A REMARK ON STOCHASTIC INTEGRATION

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Abstract

We give an example of an adapted, càdlàg process H and a martingale M such that a "stochastic integral" process $\int_0^t H_s dM_s$ makes sense but is not a semimartingale. This answers a question of Ruth Williams.

In constructing an elementary theory of stochastic integration for semimartingales, one approach is to begin with simple predictable processes and to define the integral by the obvious formula. One then easily extends the class of integrands to processes in $\mathbb{L}$ (adapted processes which are left continuous with right limits *a.s.* or "càglàd"). See for example [1], [2], [4], or [5]. A natural question is: why can one not use $\mathbb{D}$ instead of $\mathbb{L}$? ($\mathbb{D}$ = adapted processes which are right continuous with left limits *a.s.*, or "càdlàg".). A standard answer is that one cannot use $\mathbb{D}$ if one wants the stochastic integral with respect to a local martingale to be again a local martingale, and a simple example is to take N a Poisson process of parameter $\lambda = 1$, $X_t = N_t - t$, and $C_t = 1_{\{t<T\}}$ where T is the first jump time of N. Then C is in $\mathbb{D}$ and $\int_0^t C_s dX_s = -(t \wedge T)$, which is always decreasing and thus cannot be a local martingale. Note however that it is a semimartingale.

Ruth Williams [6, p.178] has posed the following question: why can one not use $\mathbb{D}$ instead of $\mathbb{L}$ for the semimartingale integral? In other words, is there a *semimartingale* justification for using predictable processes, rather than just a martingale justification as described in the previous paragraph. Maurizio Pratelli [3] has given an elegant partial answer to this question by showing that one can have a theory of stochastic integration for optional integrands with the usual dominated convergence theorem holding if and only if the semimartingale integrators satisfy $\sum_{0<s\le t} |\Delta X_s| < \infty$ a.s., each $t > 0$. (See [5] for all undefined terms and notation.)

In this note we address a different but related question: can one find a semimartingale M and a process in H in $\mathbb{D}$ such that $\int_0^t H_s dM_s$ makes sense as a stochastic process, but is not a semimartingale? If H is simple enough, the definition of $\int_0^t H_s dM_s$ should be obvious, and thus we will see that one in fact leaves the space of semimartingales quite

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readily even if one integrates processes in $\mathbb{D}$ (and not the more general optional processes). We construct such a process where M is a martingale, $H \in \mathbb{D}$, and $\int_0^t H_s dM_s$ is not a semimartingale.

Let M be a semimartingale and let $H \in \mathbb{D}$. If we could construct a coherent theory of stochastic integration with $H \in \mathbb{D}$, we would want the jump at time t of $\int H_s dM_s$ to be equal to $H_t \Delta M_t$. Therefore we would have the relation

$$(*) \qquad \int_0^t H_s dM_s = \int_0^t H_{s-} dM_s + \sum_{0<s\leq t} \Delta H_s \Delta M_s.$$

If M has no continuous martingale part, we can write this relationship as

$$\int_0^t H_s dM_s = \int_0^t H_{s-} dM_s + [H, M]_t.$$

We will construct a martingale M and an (adapted) process $H \in \mathbb{D}$ such that (*) holds, but such that the process $\int_0^t H_s dM_s$ is not a semimartingale. To construct M, let N^i be an *i.i.d.* sequence of Poisson processes with arrival rate $\lambda = 1$, so that $N_t^i - t$ is a martingale for each i. Let

$$M_t^n = \sum_{i=}^{n} \frac{1}{(i)^{3/4}} (N_t^i - t)$$
$$M_t = \sum_{i=1}^{\infty} \frac{1}{(i)^{3/4}} (N_t^i - t)$$

Note that the series converges in L^2 and that M is an L^2 martingale (M is also a Lévy process).

We now construct $H \in \mathbb{D}$, which is more complicated. Let α_n be an increasing sequence of integers, increasing at the rate $n^{5/4}$. Define increasing stopping times $(T_j) j \geq 1$ by

$$T_1 = inf \ \{t > 0 : \Delta M_t^1 > 0\}$$
$$T_j = inf \ \{t > T_{j-1} : \Delta M_t^{\alpha_j} > 0\}$$

Note that $T_j - T_{j-1}$ is exponential of parameter $\frac{1}{\alpha_j}$, and also $T_{j+1} - T_j$ is independent of $T_j - T_{j-1}$. Set $T = \lim_{j\to\infty} T_j$, and we thus have that $P(T < \infty) = 1$.

Define

$$H_t^n = \sum_{k=1}^{n} (2k-1)^{-1/16} \ 1_{[T_{2k-1}, T_{2k})}(t).$$

then H^n is in $\mathbb{D}$, and H^n converges in ucp (uniform convergence on compacts in probability) to

$$H_t = \sum_{k=1}^{\infty} (2k-1)^{-1/16} \ 1_{[T_{2k-1}, T_{2k})}(t).$$

Thus H is also in $\mathbb{D}$. Next observe that

$$\sum_{s\le T} \Delta H^n_s \Delta M_s = \sum_{k=1}^{2n} X_k,$$

where

$$X_{2k-1} = (2k-1)^{-1/16}\ \Delta M_{T_{2k-1}},$$
$$X_{2k} = -(2k-1)^{-1/16}\ \Delta M_{T_{2k}}.$$

PROPOSITION 1

$A^n_t = \sum_{s\le t} \Delta H^n_s \Delta M_s$ *converges in ucp*

Proof

Note that $H_{t\wedge T_{2n}} = H^n_{t\wedge T_{2n}}$, each n, which implies

$$\sum_{s\le T_{2n}} \Delta H_s \Delta M_s = \sum_{s\le T_{2n}} \Delta H^n_s \Delta M_s, \text{ each } n.$$

Then it suffices to show that

$$\sum_{s\le T} \Delta H^n_s \Delta M_s = \sum_{k=1}^{2n} X_k$$

converges. Since ΔM_{T_n} is uniform on $\{1, 2^{-3/4}, \ldots, \alpha_n^{-3/4}\}$, and $\alpha_n \sim n^{5/4}$, we have

$$E\{\Delta M_{T_n}\} \sim n^{-5/4} \int_1^{n^{5/4}} x^{-3/4} dx = 4(n^{-15/16} - n^{-5/4})$$

and

$$E\{(\Delta M_{T_n})^2\} \sim n^{-5/4} \int_1^{n^{5/4}} x^{-3/2} dx = 2(n^{-5/4} - n^{-15/8}).$$

Thus both $\sum_{k=1}^{2n} E\{X_k\}$ and $\sum_{k=1}^{2n}$ var (X_k) are convergent series. Since the X_k's are independent and bounded by 1, $\sum_{k=1}^{\infty} X_k$ converges a.s. and in L^2. □

One can easily check that

$$\sum_{k=1}^{\infty} |EX_k| = \infty.$$

This implies $\sum_{k=1}^{\infty} |X_k| = \infty$ a.s. and hence

$$A_t = \sum_{s\le t} \Delta H_s \Delta M_s$$

is not a process with paths of finite variation on compacts.

PROPOSITION 2

A_t is not a semimartingale.

Proof

Define

$J_t^n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{log\ (k+1)}\ 1_{(T_{k-1},T_k]}(t)$

Then J^n is in $\mathbb{L}$, and J^n converges in ucp to

$$J_t = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{log\ (k+1)}\ 1_{(T_{k-1},T_k]}(t)$$

(which is therefore also in $\mathbb{L}$).

If A were a semimartingale, then $(J^n \cdot A)$ would converge in ucp too by, for example, the Bichteler-Dellacherie theorem.

But

$$\begin{aligned}(J^n \cdot A)_T &= \sum_{k=1}^{2n} J^n_{T_{k-1}}\ (A_{T_k} - A_{T_{k-1}}) \\ &= \sum_{k=2}^{2n} \frac{1}{log\ k}\ |X_k| \to \infty\ a.s.,\end{aligned}$$

since the X_k's are independent and

$$\sum_{k=2}^{\infty} \frac{1}{log\ k}\ E\{|X_k|\} = \infty.$$

Thus A is not a semimartingale □

A final remark: The predictable σ-algebra $\mathcal{P}$ is generated by $\mathbb{L}$; that is, $\mathcal{P} = \sigma(\mathbb{L})$, while the optional σ-algebra $\mathcal{O}$ is generated by $\mathbb{D} : \mathcal{O} = \sigma(\mathbb{D})$. Since we have seen that even in the semimartingale theory (and not just the local martingale theory) one cannot go beyond $\mathbb{L}$ to $\mathbb{D}$, clearly one cannot go beyond $\mathcal{P}$ to $\mathcal{O}$ as well. Thus this example helps to clarify the standard restriction that integrands must be predictably measurable, in the general case (that is, when semimartingales can have jumps).

REFERENCES

1. C. Dellacherie, Un survol de la théorie de l'intégrale stochastique. Stochastic Processes and Their Appl. *10* (1980), 115–144.
2. G. Letta, *Martingales et Intégration Stochastique.* Scuola Normale Superiore, Pisa, 1984.
3. M. Pratelli, La classe des semimartingales qui permettent d'intégrer les processus optionnels, Séminaire de Proba. XVII, Springer Lect. Notes in Math. 986 (1983), 311–320.
4. P. Protter, Stochastic integration without tears. Stochastics *16* (1986), 295–325.
5. P. Protter, *Stochastic Integration and Differential Equations.* Springer-Verlag, Heidelberg, 1990.
6. R. J. Williams, Book Review of [5] above, Bulletin of the American Math. Soc. *25* (1991), 170–180.