1\log(-\theta_1) + \log[\Gamma(-\theta_1)]$	
$G_2(\theta_2)$	θ_2^2	$\frac{1}{\theta_2}$	$\log(\theta_2)$	$\log(\theta_2)$
Y_n	$\sum_{i=1}^n (X_i - \overline{X}_n)^2$	$\sum_{i=1}^n \frac{1}{X_i} - \frac{n}{\overline{X}_n}$	$\sum_{i=1}^n \log\left(\frac{\overline{X}_n}{X_i}\right)$	$\sum_{i=1}^n \log\left(\frac{1}{X_i}\right) - \log\left(\frac{1}{n}\sum_{j=1}^n \frac{1}{X_j}\right)$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.



——— Simulations of $\mathbb{P}_{\boldsymbol{\theta}}(\Lambda_t \leq u)$ based on 40,000 replications

– – – – $\Phi h_u = 1 - e^{-u}$

········ $\delta(u, a, \boldsymbol{\theta}) = 1 - e^{-u} - \frac{ue^{-u}}{2a}\left\{\frac{\mu^2+1.5}{\sigma^2} + \frac{11}{6}\rho^2(\boldsymbol{\theta})\right\}$

Figure 1: Simulations of $\mathbb{P}_{\boldsymbol{\theta}}(\Lambda_t \leq u)$ and approximations for $a = 8$