# Recurrence of Reinforced Random Walk on a Ladder

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## ABSTRACT

Consider reinforced random walk on a graph that looks like a doubly infinite ladder. All edges have initial weight 1, and the reinforcement convention is to add $\delta > 0$ to the weight of an edge upon first crossing, with no reinforcement thereafter. This paper proves recurrence for all $\delta > 0$. In so doing, we introduce a more general class of processes, termed multiple-level reinforced random walks.

## Section 1. Introduction and Summary

Coppersmith and Diaconis (1987) have initiated the study of a class of processes called reinforced random walks. (See also Diaconis (1988).) Take a graph with initial "weights" assigned to the edges. Then define a discrete-time, nearest-neighbor random walk on the vertices of this graph as follows. At each stage, the (conditional, given the past) probability of transition from the current vertex to an adjacent vertex is proportional to the weight currently assigned to the connecting edge. (The random walk always jumps, so these conditional transition probabilities sum to 1.) The weight of an edge can increase when the edge is crossed, with the amounts of increase depending on the "reinforcement" convention. The convention most studied by Coppersmith and Diaconis (1987) is to always add +1 to the weight of an edge each time it is crossed. In this setting, they show that a reinforced random walk on a finite graph is a mixture of stationary Markov random walks. The mixing measure is given explicitly in terms of the "loops" of the graph.

Pemantle (1988) has studied reinforced random walks with the Coppersmith-Diaconis reinforcing convention on infinite acyclic graphs. Davis (1989) obtained results for nearest-neighbor reinforced random walks on the integers $\mathbf{Z}$ with very general reinforcement schemes.

Consider nearest-neighbor reinforced random walk on the lattice $\mathbf{Z}^2$ of points in $\mathbf{R}^2$ with integer coordinates. All edges between neighboring points are assigned initial weight 1. It seems plausible (perhaps even "obvious") that any spacially homogeneous reinforcement scheme for which the process cannot "get stuck" forever on a finite set of points will be recurrent, that is, will visit each point of the lattice infinitely often. However, to the author's knowledge no one has been able to prove or disprove recurrence of reinforced random walk on $\mathbf{Z}^2$ for any such reinforcement scheme. Michael Keane has proposed the following simpler variant: consider nearest-neighbor reinforced random walk on the points of $\mathbf{Z}^2$ with $y$ coordinate 1 or 2 (and starting at (0, 1), say). If one draws in the edges between nearest neighbors, one of course gets an infinite horizontal "ladder." Again, all initial weights are taken to be +1. For the reinforcement scheme, Keane suggested that edges be reinforced by $\delta = 1$ the first time they are crossed and then never again. This paper will show that Keane's reinforced random walk on a ladder with "one-time" reinforcement is recurrent for any positive reinforcement parameter $\delta$.

Section 2 will introduce a class of processes termed "'multiple-level reinforced random walks", or MLRRW's, which generalize Keane's reinforced random walk on a ladder. The set of possible positions in an MLRRW with $d$ levels is $\mathbb{Z} \times \{1,2,\ldots,d\}$. The horizontal motion is reinforced random walk, but the vertical movement between horizontal steps is arbitrary (though "nonanticipating" with respect to the horizontal motion). Section 3 will show that the $X$-coordinate of $d$-level MLRRW is recurrent when $\delta < 1/(d-1)$, implying that Keane's reinforced random walk on a ladder is recurrent for $\delta < 1$. (It should be obvious that Keane's process visits all sites infinitely often if its $X$-coordinate is recurrent.) The technicalities are a bit annoying, but the idea of the argument is very simple. Let $X_n$ be the horizontal position after $n$ horizontal steps. Suppose $X_{n_0} = k > 0$, and consider the problem of bounding $P\{X_T = M|\mathcal{F}_{n_0}\}$, where $T$ is the first time after $n_0$ that $X_i$ hits either 0 or $M$, $M \gg k$, and $\mathcal{F}_n$ is the $\sigma$-field of everything that happens up through time $n$. Since $X_T$ must equal either 0 or $M$,

$$P\{X_T = M \Big| \mathcal{F}_{n_0}\} = \frac{1}{M} E(X_T|\mathcal{F}_{n_0}). \tag{1.1}$$

If $X_i$ were a martingale, we would have $E(X_T|\mathcal{F}_{n_0}) = X_{n_0}$ and hence $P\{X_T = M|\mathcal{F}_{n_0}\} = X_{n_0}/M$. But of course $X_i$ is not a martingale. When the edges to the left and right are unequally reinforced, the next horizontal step has a bias, or "drift." However, the total expected cumulative bias which can arise over time at a particular horizontal location $x > 0$ turns out to be no larger than $(d-1)\delta$. Summing over $x, 0 < x < M$, we get that the total expected cumulative bias between times $n_0$ and $T$ is no larger than $(M-1)(d-1)\delta$. It follows that

$$E(X_T|\mathcal{F}_{n_0}) \leq X_{n_0} + (M-1)(d-1)\delta, \tag{1.2}$$

and hence from (1.1) that

$$P\{X_T = M|\mathcal{F}_{n_0}\} \leq \frac{X_{n_0}}{M} + (d-1)\delta. \tag{1.3}$$

Thus, if $(d-1)\delta < 1$, we can make the probability of hitting $M$ before next hitting 0 less than $(d-1)\delta + \varepsilon < 1$ by choosing $M > \varepsilon^{-1} X_{n_0}$. It follows easily that $X_n$ (or $|X_n|$) cannot drift off to $\infty$ without repeatedly coming back to 0.

Section 4 shows that $X_n$ is recurrent for *all* 2-level MLRRW's, so that Keane's process is in fact recurrent for all $\delta > 0$. The basic structure of the argument is as in Section 3, but the bookkeeping to keep track of the cumulative bias must be more careful.

Section 5 briefly indicates how the results of Section 4 generalize to MLRRW's with more than 2 levels.

## Section 2. Notation for MLRRW

Let $\{(X_n, Y_n)\}_{n=0}^{\infty}$ be a sequence of random points in $\mathbb{Z}^2$. Let $\mathcal{F}_0 < \mathcal{F}_1 < \dots$ be $\sigma$-fields. We assume that

(i) $(X_0, Y_0) = (0, 1)$.

(ii) $\{X_n\}_{n=0}^{\infty}$ is a nearest neighbor random motion on $\mathbb{Z}$, i.e., $P\{|X_{n+1} - X_n| = 1\} = 1 \ \forall n$.

(iii) $Y_n \varepsilon \{1, \dots, d\}$ for some positive integer $d$.

(iv) $(X_n, Y_n)$ is $\mathcal{F}_n$-measurable.

(v) $P\{X_{n+1} = X_n + 1 | \mathcal{F}_n\} = \frac{W_n(X_n, Y_n)}{W_n(X_n, Y_n) + W_n(X_n - 1, Y_n)}$.

Here, $W_n(x, y)$ is the "weight" at time $n$ of the horizontal unit segment to the right of $(x, y)$. Our reinforcement convention is that $W_n(x, y)$ equals $1 + \delta$ if for some $i, 0 \le i \le n$, either $(X_i, Y_i) = (x, y)$ and $X_{i+1} = x + 1$, or $(X_i, Y_i) = (x + 1, y)$ and $X_{i+1} = x$. Otherwise, $W_n(x, y) = 1$. We always assume $\delta > 0$.

For lack of a better name, the process just described will be called MLRRW, standing for "multiple-level reinforcing random walk." The way to think about it is that we first move horizontally from $(X_n, Y_n)$ according to the rules of reinforced random walk, and then we can move vertically in an arbitrary way before the next horizontal move.

To make Keane's reinforced random walk on a ladder into an MLRRW, take $\mathcal{F}_n$ to be the $\sigma$-field generated by the process up to just before the $(n + 1)^{st}$ horizontal step, together with the knowledge that the next step will be horizontal. It is easy to show that the conditions for MLRRW are satisfied by this choice of $\{\mathcal{F}_n\}_{n=0}^{\infty}$.

Let $p = (1 + \delta)/(2 + \delta)$ and $q = 1 - p = 1/(2 + \delta)$. Note that $p$ is the probability of crossing the reinforced edge when the choice is between one reinforced edge and one unreinforced edge.

In the calculations of subsequent sections, a sum $\sum_{i=j}^{m} a_i$ will be taken to equal 0 when $m < j$. $I(A)$ is of course the indicator function of event $A$.

## Section 3. Recurrence of MLRRW when $\delta$ is small

**Theorem 3.1** *If $\delta < (d-1)^{-1}$ in a MLRRW with d levels, then $X_n$ is recurrent, i.e., visits every integer infinitely often, a.s.*

PROOF

If $X_n = 0$ and $k$ is a positive integer, then

$$P\{X_{n+k} = k|\mathcal{F}_n\} \geq q^k, \text{ and } P\{X_{n+k} = -k|\mathcal{F}_n\} \geq q^k.$$

Thus, it is easy to see that it is enough to show that $X_n$ visits 0 infinitely often, a.s.

Define the "compensator" of the increment $X_{n+1} - X_n$ by

$$C_n = E(X_{n+1} - X_n|\mathcal{F}_n).$$

Then obviously $X_n - \sum_{i=1}^{n-1} C_i$ is an $\mathcal{F}_n$-martingale. For integers $n_0 \geq 0$ and $M \geq 1$, define

(3.2) $$T(n_0, M) =: \inf\{i \geq n_0 : |X_i| = 0 \text{ or } |X_i| = M\}.$$

**Lemma 3.3** *Fix $n_0$ and $M$, and write $T = T(n_0, M)$.* If $|X_{n_0}| \leq M$ in a $d$-level MLRRW, then

$$P\{|X_T| = M|\mathcal{F}_{n_0}\} \leq \frac{|X_{n_0}|}{M} + (d-1)\delta.$$

**Claim 3.4** *Under the conditions of Lemma 3.3, we have for $0 < x < M$ that*

$$E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\}|\mathcal{F}_{n_0}\right] \leq (d-1)\delta.$$

PROOF OF CLAIM 3.4

Fix $x$. With $n_0$ and $M$ fixed, let

(3.5) $$\tau_x =: (T+1) \wedge \inf\{i : i \geq n_0 \text{ and } X_i = x\}.$$

Since $\mathcal{F}_{\tau_x} \supset \mathcal{F}_{n_0}$, it suffices to prove the claim with $\mathcal{F}_{\tau_x}$ in place of $\mathcal{F}_{n_0}$. If $\tau_x = T+1$, then the conditional expectation given $\mathcal{F}_{\tau_x}$ is 0. Thus, we need to show

(3.6) $$E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\}|\mathcal{F}_{\tau_x}\right] \leq (d-1)\delta$$

when $\tau_x \le T$.

If $\tau_x \le T$, then $W_{\tau_x}(x-1, y) = 1+\delta$ for at least one ($\mathcal{F}_{\tau_x}$-measurable) value of $y$ (namely, $Y_{\tau_x - 1}$) which we may without loss of generality (wlog) assume to be $y = 1$. Then

$$E\left[\sum_{i=n_0}^{T-1} C_i I\{(X_i, Y_i) = (x,1)\} | \mathcal{F}_{\tau_x}\right] \le 0, \tag{3.7}$$

since there can never be a positive bias at $(x, 1)$.

For other values of $y$, it may happen for $i > \tau_x$ that $(X_i, Y_i) = (x, y), W_i(x-1, y) = 1$, and $W_i(x, y) = 1 + \delta$, in which case

$$C_i I\{(X_i, Y_i) = (x,y)\} = p - q = \frac{\delta}{2+\delta}.$$

However, the number of times that $(X_i, Y_i)$ visits a particular site $(x, y)$ before $X_{i+1} = x-1$ is stochastically bounded (conditionally on $\mathcal{F}_{\tau_x}$) by a geometric ($q$) random variable (with expectation $q^{-1} = 2+\delta$). Thereafter, there will be no positive bias at site $(x, y)$. Thus,

$$E\left\{\sum_{i=n_0}^{T-1} C_i I\{(X_i, Y_i) = (x,y)\} | \mathcal{F}_{\tau_x}\right\} \le (p-q)q^{-1} = \delta \tag{3.8}$$

for all values of $y$. Applying both (3.7) and (3.8) yields

$$\begin{aligned} &E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\} | \mathcal{F}_{\tau_x}\right] \\ &= E\left[\sum_{y=1}^{d} \sum_{i=n_0}^{T-1} C_i I\{(X_i, Y_i) = (x,y)\} | \mathcal{F}_{\tau_x}\right] \\ &\le (d-1)\delta, \end{aligned}$$

which establishes (3.6) and therefore the claim. □

PROOF OF LEMMA 3.3

Suppose $0 \le X_{n_0} \le M$. (The argument for $0 > X_{n_0} \ge -M$ is the same). Note that

$$X_{n\wedge T} - \sum_{i=}^{(n\wedge T)-1} C_i, \qquad n \ge 0, \tag{3.9}$$

is an $L^2$-bounded martingale, since the increments are bounded and $ET^2 < \infty$. Hence,

$$\begin{aligned} E(X_T - X_{n_0} | \mathcal{F}_{n_0}) &= E\left(\sum_{i=n_0}^{T-1} C_i | \mathcal{F}_{n_0}\right) \\ &= \sum_{x=1}^{M-1} E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\} | \mathcal{F}_{n_0}\right] \\ &\le (M-1)(d-1)\delta, \end{aligned} \tag{3.10}$$

where the inequality follows from Claim 3.4. But $X_T$ is either 0 or $M$, so by (3.10)

$$P\{X_T = M|\mathcal{F}_{n_0}\} = \frac{1}{M}E(X_T|\mathcal{F}_{n_0}) \leq \frac{X_{n_0}}{M} + (d-1)\delta, \tag{3.11}$$

which proves Lemma 3.3 when $X_{n_0} \geq 0$. □

PROOF OF THEOREM 3.1

Let $\alpha$ be large enough so that $2\alpha^{-1} + (d-1)\delta < 1$. Let $\varepsilon = 1 - 2\alpha^{-1} - (d-1)\delta$. For $n = 1, 2, \ldots$, let $\tau^*(n)$ be the first time that $|X_i|$ exceeds $\alpha^n$. It is obvious that $\limsup |X_n| = \infty$, so these stopping times $\tau^*(n)$ are all finite, a.s. By Lemma 3.3, the conditional probability, given $\mathcal{F}_{\tau^*(n)}$, that $X_i$ visits 0 between times $\tau^*(n)$ and $\tau^*(n+1)$ is greater than $\varepsilon$ for all $n$, since $\frac{\alpha^n+1}{\alpha^{n+1}} < 2\alpha^{-1}$ for all $n$. It follows easily (for instance from the strong law of large numbers for martingales in Neveau (1965), page 148) that $X_i$ visits 0 in infinitely many of the time intervals $[\tau^*(n), \tau^*(n+1)]$, a.s. □

## Section 4. Recurrence of 2-level MLRRW

**Theorem 4.1.** *For a MLRRW as defined in Section 2 with $d = 2, X_n$ is recurrent.*

The goal will again be to show that the expected "cumulative bias"

$$E\left[\sum_{i=0}^{\infty} C_i I\{X_i = x\}\right] \tag{4.2}$$

at each horizontal location $x$ is too small (e.g., less than $1-\varepsilon$, some $\varepsilon > 0$, for all positive $x$) to prevent recurrence. Fix a positive integer $x$, and suppose you are trying to use the vertical motion so as to *maximize* (4.2), with the goal of preventing $X_i$ from ever returning to zero. When horizontal location $x$ is first reached, the transition has to have been either from $(x-1, 1)$ or from $(x-1, 2)$. Assume wlog that it was from $(x-1, 1)$. Then $C_i$ can never be positive when $(X_i, Y_i) = (x, 1)$, since the edge to the left has been reinforced. So you'll try to make $\sum_{i=0}^{\infty} C_i I\{(X_i, Y_i) = (x, 2)\}$ big. For $C_i I\{(X_i, Y_i) = (x, 2)\}$ to be positive, $W_i(x, 2)$ must be $1+\delta$ and $W_i(x-1, 2)$ must be 1. This situation can arise in two ways, depending on whether the edge from $(x, 2)$ to $(x+1, 2)$ is reinforced by a right-to-left crossing or by a left-to-right crossing. Suppose you decide to reinforce this edge by a right-to-left crossing. For this, $X_i$ must first hit $x+1$ by crossing the edge from $(x, 1)$ to $(x+1, 1)$. Doing this will, in expectation, produce a cumulative bias of $-\delta$ at $(x, 1)$, since the number of visits to $(x, 1)$ needed to achieve "success" [=crossing the edge from $(x, 1)$ to $(x+1, 1)$] is geometric $(q)$, and each such visit has $C_i I\{(X_i, Y_i) = (x,1)\} = -(p-q)$. (Compare with (3.8).) After $X_i$ hits $x+1$, you can (maybe) come back to $x$ across the edge from $(x+1,2)$ to $(x,2)$. Subsequent visits to $(x,2)$ have $C_i I\{(X_i, Y_i) = (x,2)\} = p-q$ until the edge from $(x-1,2)$ to $(x,2)$ gets reinforced, which will happen after a geometric $(q)$ number of visits. In expectation, one has at best a "cumulative bias" of $\delta$ at $(x,2)$, assuming $(x,2)$ is visited often enough to exhaust the positive bias there. (Again, compare with (3.8).) Adding the expected cumulative biases at $(x,1)$ and at $(x,2)$ gives a value of zero (at best) for (4.2), so this strategy does not seem effective. The other strategy is to try to reinforce the edge from $(x,2)$ to $(x+1,2)$ by crossing from left to right. So you set $Y_i = 2$ the first time that $X_i = x$, causing $X_{i+1} = x+1$ and $X_{i+1} = x-1$ to both have conditional probability $\frac{1}{2}$. If $X_{i+1} = x+1$, you can get an expected cumulative bias of $\delta$ at $(x,2)$, assuming you come back to $(x,2)$ often enough. However, if $X_{i+1} = x-1$, then subsequent visits to either $(x,1)$ or $(x,2)$ will have $C_i I\{X_i = x\} = -(p-q)$ until you get to $x+1$, and the expected cumulative bias at $x$ will be near $-\delta$ if the probability of eventually getting to $x+1$ is near 1. Thus, with this second strategy, the only way for (4.2) to be $\geq 1$ is for there to be a sufficiently large conditional probability of never reaching $x+1$ after reaching $x$. Thus, neither strategy looks promising for making $X_i$ converge to $+\infty$.

The observations of the above paragraph do not quite constitute a proof of Theorem 4.2, since "randomized" strategies for choosing $Y_i$ when $X_i = x$ were not considered. Turning the above paragraph into a rigorous proof is just a matter of careful bookkeeping, however.

Again, let $T = T(n_0, M)$ be defined as in (3.2) for integers $n_0 \geq 0$ and $M \geq 1$.

**Lemma 4.3** *If* $\max_{0 \leq i \leq n_0} |X_i| = m < M$ *in a 2-level MLRRW, then*

$$P\{|X_T| = M | \mathcal{F}_{n_0}\} \leq 1 - \frac{M - m - m\delta}{M + (M-m)\delta}.$$

**Claim 4.4** *Under the conditions of Lemma 4.3, for* $m < x < M$ *we have*

$$E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\} | \mathcal{F}_{n_0}\right] \leq \delta P\{X_T = 0 | \mathcal{F}_{n_0}\}.$$

Lemma 4.3 follows from Claim 4.4 and a little algebra. (See the proof of Lemma 4.3 below.)

PROOF OF CLAIM 4.4

As in Section 3, for $0 < x \leq M$ (and fixed $n_0 \geq 0, M \geq 1$, and $0 < m < M$), let

$$\tau_x =: (T+1) \wedge \inf\{i: i \geq n_0 \text{ and } X_i = x\}.$$

For this proof, fix a value of $x$ for which $m < x < M$. Since $\mathcal{F}_{\tau_x} \supset \mathcal{F}_{n_0}$, it suffices to prove the claim with $\mathcal{F}_{\tau_x}$ in place of $\mathcal{F}_{n_0}$.

On the event $\{\tau_x = T+1\}$,

$$E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\} | \mathcal{F}_{\tau_x}\right] = 0, \tag{4.5}$$

so we can concentrate on showing

$$E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\} | \mathcal{F}_{\tau_x}\right] \leq \delta P\{X_T = 0 | \mathcal{F}_{\tau_x}\} \tag{4.6}$$

on the set $\{\tau_x \leq T\}$. There, $\tau_x$ is the very first time that $X_i = x$, so exactly one of the two edges between horizontal position $x-1$ and $x$ is reinforced at time $\tau_x$. Assume wlog that the bottom edge, between $(x-1, 1)$ and $(x, 1)$, was reinforced. Let $\tau_x^*$ be the first time at or before time $T$ that one of the *remaining* 3 horizontal edges with ends at horizontal position $x$ is reinforced. If none of these 3 edges is reinforced by time $T$, set $\tau_x^* = T+1$. Let $B$ be the event that the other "bottom" edge [the one between $(x, 1)$ and $(x+1, 1)$] is the one reinforced at time $\tau_x^*$. Let $A^L$ be the event that the edge reinforced at time $\tau_x^*$ is the one "above" and to the left, i.e., between $(x-1, 2)$ and $(x, 2)$. Similarly, $A^R$ is the event that the edge "above" and to the right is reinforced at time $\tau_x^*$. Note that $B, A^L$, and $A^R$ do not exhaust the possibilities, since $(A^L \cup A^R \cup B)^C$ is the event that $\tau_x^* = T+1$.

Let $B_i$ be the event that $B$ occurs and $\tau_x^* = i+1$. Then

$$P\{B_i | \mathcal{F}_i\} = qI\{(X_i, Y_i) = (x, 1) \text{ and } \tau_x^* > i\}, \tag{4.7}$$

since $q = (2+\delta)^{-1}$ is the probability of moving against the bias. Now $B$ is the disjoint union of the $B_i$'s, so

$$P\{B | \mathcal{F}_{\tau_x}\} = \sum_{i=n_0}^{\infty} E[P\{B_i | \mathcal{F}_i\} | \mathcal{F}_{\tau_x}] \tag{4.8}$$

$$= qE\left[\sum_{i=n_0}^{\tau_x^*-1} I\{(X_i,Y_i)=(x,1)\}|\mathcal{F}_{\tau_x}\right]$$
$$= -\delta^{-1}E\left[\sum_{i=n_0}^{\tau_x^*-1} C_i I\{(X_i,Y_i)=(x,1)\}|\mathcal{F}_{\tau_x}\right],$$

since $C_i = -\frac{\delta}{2+\delta}$ when the bias is to the left. Thus,

$$-\delta P\{B|\mathcal{F}_{\tau_x}\} = E\left[\sum_{i=n_0}^{\tau_x^*-1} C_i I\{(X_i,Y_i)=(x,1)\}|\mathcal{F}_{\tau_x}\right]. \tag{4.9}$$

Let $A_i^L$ be the event that $A^L$ occurs and $\tau_x^* = i+1$, with $A_i^R$ defined similarly. For $A_i^R$ to occur, we *must* have $(X_i,Y_i)=(x,2)$ and $\tau_x^* > i$. If $(X_i,Y_i)=(x,2)$ and $\tau_x^* > i$, then $\tau_x^* = i+1$ is a sure thing, and $A_i^L$ and $A_i^R$ each have conditional probability $\frac{1}{2}$. Thus,

$$\begin{aligned} P\{A_i^R|\mathcal{F}_{\tau_x}\} &= \frac{1}{2}P\{(X_i,Y_i)=(x,2),\tau_x^*>i|\mathcal{F}_{\tau_x}\} \\ &\le P\{A_i^L|\mathcal{F}_{\tau_x}\} \end{aligned} \tag{4.10}$$

[Note that $A_i^L$ can also occur when $(X_i,Y_i)=(x-1,2)$ and $\tau_x^* > i$.]. Summing over $i$ yields

$$P\{A^R|\mathcal{F}_{\tau_x}\} \le P\{A^L|\mathcal{F}_{\tau_x}\}. \tag{4.11}$$

Note that $C_i I\{(X_i,Y_i)=(x,2)\} = 0$ for $i < \tau_x^*$. Also, by the same argument as for (3.8),

$$E\left[\sum_{i=\tau_x^*}^{T-1} C_i I\{(X_i,Y_i)=(x,2)\}|\mathcal{F}_{\tau_x^*}\right] \le \delta. \tag{4.12}$$

Since we are assuming that the edge from $(x-1,1)$ to $(x,1)$ is the one reinforced at time $\tau_x, C_i I\{(X_i,Y_i)=(x,1)\} \le 0$ for all $i$.

Let $A^{L+}$ be the event that $A^L$ and $\tau_{x+1} < T+1$ both occur. Let $A_j^{L+}$ be the event that $A^{L+}$ occurs and $\tau_{x+1} = j+1$. Then

$$P\{A_j^{L+}|\mathcal{F}_j\} = qI\{\tau_x^* \le j < \tau_{x+1}, A_L, X_j = x\}, \tag{4.13}$$

so that

$$(4.14)\qquad P\{A^{L+}|\mathcal{F}_{\tau_x^*}\} = \sum_{j=\tau_x^*}^{\infty} E[P\{A_j^{L+}|\mathcal{F}_j\}|\mathcal{F}_{\tau_x^*}]$$
$$= qI\{A^L\}E\left[\sum_{j=\tau_x^*}^{\tau_{x+1}-1} I\{X_j = x\}|\mathcal{F}_{\tau_x^*}\right]$$
$$= -\delta^{-1}I\{A^L\}E\left[\sum_{j=\tau_x^*}^{\tau_{x+1}-1} C_jI\{X_j = x\}|\mathcal{F}_{\tau_x^*}\right]$$
$$\leq -\delta^{-1}I\{A^L\}E\left[\sum_{j=\tau_x^*}^{T-1} C_jI\{X_j = x\}|\mathcal{F}_{\tau_x^*}\right],$$

the last inequality following from the observation that $C_jI\{X_j = x\} \leq 0$ for $j \geq \tau_x^*$ on $A^L$, since any bias at $x$ after $\tau_x^*$ will be to the left. Now $A^{L+} \supset A^L \cap \{X_T = M\}$, so it follows from (4.14) that

$$(4.15)\qquad -\delta I\{A^L\}P\{X_T = M|\mathcal{F}_{\tau_x^*}\} \geq I\{A^L\}E\left[\sum_{j=\tau_x^*}^{T-1} C_jI\{X_j = x\}|\mathcal{F}_{\tau_x^*}\right].$$

Hence,

$$(4.16)\qquad I\{A^L\}E\left[\sum_{j=\tau_x^*}^{T-1} C_jI\{X_j = x\}|\mathcal{F}_{\tau_x^*}\right]$$
$$\leq -\delta I\{A_L\} + \delta I\{A^L\}P\{X_T = 0|\mathcal{F}_{\tau_x^*}\}.$$

Now we have all our ducks in a row. On the set $\{\tau_x \leq T\}$ for $m < x < M$,

$$(4.17)\qquad E\left[\sum_{i=n_0}^{T-1} C_iI\{X_i = x\}|\mathcal{F}_{\tau_x}\right]$$
$$= E\left[\sum_{i=n_0}^{\tau_x^*-1} C_iI\{(X_i, Y_i) = (x, 1)\}|\mathcal{F}_{\tau_x}\right]$$
$$+ E\left[\sum_{i=n_0}^{\tau_x^*-1} C_iI\{(X_i, Y_i) = (x, 2)\}|\mathcal{F}_{\tau_x}\right]$$
$$+ E\left\{I\{B \cup A^R\}E\left[\sum_{i=\tau_x^*}^{T-1} C_iI\{(X_i, Y_i) = (x, 1)\}|\mathcal{F}_{\tau_x^*}\right]|\mathcal{F}_{\tau_x}\right\}$$

$$
\begin{aligned}
&+E\left\{I\{B\}E\left[\sum_{i=\tau_x^*}^{T-1} C_i I\{(X_i,Y_i)=(x,2)\}|\mathcal{F}_{\tau_x^*}\right]|\mathcal{F}_{\tau_x}\right\}\\
&+E\left\{I\{A^R\}E\left[\sum_{i=\tau_x^*}^{T-1} C_i I\{(X_i,Y_i)=(x,2)\}|\mathcal{F}_{\tau_x^*}\right]|\mathcal{F}_{\tau_x}\right\}\\
&+E\left\{I\{A^L\}E\left[\sum_{i=\tau_x^*}^{T-1} C_i I\{X_i=x\}|\mathcal{F}_{\tau_x^*}\right]|\mathcal{F}_{\tau_x}\right\}\\
\le& -\delta P\{B|\mathcal{F}_{\tau_x}\} \qquad \text{[See (4.9).]}\\
&+0 \qquad \text{[See remark just before (4.12).]}\\
&+0 \qquad \text{[See remark just after (4.12).]}\\
&+\delta P\{B|\mathcal{F}_{\tau_x}\} \qquad \text{[See (4.12).]}\\
&+\delta P\{A^R|\mathcal{F}_{\tau_x}\} \qquad \text{[See (4.12).]}\\
&-\delta P\{A^L|\mathcal{F}_{\tau_x}\}+\delta P\{X_T=0|\mathcal{F}_{\tau_x}\}. \qquad \text{[See (4.16).]}\\
=&\ \delta[P\{A^R|\mathcal{F}_{\tau_x}\}-P\{A^L|\mathcal{F}_{\tau_x}\}]+\delta P\{X_T=0|\mathcal{F}_{\tau_x}\}\\
\le&\ \delta P\{X_T=0|\mathcal{F}_{\tau_x}\} \qquad \text{[by (4.11).],}
\end{aligned}
$$

which establishes (4.6) and hence Claim 4.4. □

PROOF OF LEMMA 4.3

Suppose $X_{n_0} \ge 0$. (Again, the argument is the same for $X_{n_0} < 0$.) As in the proof of Lemma 3.3, expression (3.9) is an $L^2$-bounded martingale, and $X_T$ is either 0 or $M$, so that

(4.18)
$$
\begin{aligned}
P\{X_T=M|\mathcal{F}_{n_0}\} &= \frac{1}{M}E(X_T|\mathcal{F}_{n_0})\\
&= \frac{1}{M}\left\{X_{n_0}+E\left(\sum_{i=n_0}^{T-1} C_i|\mathcal{F}_{n_0}\right)\right\}\\
&\le \frac{1}{M}\left\{m+E\left(\sum_{i=n_0}^{T-1} C_i|\mathcal{F}_{n_0}\right)\right\}
\end{aligned}
$$

But

(4.19)
$$
E\left(\sum_{i=n_0}^{T-1} C_i|\mathcal{F}_{n_0}\right)=\sum_{x=1}^{m} E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i=x\}|\mathcal{F}_{n_0}\right]
$$

$$+\sum_{x=m+1}^{M-1} E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\}|\mathcal{F}_{n_0}\right].$$
$$\leq m\delta + (M-m)\delta P\{X_T = 0|\mathcal{F}_{n_0}\},$$

where the inequality follows from applying Claim 3.4 (still valid here with $d = 2$) to the first summation and Claim 4.4 to the second summation. Substituting (4.19) into (4.18) and using $P\{X_T = 0|\mathcal{F}_{n_0}\} = 1 - P\{X_T = M|\mathcal{F}_{n_0}\}$ yields the inequality of Lemma 4.3 after a little algebra. □

PROOF OF THEOREM 4.1

The argument is the same as for Theorem 3.1. The bound in Lemma 4.3 depends only on the ratio of $M$ and $m$, and by making this ratio sufficiently large we can make the bound as close as we like to $\delta/(1+\delta)$. Thus, there is an $\alpha > 1$ so that $M > m\alpha/2$ implies in Lemma 4.3 that $P\{|X_T| = 0|\mathcal{F}_{n_0}\} > \varepsilon$, where $\varepsilon = (2+\delta)^{-1}$. Again define $\tau^*(n)$ to be the first time that $|X_i|$ exceeds $\alpha^n$. It follows as before that $X_i$ visits 0 in infinitely many of the time intervals $[\tau^*(n), \tau^*(n+1)]$, a.s. □

## Section 5. Generalization of Section 4 to $d$>2

By working a little harder with the techniques of Section 4, one can obtain Theorem 5.1, which improves Theorem 3.1 and generalizes Theorem 4.1.

**Theorem 5.1** *If $\delta < (d-2)^{-1}$ in a MLRRW with $d$ levels, then $X_n$ is recurrent.*

Theorem 5.1 follows from Lemma 5.2, by the same argument used to get Theorem 4.1 from Lemma 4.3.

**Lemma 5.2** *If $\max_{0\leq i\leq n_0} |X_i| = m < M$ in a d-level MLRRW and $T = T(n_0, M)$, then*

$$P\{|X_t| = M|\mathcal{F}_{n_0}\} \leq 1 - \frac{(M-m)\{1-(d-2)\delta\} - m(d-1)\delta}{M + (M-m)\delta}.$$

Lemma 5.2 follows by algebra from

**Claim 5.3** *Under the conditions of Lemma 5.2, we have for $m < x < M$ that*

$$E\left[\sum_{i=n_0}^{T-1} C_i I\{X_i = x\}|\mathcal{F}_{n_0}\right] \leq \delta P\{X_T = 0|\mathcal{F}_{n_0}\} + (d-2)\delta.$$

Claim 5.3, which generalizes Claim 4.4, has a similar but slightly more complicated proof.

Theorem 5.1 is sharp in the sense that a $d$-level MLRRW can be transient if $\delta > (d-2)^{-1}$. A rule for choosing the vertical level at each stage which makes MLRRW transient (with $X_n \to \infty$, a.s.) is the following "randomized enlightened greedy algorithm". At each stage, choose $Y_i$ to make $C_i$ positive if you can. If $C_i$ cannot be made positive, make it zero if you can, but giving preference to sites where the edges to either side are already reinforced. (This preference is the enlightened part of the greed.) The bias $C_i$ is negative only if this cannot be avoided. "Randomized" means that $Y_i$ is chosen randomly (with equal probabilities) from among the permissible values. The proof of transience is similar to arguments to be found in Sellke (1993). The gist is as follows. First of all, it is easy to show that $X_n$ is transient if and only if $X_n^+$, the positive part of $X_n$, is transient. (Note that $C_i$ is never negative when $X_i$ is negative.) So consider the process $X_n^+$. A zero-one law argument shows that $X_n^+$ is either almost surely transient or almost surely recurrent. If $X_n^+$ were almost surely recurrent, we could find an $M$ large enough so that, for the overwhelming majority of $x$ values between 0 and $M$, the probability is near 1 that all horizontal edges at $x$ are reinforced before $X_n^+$ hits $M$. One then shows that, for the randomized enlightened greedy algorithm, the expected cumulative bias at a positive $x$ is $(d-2)\delta$ if $X_n^+$ visits $x$ often enough. Consequently, the expected cumulative bias accumulated by $X_n^+$ by the time $T_M$ that $M$ is finally hit can be shown to be greater than $M$. But this would imply $E(X_{T_M}) > M$, which contradicts $X_{T_M} \equiv M$.

In the critical case $\delta = (d-2)^{-1}$, the randomized enlightened greedy algorithm can be shown to produce recurrence, again by arguments similar to those in Sellke (1993).


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## References


Coppersmith, D. and Diaconis, P. (1987). Random walk with reinforcement. Unpublished.

Davis, B. (1990). Reinforced random walk. *Probab. Th. Rel. Fields* **84**, 203–229.

Diaconis, P. (1988). Recent progress on de Finetti's notions of exchangeability. *Bayesian Statistics* **3**, 111–125, J.M. Bernardo, M.H. DeGroot, D.V. Lindley, and A.F.M. Smith, Eds. Oxford University Press.

Neveau, J. (1965). *Mathematical Foundations of the Calculus of Probabilities*, Holden-day, San Francisco.

Pemantle, R. (1988). Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* **16**, 1229–1241.

Sellke, T. (1993). Nearest-neighbor random walk in a changing environment. To appear.