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Holger Dette and William J. Studden
Universität Göttingen Purdue University

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SOME NEW ASYMPTOTIC PROPERTIES FOR THE ZEROS OF JACOBI, LAGUERRE AND HERMITE POLYNOMIALS

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Holger Dette
Institut für Mathematische Stochastik
Universität Göttingen
Lotzestr. 13
3400 Göttingen
GERMANY

and

William J. Studden*
Department of Statistics
Purdue University
1399 Mathematical Sciences Bldg.
West Lafayette, IN 47907-1399
U.S.A.

ABSTRACT

For the generalized Jacobi, Laguerre and Hermite polynomials $P_n^{(\alpha_n,\beta_n)}(x), L_n^{(\alpha_n)}(x), H_n^{(\gamma_n)}(x)$ the limit distributions of the zeros are found, when the sequences α_n or β_n tend to infinity with a larger order than n. The derivation uses special properties of the sequences in the corresponding recurrence formulae. The results are used to give second order approximations for the largest and smallest zero which improve (and generalize) the limit statements in a paper of Moak, Saff and Varga (1979).

1. Introduction. The aim of this paper is to give some new asymptotic results for the zeros of the classical orthogonal polynomials. To be precise let $P_n^{(\alpha,\beta)}(x), L_n^{(\alpha)}(x)$ and $H_n^{(\gamma)}(x)$ denote the Jacobi, Laguerre and Hermite polynomials orthogonal with respect to the measures $(1-x)^\alpha(1+x)^\beta dx$ $(\alpha,\beta > -1, x \in [-1,1]), x^\alpha e^{-x}dx$ $(\alpha > -1, x \in [0,\infty))$ and $|x|^{2\gamma}e^{-x^2}dx$ $(\gamma > -\frac{1}{2}, x \in (-\infty,\infty))$, respectively (see Szegö (1975)). In the following we will allow the weight functions to depend on the degree of the polynomials and consider the "generalized" classical orthogonal polynomials $P_n^{(\alpha_n,\beta_n)}(x), L_n^{(\alpha_n)}(x)$ and $H_n^{(\gamma_n)}(x)$ where $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ are sequences of real numbers $(\alpha_n, \beta_n > -1, \gamma_n > -\frac{1}{2})$. These polynomials are, in general, not orthogonal on the corresponding intervals, which reduces the number of methods for investigating their properties. If α_n, β_n are linear functions of n various asymptotic results for the generalized Jacobi polynomials

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can be found in papers of Moak, Saff and Varga (1979), Gonchar and Rakhmanov (1984), Mhaskar and Saff (1984), Gawronski and Shawyer (1991), Chen and Ismail (1991) and Ismail and Li (1992). A different approach (fixing the degree of the polynomials and varying the parameters) was discussed in Elbert and Laforgia (1987). Generalized Laguerre and Hermite polynomials with parameters depending linearly on n have been studied in Chen and Ismail (1991) and Gawronski (1993). In this paper we will investigate the asymptotic behaviour of the zeros of the generalized classical orthogonal polynomials when the sequences of the parameters tend to infinity with a larger order than n.

In Section 2 we find the limit distribution for the (suitable standardized) zeros of the generalized Jacobi polynomials while Section 3 states similar results for the generalized Laguerre and Hermite polynomials. It turns out that there exist essentially three types of limit distributions (Corollary 2.3, 2.4, 2.5 and 2.6) depending on the order of α_n, β_n and γ_n. Section 4 deals with second order approximations for the largest and smallest zero of the classical orthogonal polynomials which improve the statements of Moak, Saff and Varga (1979) for the Jacobi case. The results of this paper are based on certain characterizing properties of the chain sequences of the Jacobi, Laguerre and Hermite measure which have recently been established in a paper of Dette and Studden (1992). This approach allows a very simple derivation of asymptotic properties for the zeros of the corresponding "generalized" orthogonal polynomials.

2. The asymptotic distribution of the zeros of the generalized Jacobi polynomials.

Let $\alpha_n, \beta_n > -1$ for all $n \in \mathbb{N}$, then the zeros of $P_n^{(\alpha_n,\beta_n)}(x)$ are all simple and located in the interval $(-1,1)$. For $\xi \in \mathbb{R}$ we define

$$(2.1) \qquad N^{(\alpha_n,\beta_n)}(\xi) := \#\{x \mid P_n^{(\alpha_n,\beta_n)}(x) = 0,\ x \le \xi\}$$

as the number of zeros of $P_n^{(\alpha_n,\beta_n)}(x)$ that are less or equal than ξ and $\mu^{(\alpha_n,\beta_n)}$ as the (discrete) uniform distribution on the set $\{x | P_n^{(\alpha_n,\beta_n)}(x) = 0\}$, i.e. $\mu^{(\alpha_n,\beta_n)}(\xi) = \frac{1}{n}N^{(\alpha_n,\beta_n)}(\xi)$. For every probability measure on the interval $[-1,1]$ let $\Phi_\mu(z)$ denote the Stieltjes transform of μ with corresponding continued fraction expansion

$$(2.2) \qquad \Phi_\mu(z) := \int_{-1}^{1} \frac{d\mu(x)}{z-x} = \frac{1 \quad |}{|z \ + \ 1} - \frac{2\zeta_1|}{|\ 1} - \frac{2\zeta_2 \ |}{|z+1} - \ldots$$

$$= \frac{1}{|z + 1 - 2\zeta_1} \Big| - \frac{4\zeta_1\zeta_2}{|z + 1 - 2(\zeta_2 + \zeta_3)} \Big| - \frac{4\zeta_3\zeta_4}{|z + 1 - 2(\zeta_4 + \zeta_5)} \Big| - \cdots$$

where $\zeta_1 = p_1, \zeta_j = (1 - p_{j-1})p_j$ $(j \geq 2)$ and the quantities p_j in the chain sequences $(\zeta_j)_{j \in \mathbb{N}}$ are bijective continuous functions of the moments of the measure μ (see Perron (1954), Wall (1948) and Skibinsky (1986)). The following result characterizes the uniform distribution $\mu^{(\alpha_n, \beta_n)}$ in terms of the quantities p_j and is an immediate consequence from Lemma 2.1, Lemma 2.2 and Theorem 3.1 of Dette and Studden (1992).

Proposition 2.1. The uniform distribution $\mu^{(\alpha_n, \beta_n)}$ on the set $\{x | P_n^{(\alpha_n, \beta_n)}(x) = 0\}$ is uniquely determined by the chain sequence

$$\text{(2.3)} \qquad \begin{aligned} p_{2i}^{(n)} &= \frac{n - i}{2(n - i) + 1 + \alpha_n + \beta_n} \\ p_{2i-1}^{(n)} &= \frac{\beta_n + n - i + 1}{2(n - i + 1) + \alpha_n + \beta_n} \end{aligned} \qquad i = 1, \ldots, n$$

in the corresponding continued fraction expansion (2.2).

Theorem 2.2. Let $N^{(\alpha_n, \beta_n)}(\xi)$ be defined in (2.1) and assume that there exists sequences $(\delta_n)_{n \in \mathbb{N}}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ $(\delta_n > 0)$ and constants $a_1, a_2 \in \mathbb{R}, b_1, b_2 > 0$ such that the limits

$$\lim_{n \to \infty} \frac{1}{\delta_n} \left(\frac{n + \beta_n}{2n + \alpha_n + \beta_n} - \varepsilon_n \right) = \frac{a_1}{2}$$

$$\lim_{n \to \infty} \frac{1}{\delta_n} \left(\frac{n(n + \alpha_n) + (n + \beta_n)(n + \alpha_n + \beta_n)}{(2n + \alpha_n + \beta_n)^2} - \varepsilon_n \right) = \frac{a_2}{2}$$

$$\lim_{n \to \infty} \frac{1}{\delta_n^2} \frac{(n + \beta_n)(n + \alpha_n)n}{(2n + \alpha_n + \beta_n)^3} = \frac{b_1}{4}$$

$$\lim_{n \to \infty} \frac{1}{\delta_n^2} \frac{(n + \beta_n)(n + \alpha_n)(n + \alpha_n + \beta_n)n}{(2n + \alpha_n + \beta_n)^4} = \frac{b_2}{4}$$

then

$$\lim_{n \to \infty} \frac{1}{n} N^{(\alpha_n, \beta_n)}(\delta_n \xi + 2\varepsilon_n - 1) = \int_{a_2 - 2\sqrt{b_2}}^{\xi} f^{(a_1, a_2, b_1, b_2)}(x) dx$$

where the limiting density is given by

$$f^{(a_1,a_2,b_1,b_2)}(x) :=$$

$$\begin{cases} \dfrac{b_1}{2\pi} \dfrac{\sqrt{4b_2-(x-a_2)^2}}{(b_2-b_1)x^2+(b_1a_2+b_1a_1-2b_2a_1)x+b_2a_1^2-a_1a_2b_1+b_1^2} & \text{if } |x-a_2| \le 2\sqrt{b_2} \\ 0 & \text{else.} \end{cases}$$

Proof: Let μ_n denote the uniform distribution on the set

$$\left\{ \frac{x-2\varepsilon_n+1}{\delta_n} \,\middle|\, P_n^{(\alpha_n,\beta_n)}(x)=0 \right\}$$

then we obtain from (2.2) and Proposition 2.1 for the corresponding Stieltjes transform

$$\begin{aligned} \Phi_{\mu_n}(z) &= \int \frac{d\mu_n(x)}{z-x} = \delta_n \int \frac{d\mu^{(\alpha_n,\beta_n)}(x)}{\delta_n z+2\varepsilon_n-1-x} \\ &= \frac{1 \mid}{|z-\eta_1^{(n)}} - \frac{\lambda_1^{(n)} \mid}{|z-\eta_2^{(n)}} - \ldots - \frac{\lambda_{n-1}^{(n)} \mid}{|z-\eta_n^{(n)}} \end{aligned}$$

where $p_0^{(n)}=0,\ p_{-1}^{(n)}=0$,

$$\lambda_j^{(n)} = \frac{4}{\delta_n^2}(1-p_{2j-2}^{(n)})p_{2j-1}^{(n)}(1-p_{2j-1}^{(n)})p_{2j}^{(n)} \qquad (j=1,\ldots,n-1),$$

$$\eta_j^{(n)} = \frac{2}{\delta_n}\left\{(1-p_{2j-3}^{(n)})p_{2j-2}^{(n)}+(1-p_{2j-2}^{(n)})p_{2j-1}^{(n)}-\varepsilon_n\right\} \qquad (j=1,\ldots,n)$$

and the $p_j^{(n)}$ are defined in (2.3). From the limit assumptions of the theorem and (2.3) it can easily be seen that

$$\begin{aligned} \lim_{n\to\infty}\lambda_1^{(n)} &= b_1 \qquad \lim_{n\to\infty}\lambda_j^{(n)} = b_2 \qquad (j\ge 2), \\ \lim_{n\to\infty}\eta_1^{(n)} &= a_1 \qquad \lim_{n\to\infty}\eta_j^{(n)} = a_2 \qquad (j\ge 2). \end{aligned}$$

Because the quantities in the continued fraction expansion of the Stieltjes transform are bijective continuous functions of the moments of the given distribution we obtain that the moments of μ_n converge to the moments of a distribution μ with Stieltjes transform

$$\begin{aligned} \Phi_\mu(z) &= \frac{1 \mid}{|z-a_1} - \frac{b_1 \mid}{|z-a_2} - \frac{b_2 \mid}{|z-a_2} - \frac{b_2 \mid}{|z-a_2} - \ldots \\ &= \frac{1}{2}\frac{(2b_2-b_1)z+(b_1a_2-2b_2a_1)-b_1\sqrt{(z-a_2)^2-4b_2}}{(b_2-b_1)z^2+z(b_1a_2+b_1a_1-2b_2a_1)+b_2a_1^2-a_1a_2b_1+b_1^2} \end{aligned}$$

(here the last equality follows by straightforward algebra). From Theorem 40 of Nevai (1979, p. 143) we have that μ has a density in the interval $[a_2 - 2\sqrt{b_2}, a_2 + 2\sqrt{b_2}]$ and by the inversion formula for Stieltjes transforms (see e.g. Perron (1954)) this density is given by

$$-\frac{1}{\pi}\mathrm{Im}(\Phi_\mu(x)) = f^{(a_1,a_2,b_1,b_2)}(x) \qquad \text{if } x \in [a_2 - 2\sqrt{b_2}, a_2 + 2\sqrt{b_2}].$$

Therefore the limit distribution μ is determined by its moments and it follows from well known results of probability theory (see e.g. Feller (1966), p. 263) that μ_n converges weakly with limit μ, that is

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n,\beta_n)}(\delta_n \xi + 2\varepsilon_n - 1) = \lim_{n\to\infty} \mu_n(\xi) = \mu(\xi) = \int_{a_2 - 2\sqrt{b_2}}^{\xi} f^{(a_1,a_2,b_1,b_2)}(x)dx$$

■

In the following we will apply Theorem 2.2 to special sequences α_n, β_n which might be of interest in applications. All results can be proved by straightforward (but sometimes tedious) calculations checking the conditions of Theorem 2.2 and the proofs are therefore omitted. Corollary 2.3 has already been established in Gawronski and Shawyer (1991) as an application of strong asymptotic results for the Jacobi polynomials, the other results are new.

Corollary 2.3. Let $\lim_{n\to\infty} \frac{\alpha_n}{n} = a$ and $\lim_{n\to\infty} \frac{\beta_n}{n} = b$ $(a, b \geq 0)$ then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n,\beta_n)}(\xi) = \frac{2+a+b}{2\pi} \int_{r_1}^{\xi} \frac{\sqrt{(r_2 - x)(x - r_1)}}{1 - x^2} dx \qquad r_1 \leq \xi \leq r_2$$

where

$$r_{1/2} := \frac{b^2 - a^2 \pm 4\sqrt{(a+1)(b+1)(a+b+1)}}{(2+a+b)^2} .$$

Corollary 2.4. Let $\lim_{n\to\infty} \frac{\alpha_n}{n} = \infty$, $\lim_{n\to\infty} \frac{\beta_n}{n} = \infty$ and $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = c > 0$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n,\beta_n)} \left(\sqrt{\frac{n}{\alpha_n}} \xi - \frac{\alpha_n - \beta_n}{\alpha_n + \beta_n} \right) = \frac{2}{\pi\sigma^2} \int_{-\sigma}^{\xi} \sqrt{\sigma^2 - x^2} dx \qquad |\xi| \leq \sigma$$

where $\sigma = 4c/(1+c)^{3/2}$.

Corollary 2.5. Let $\lim_{n\to\infty} \frac{\alpha_n}{n} = \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{n} = b \geq 0$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n,\beta_n)}(\frac{n}{\alpha_n}\xi - 1) = \frac{1}{4\pi} \int_{s_1}^{\xi} \frac{\sqrt{(s_2-x)(x-s_1)}}{x} dx \qquad s_1 \leq \xi \leq s_2$$

where $s_{1/2} = 2(2+b) \pm 4\sqrt{1+b}$.

Corollary 2.6. Let $\lim_{n\to\infty} \frac{\alpha_n}{n} = \infty$, $\lim_{n\to\infty} \frac{\beta_n}{n} = \infty$ and $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n,\beta_n)} \left(\frac{\sqrt{n\beta_n}}{\alpha_n}\xi - \frac{\alpha_n + 2\sqrt{n\beta_n} - \beta_n}{2n + \alpha_n + \beta_n} \right) = \frac{1}{8\pi} \int_{-2}^{\xi} \sqrt{(6-x)(x+2)} dx$$

for all $-2 \leq \xi \leq 6$.

Example 2.7. Letting $\alpha_n = n^4, \beta_n = n^3$ in Corollary 2.6 we have that

$$\lim_{n\to\infty} \frac{1}{n} N^{(n^5,n^3)} \left(\frac{\xi}{n^2} - \frac{n^3 - n^2 + 2n}{n^3 + n^2 + 2} \right) = \frac{1}{8\pi} \int_{-2}^{\xi} \sqrt{(6-x)(x+2)}\, dx$$

for all $-2 \leq \xi \leq 6$ which can be easily rewritten as

$$\lim_{n\to\infty} \frac{1}{n} N^{(n^5,n^3)} \left(\frac{4}{n^2}\xi + \frac{2}{n^2} - \frac{n^3 - n^2 + 2n}{n^3 + n^2 + 2} \right) = \frac{2}{\pi} \int_{-1}^{\xi} \sqrt{1-x^2} dx. \tag{2.4}$$

It should be noted that the location sequences $2\varepsilon_n - 1$ in Theorem 2.2 can be changed by an addition of an amount g_n, and maintain the same limit, only if $g_n/\delta_n \to 0$. This means that the limit distribution of $\frac{1}{n}N^{(n^5,n^3)}(\frac{4}{n^2}\xi - 1)$ is not the same as (2.4). We can however simplify this to

$$\lim_{n\to\infty} \frac{1}{n} N^{(n^5,n^3)} \left(\frac{4\xi}{n^2} - 1 + \frac{2n-2}{n^2} \right) = \frac{2}{\pi} \int_{-1}^{\xi} \sqrt{1-x^2} dx$$

■

Remark 2.8 The results of Corollary 2.3 – 2.6 can be motivated heuristically in the following way. As an example consider the situation of Corollary 2.4 for $\alpha_n = \beta_n$ (i.e. $c = 1$). Because the polynomials $P_n^{(\alpha_n,\alpha_n)}(\sqrt{n/\alpha_n}x)$ are orthogonal on the interval $[-\sqrt{\alpha_n/n}, \sqrt{\alpha_n/n}]$ with respect to the weight function

$$\left(1 - \frac{n}{\alpha_n}x^2\right)^{\alpha_n} \approx e^{-nx^2}$$

we may expect that the uniform distribution on the set

$$\left\{\sqrt{\frac{\alpha_n}{n}}x \mid P_n^{(\alpha_n,\alpha_n)}(x) = 0\right\}$$

has the same limit behaviour as the uniform distribution on the set

$$\{\frac{x}{\sqrt{n}} \mid H_n(x) = 0\}$$

(here $H_n(x)$ denotes the n-th Hermite polynomial). The weak limit of this distribution is known to be the measure with density $\pi^{-1}\sqrt{2-x^2}$ on the interval $[-\sqrt{2}, \sqrt{2}]$ (see e.g. Dette and Studden (1992)).

The case $\alpha_n \neq \beta_n$ is similar. Thus the Jacobi weight function when properly scaled is approximately the Hermite weight function, under the conditions of Corollary 2.4. If α_n and β_n go to infinity faster than n we might except the zeros of $P_n^{(\alpha_n \beta_n)}(x)$ to behave like the Hermite polynomials when properly scaled.

If a and b are both zero in Corollary 2.3 the limit density is the classical arc-sin distribution. So if α_n and β_n both go to infinity slower than n as in Corollary 2.3, then the zero of $P_n^{(\alpha_n,\beta_n)}(x)$ behave like the arc-sin distribution. The limit distributions in Corollary 2.5 and 2.6 are somewhat similar. The limit distribution in Corollary 2.5 is like the Laguerre case and Corollary 2.6 is like the Hermite. ■

3. The asymptotic distribution of the zeros of the generalized Laguerre and Hermite polynomials.

Throughout this section let

$$N^{(\alpha_n)}(\xi) := \#\left\{ x \mid L_n^{(\alpha_n)}(x) = 0,\ x \leq \xi\right\}$$
$$N^{(\gamma_n)}(\xi) := \#\left\{ x \mid H_n^{(\gamma_n)}(x) = 0,\ x \leq \xi\right\}$$

denote the number of zeros of the generalized Laguerre and Hermite polynomials less or equal than ξ, respectively. The following theorem can be proved by similar arguments as in Secton 2 using the corresponding characterization for the Laguerre polynomials in Dette and Studden (1992). Its first part was also derived by Gawronski (1993) as an application of strong asymptotics for the generalized Laguerre polynomials and is given here for the sake of completeness.

Theorem 3.1.

a) Let $\lim_{n\to\infty} \frac{\alpha_n}{n} = a \geq 0$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n)}(n\xi) = \frac{1}{2\pi} \int_{r_1}^{\xi} \frac{\sqrt{(r_2 - x)(x - r_1)}}{x} dx \qquad r_1 \leq \xi \leq r_2$$

where $r_{1/2} = 2 + a \pm 2\sqrt{1+a}$

b) Let $\lim_{n\to\infty} \frac{\alpha_n}{n} = \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\alpha_n)}(\sqrt{n\alpha_n}\xi + \alpha_n) = \frac{1}{2\pi} \int_{-2}^{\xi} \sqrt{4 - x^2} dx \qquad |\xi| \leq 2.$$

The corresponding results for the generalized Hermite polynomials can easily be derived from Theorem 3.1 and the relations ($n \geq 0$)

$$\begin{aligned} H_{2n}^{(\gamma_n)}(x) &= (-1)^k 2^{2k} k! L_k^{(\gamma_n - 1/2)}(x^2) \\ H_{2n+1}^{(\gamma_n)}(x) &= (-1)^k 2^{2k+1} k! x L_k^{(\gamma_n - 1/2)}(x^2) \end{aligned}$$

(see e.g. Chihara (1978)).

Theorem 3.2.

a) Let $\lim_{n\to\infty} \frac{\gamma_n}{n} = c \geq 0$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\gamma_n)}(\sqrt{n}\xi) = \int_{-\sigma}^{\xi} f(x) dx \qquad -\sigma \leq \xi \leq \sigma$$

where $\sigma = \sqrt{1 + c + \sqrt{1+2c}}$, $\rho = \sqrt{1 + c - \sqrt{1+2c}}$ and

$$f(x) = \begin{cases} \frac{1}{\pi} \frac{\sqrt{(x^2 - \rho^2)(\sigma^2 - x^2)}}{|x|} & \text{if } \rho < |x| < \sigma \\ 0 & \text{else} \end{cases}$$

b) Let $\lim\limits_{n\to\infty} \frac{\gamma_n}{n} = \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} N^{(\gamma_n)} \left(\sqrt{\sqrt{\gamma_n n}\xi^2 + \gamma_n}\right) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\xi} x\sqrt{2 - x^4}dx \qquad 0 \le \xi \le \sqrt[4]{2}\,.$$

Remark 3.3. The first part of Theorem 3.2 can also be found in Gawronski (1993). It is also worthwile to mention that in contrary to all other results in this paper the normalizing sequence in Theorem 3.2b) is nonlinear in the sense that the argument of $N^{(\gamma_n)}(\cdot)$ is not a linear function of ξ. ■

4. Asymptotics for the largest and smallest zero. Using the Sturm Comparison Theory, Moak, Saff and Varga (1979) proved the following limit theorem for the largest and smallest zero of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$ (a different proof can be found in a recent paper of Ismail and Li (1992) using general results for bounds of the extreme zeros of orthogonal polynomials).

Theorem (Moak, Saff, Varga (1979)) Let $r_n^{(\alpha_n,\beta_n)}$ and $s_n^{(\alpha_n,\beta_n)}$ denote, respectively, the smallest and largest zeros of the generalized Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$. If

$$\lim_{n\to\infty} \frac{\alpha_n}{2n+\alpha_n+\beta_n} = A \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_n}{2n+\alpha_n+\beta_n} = B,$$

then

$$\lim_{n\to\infty} r_n^{(\alpha_n,\beta_n)} = r_{A,B} \quad \text{and} \quad \lim_{n\to\infty} s_n^{(\alpha_n,\beta_n)} = s_{A,B}$$

where

$$r_{A,B}/s_{A,B} = B^2 - A^2 \pm \sqrt{(A^2+B^2-1)^2 - 4A^2B^2}$$

Furthermore the zeros of the sequence $\{P_n^{(\alpha_n,\beta_n)}(x)\}_{n=0}^{\infty}$ are dense in the interval $[r_{a,b}, s_{a,b}]$.

In the cases considered in Corollary 2.4, 2.5 and 2.6 it can easily be shown that $A+B = 1$ and the interval $[r_{A,B}, s_{A,B}]$ degenerates to the points -1, $\frac{1-c}{1+c}$ and -1, respectively. The results of the previous sections suggest that we can find similar statements as in the theorem of Moak, Saff and Varga (1979) when the sequences α_n and/or β_n converge to infinity with a larger order than n, provided that the zeros of the orthogonal polynomials are standardized appropriately. In fact we have the following second order approximations for the zeros of the generalized Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$ when $\alpha_n/n \to +\infty$ and/or $\beta_n/n \to +\infty$.

Theorem 4.1. Let $r_n^{(\alpha_n,\beta_n)}$ and $s_n^{(\alpha_n,\beta_n)}$ denote, respectively, the smallest and largest zero of the generalized Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, assume that $\lim\limits_{n\to\infty} \frac{\alpha_n}{n} = \infty$, $\lim\limits_{n\to\infty} \frac{\beta_n}{n} = \infty$ and $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = c > 0$, then

$$\lim_{n\to\infty} \sqrt{\frac{\alpha_n}{n}} \left\{ r_n^{(\alpha_n,\beta_n)} + \frac{\alpha_n - \beta_n}{\alpha_n + \beta_n} \right\} = -\frac{4c}{(1+c)^{3/2}}$$

$$\lim_{n\to\infty} \sqrt{\frac{\alpha_n}{n}} \left\{ s_n^{(\alpha_n,\beta_n)} + \frac{\alpha_n - \beta_n}{\alpha_n + \beta_n} \right\} = \frac{4c}{(1+c)^{3/2}}.$$

Furthermore, the zeros of

$$P_n^{(\alpha_n,\beta_n)}\left(\sqrt{\frac{n}{\alpha_n}}\left[x + \frac{\beta_n - \alpha_n}{\beta_n + \alpha_n}\right]\right)$$

become dense in the interval $[-4c(1+c)^{-3/2}, 4c(1+c)^{-3/2}]$.

Proof: By Theorem 2 of Ismail and Li (1992) we obtain for the largest zero $s_n^{(\alpha_n,\beta_n)}$ of $P_n^{(\alpha_n,\beta_n)}(x)$ the upper bound

$$s_n^{(\alpha_n,\beta_n)} \leq \frac{\beta_n - \alpha_n}{\beta_n + \alpha_n} + \frac{1}{2 + \alpha_n + \beta_n}\sqrt{g(\alpha_n,\beta_n) + h(\alpha_n,\beta_n)} \tag{4.1}$$

where

$$g(\alpha_n,\beta_n) := \left(\frac{\beta_n - \alpha_n}{\beta_n + \alpha_n}\right)^2$$

and

$$h(\alpha_n,\beta_n) := \frac{16n(n+\alpha_n)(n+\beta_n)(n+\alpha_n+\beta_n)}{(\alpha_n+\beta_n+1)(\alpha_n+\beta_n+3)}.$$

Straightforward algebra now yields that

$$\lim_{n\to\infty} \frac{\alpha_n\beta_n}{(2+\alpha_n+\beta_n)^2} \cdot g(\alpha_n,\beta_n) = \frac{c(c-1)^2}{(c+1)^4} \quad , \tag{4.2}$$

$$\lim_{n\to\infty} \frac{\alpha_n}{n}\frac{1}{(2+\alpha_n+\beta_n)^2} \cdot h(\alpha_n,\beta_n) = \frac{16c^2}{(1+c)^3} \quad , \tag{4.3}$$

and combining (4.1), (4.2) and (4.3) we thus obtain

$$\limsup_{n\to\infty} \sqrt{\frac{\alpha_n}{n}} \left\{ s_n^{(\alpha_n,\beta_n)} + \frac{\alpha_n - \beta_n}{\alpha_n + \beta_n} \right\} \leq \frac{4c}{(1+c)^{3/2}}. \tag{4.4}$$

It is clear from Corollary 2.4 that equality must hold in (4.4); which proves the assertion regarding the largest zero. The corresponding statement for the smallest zero $r_n^{(\alpha_n,\beta_n)}$ follows from the well known relation $P_n^{(\alpha,\beta)}(x) = P_n^{(\beta,\alpha)}(-x)$ and the first part. Finally the statement regarding the denseness of the zeros also follows from Corollary 2.4.
■

In the following theorems we consider the remaining cases of Corollary 2.5 and 2.6 for the Jacobi polynomials and the corresponding results for the generalized Laguerre and Hermite polynomials. The proofs are similar to that of Theorem 4.1 and therefore omitted.

Theorem 4.2. Let $r_n^{(\alpha_n,\beta_n)}$ and $s_n^{(\alpha_n,\beta_n)}$ denote, respectively, the smallest and largest zero of the generalized Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$ and assume that $\lim_{n\to\infty} \frac{\alpha_n}{n} = \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{n} = b \geq 0$, then

$$\lim_{n\to\infty} \frac{\alpha_n}{n} \{ s_n^{(\alpha_n,\beta_n)} + 1 \} = 2(2+b) + 4\sqrt{1+b}$$

$$\lim_{n\to\infty} \frac{\alpha_n}{n} \{ r_n^{(\alpha_n,\beta_n)} + 1 \} = 2(2+b) - 4\sqrt{1+b}$$

Furthermore, the zeros of

$$P_n^{(\alpha_n,\beta_n)}\left(\frac{n}{\alpha_n}[x-1] \right)$$

become dense in the interval $[2(2+b) - 4\sqrt{1+b}, 2(2+b) + 4\sqrt{1+b}]$.

Theorem 4.3. Let $r_n^{(\alpha_n,\beta_n)}$ and $s_n^{(\alpha_n,\beta_n)}$ denote, respectively, the smallest and largest zero of the generalized Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, assume that $\lim_{n\to\infty} \frac{\alpha_n}{n} = \infty$, $\lim_{n\to\infty} \frac{\beta_n}{n} = \infty$, $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = \infty$ and let $\varepsilon_n = (\alpha_n + 2\sqrt{n\beta_n} - \beta_n)/(2n + \alpha_n + \beta_n)$, then

$$\lim_{n\to\infty} \frac{\alpha_n}{\sqrt{n\beta_n}} \{ s_n^{(\alpha_n,\beta_n)} + \varepsilon_n \} = 6$$

$$\lim_{n\to\infty} \frac{\alpha_n}{\sqrt{n\beta_n}} \{ r_n^{(\alpha_n,\beta_n)} + \varepsilon_n \} = -2.$$

Furthermore, the zeros of the sequence

$$P_n^{(\alpha_n,\beta_n)}\left(\frac{\sqrt{n\beta_n}}{\alpha_n}[x-\varepsilon_n]\right)$$

become dense in the interval $[-2,6]$.

Theorem 4.4. Let $r_n^{(\alpha_n)}$ and $s_n^{(\alpha_n)}$, respectively, denote the smallest and largest zero of the generalized Laguerre polynomials $L_n^{(\alpha_n)}(x)$.

a) Assume that $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=a\geq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\frac{s_n^{(\alpha_n)}}{n} &= 2+a+2\sqrt{1+a}\\ \lim_{n\to\infty}\frac{r_n^{(\alpha_n)}}{n} &= 2+a-2\sqrt{1+a}\end{aligned}$$

Furthermore, the zeros of $L_n^{(\alpha_n)}(nx)$ are dense in the interval $[2+a-2\sqrt{1+a},2+a+2\sqrt{1+a}]$.

b) Assume that $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=\infty$, then

$$\begin{aligned}\lim_{n\to\infty}\frac{s_n^{(\alpha_n,\beta_n)}-\alpha_n}{\sqrt{n\alpha_n}} &= 2\\ \lim_{n\to\infty}\frac{r_n^{(\alpha_n,\beta_n)}-\alpha_n}{\sqrt{n\alpha_n}} &= -2\end{aligned}$$

Furthermore, the zeros of $L_n^{(\alpha_n)}(\sqrt{n\alpha_n}x+\alpha_n)$ become dense in the interval $[-2,2]$.

Theorem 4.5. Let ρ_n^+ (ρ_n^-) and σ_n^+ (σ_n^-) denote, respectively, the smallest and largest positive (negative) zero of the generalized Hermite polynomials $H_n^{(\gamma_n)}(x)$.

a) Assume that $\lim\limits_{n\to\infty}\frac{\gamma_n}{n}=c\geq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\frac{\sigma_n^+}{\sqrt{n}} &= -\lim_{n\to\infty}\frac{\rho_n^-}{\sqrt{n}} = \sqrt{1+c+\sqrt{1+2c}} = \sigma\\ \lim_{n\to\infty}\frac{\rho_n^+}{\sqrt{n}} &= -\lim_{n\to\infty}\frac{\sigma_n^-}{\sqrt{n}} = \sqrt{1+c-\sqrt{1+2c}} = \rho\end{aligned}$$

Furthermore, the zeros of $H_n^{(\gamma_n)}(\sqrt{n}x)$ become dense in $[-\sigma,-\rho]\cup[\rho,\sigma]$.

b) Assume that $\lim_{n\to\infty} \frac{\gamma_n}{n} = \infty$, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\sigma_n^+)^2 - \gamma_n}{\sqrt{n\gamma_n}} &= \lim_{n\to\infty} \frac{(\rho_n^-)^2 - \gamma_n}{\sqrt{n\gamma_n}} = \sqrt{2} \\
\lim_{n\to\infty} \frac{(\sigma_n^-)^2 - \gamma_n}{\sqrt{n\gamma_n}} &= \lim_{n\to\infty} \frac{(\rho_n^-)^2 - \gamma_n}{\sqrt{n\gamma_n}} = 0
\end{aligned}$$

Furthermore, the zeros of $H_n(\sqrt{\sqrt{n\gamma_n}x^2 + \gamma_n})$ become dense in the interval $[-\sqrt{2}, \sqrt{2}]$.

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