A REVIEW OF THE BOOTSTRAP FOR DEPENDENT SAMPLES

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Abstract

This paper reviews the current state of the art of the bootstrap procedure as it applies primarily to dependent models. There is no unique way of implementing the bootstrap paradigm in dependent situations. We take a look at several approaches, many of which are extremely recent and have proven useful in estimating even parameters of the infinite dimensional stationary distribution, such as the spectral density function evaluated at a point. We provide a reasonably extensive bibliography and offer possible directions for future studies in this exciting and often technically difficult area of research.

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1. Introduction to bootstrap procedures

Bootstrap perhaps begun as a method to estimate standard error of estimates but has evolved into a method of estimating any particular feature of a statistic (or more generally any functional of the data and the underlying distribution) including the entire sampling distribution. To understand the procedure we will first concentrate on the situation where the observations are univariate, independent, and identically distributed (i.i.d.).

Suppose $X_1, \ldots, X_n$ are i.i.d. from a distribution F which may or may not belong in some parametric family of distributions. Let $T_n = T(X_1, \ldots X_n)$ be any statistic estimating the parameter $T(F)$. The function T may be as complicated as we please. Suppose the problem is to estimate some feature of T_n, such as the variance, percentile or even its entire probability law which we will denote by $L_T(F)$ as it depends on F.

Let F_n be some reasonable estimate of F. For example if we know that F is normal then F_n could be $N(\bar{x}, s^2)$ where $\bar{x}$ is the sample mean and s^2 is the sample variance. If on the other hand we are in the so-called nonparametric situation and no distributional properties of F are known then F_n could be the *empirical distribution function (edf)* of $X_1, \ldots X_n$. That is, F_n puts mass $1/n$ at each X_i. If the smoothness of F is known then we may use a smoothened form of the *edf* as F_n. The bootstrap approximation of $L_T(F)$ is just $L_T(F_n)$. To explain this, let $X_1^*, \ldots X_n^*$ be i.i.d. observations drawn from F_n. This is usually called the *bootstrap* sample. (Note that this sample can be drawn on the computer). In particular when F_n is the *edf* then there are n^n possible values of the bootstrap sample. Consider the distribution of $T^* = T(X_1^*, \ldots X_n^*)$ *given* $X_1, \ldots X_n$. In principle this distribution is completely known. Now, any distributional feature of the statistic T_n may be approximated by that of T^*. As a case in point, the variance of T_n may be estimated by the variance of T^*. In general, for future use, we mention that *the bootstrap quantities will always be denoted with an asterisk* *. (We may mention that in some situations it is appropriate to choose the bootstrap sample size to be different from n. This issue has deep theoretical significance, apart from the practical effect on the bootstrap algorithm. See Politis and Romano (1992b) for some situations where a different bootstrap sample size is necessitated). However, to completely compute it may require enormous computer time. For example, in the nonparametric case, this requires n^n computations. The following repeated sampling is in turn used to approximate the distribution of T^*. Draw m independent bootstrap samples as described above and compute m values of T^*, say $T_1^*, \ldots, T_m^*$. The empirical distribution of these values approximates the distribution of T^*. For example the variance of T_n is approximated by that of T^*. This, in turn, is approximated by $\sum_{i=1}^{m}(T_i^* - \bar{T}^*)^2/(m-1)$. Note that the method involves *resampling* from the data and is indeed one of the many methods known to use the sample values to create pseudovalues of T for estimation or inference purposes. Some other methods include the jackknife, halfsampling, random subsampling, etc.; see Efron (1982) for description and examples of these other methods.. It was the work of Efron (1979, 1982) that really spurred a detailed look at the procedure, both from theoretical and practical standpoints.

It is clear from the above description that the basic recipe for implementing the bootstrap procedure may be described as follows:

(a) Express T as a function of $X_1, \ldots, X_n$.

(b) Find an estimate of F, say F_n.

(c) Draw m replicated i.i.d. samples of size n from F_n and compute m pseudovalues $T_1^* \ldots, T_m^*$.

An estimate of the desired quantity is obtained by using the "pseudovalues" from (c) pretending that these are values of T.

It is clear that the success of this approach depends on the following:

(A) How many replications m have been taken?

(B) How good is F_n as an estimate of F?

(C) How smooth is $L_T(F)$ as a function of F?

Generally speaking, the issue in (A) is an issue of the law of large numbers taking hold. If $m = \infty$ then the error due to replication vanishes. So the larger the value of m the better. However, the choice of m will often be dictated by the available computer time. Moreover, in practice even moderate values of m yield good approximations. See Hall (1989a) for more information on this aspect of implementation of the bootstrap algorithm.

The problem in (B) has been well studied in the literature. If we have a parametric family then, of course, parametric estimates are better. In general, care has to be taken to ensure that the bootstrap world imitates the original world as closely as possible. The issue becomes more involved in dependent situations since in such situations there may be several apparently natural ways of generating the bootstrap samples; see section 2.

Example 1. Suppose F is uniform $(0, \theta)$ where θ is unknown. Let $X_1, \ldots X_n$ be i.i.d. observations from F. The maximum likelihood estimate of θ is the maximum order statistic $X_{(n)}$. To find a confidence interval for θ we may be interested in finding the distribution of $X_{(n)}$. Note that here the natural pivot is $T_n = n(\theta - X_{(n)})/\theta$ which is asymptotically distributed as the standard exponential. (T will be called a *pivot* if its distribution is free of parameters for every fixed n; or, more generally as n tends to ∞). How should we bootstrap T in this case? It can shown that if we bootstrap by using the *edf*, it fails. See Bickel and Freedman (1981). However if we resample by using the special nature of F, it works. Let $X_1^*, \ldots X_n^*$ be i.i.d. uniform $(0, X_{(n)})$. This choice is natural since $X_{(n)}$ is the m.l.e. of θ. A slight reflection will convince the reader that we should take $T^* = n(X_{(n)} - X_{(n)}^*)/X_{(n)}$ where $X_{(n)}^*$ is the maximum order statistic of the bootstrap sample. The reader can verify that in this case the distribution of T is exactly equal to that of T^* (given the sample $X_1, \ldots X_n$). Note that here by choosing m large we get as close to the distribution of T as we please. □

The problem in (C) has been looked into from theoretical and the applied standpoints. Simulation results have vindicated the validity of this step in a variety of situations. The accuracy question has been studied from the theoretical point in the asymptotic sense (as n tends to ∞). To explain this approach consider the following example.

Example 2. Suppose the problem is to find a confidence interval for the mean μ of an unknown F and for simplicity assume that the variance of F is known and equals σ. The approximate confidence interval based on the normal approximation for the sample mean has one obvious shortcoming, namely that it is always symmetric and cannot correct for a possible presence of skewness in the distribution of the sample mean. The bootstrap algorithm here is implemented as follows. Let $\bar{X}^*$ be the mean of the bootstrap sample. Then the distribution of $T_n = n^{1/2}(\bar{X}_n - \mu)/\sigma$ is approximated by that of $T^* = n^{1/2}(\bar{X}_n^* - \bar{X}_n)/s$ where s is the sample standard deviation. Note that s is the "population" standard deviation of the *edf* and it would be inappropriate to use the divisor σ in the description of the bootstrap approximation. We may use this distribution to set the limits of the interval instead of the normal percentiles. Note that for a fixed n this distribution may be skewed. Indeed, it has been recognized that this method captures the skewness of the population well and yields better confidence intervals. For further information on bootstrap confidence intervals and how they perform relative to other methods see Hall (1986, 1988) and Babu and Bose (1988). In the above example, it can be shown that under certain assumptions as n tends to ∞ the (conditional) distribution of T^* converges to the standard normal *almost surely*. Thus, in the limit, the bootstrap does give correct result. □

In general, if the distribution of T and T^* agree in the limit then the bootstrap procedure may be termed *consistent*. Clearly, an inconsistent bootstrap procedure is useless from our point of view. We may mention that for many situations this consistency is present for the bootstrap but not for other methods.

Example 3. The sample median may be used as an estimate of an unknown population median. Suppose T_n is the sample median on n i.i.d. observations from a distribution F, having a density f. It is desired to estimate the standard error of T_n. It may be shown that under some conditions (cf. Ghosh *et al.* (1984)) the bootstrap produces a consistent estimate whereas the (delete one) jackknife method does not. (The jackknife method can be modified to yield consistency in this problem). Note that here the asymptotic variance of T_n involves the unknown density at the population median and hence, conventional method of estimating the standard error boils down to estimating a density, which is a particularly nasty problem. □

It is clear that consistency will hold if steps (B) and (C) above go through. This is indeed the case in most situations; see Bickel and Freedman (1981). In particular it is consistent for smooth functions of sample means (for example, the mean, t-statistic, sample correlation, etc.). under moment assumptions. If these moment assumptions are violated things may go severely wrong. See Athreya (1987) for more information. In some instances the situation may be salvaged partly by choosing the resample size to be different from n, and/or by sampling *without* replacement (*subsampling*); see Politis and Romano (1992b).

However, it came as a surprise to researchers when it was established that more than consistency is true. To explain this let us look at Example 2 again. It is known that the normal approximation has error of order $O(n^{-1/2})$; this is the celebrated Berry-Esseen Theorem, see for example Feller (1968). Under sufficient assumptions (see for example Singh (1981)) it may be shown that the bootstrap approximation has an error of order $o(n^{-1/2})$ *almost surely*, that is for almost all sequences of the observations. Thus the bootstrap in this sense "beats" the normal approximation. It may be noted that the result is asymptotic and the improvement is only from $O(n^{-1/2})$ to $o(n^{-1/2})$. For more general results of this nature, see Babu and Singh (1984). However, as many simulation results have shown, the approximation is remarkably accurate in finite samples. See Efron (1982) for some examples. The probability of obtaining a sample for which the bootstrap is not accurate tends to zero since the above error bounds hold for almost all sample paths. Bose and Babu (1991) show that this probability tends to zero very fast.

It is interesting to note that the order results are typically known only in situations where the limiting distribution exists and is, in addition, normal; see Babu (1986) for one example of a nonnormal situation. A recent unpublished paper of S. Lahiri has an example where the limiting distribution does not exist but a consistency result is still valid in the sense that the difference between the bootstrap and the original distribution approaches zero as n tends to ∞.

In dependent or non-i.i.d. situations, it is harder to do conventional asymptotics. Even more critical is the fact that these asymptotic approximations often perform poorly in finite samples. Thus the bootstrap emerges as a potentially powerful tool in dependent situations. The consistency of the bootstrap and accuracy results as above are of course harder to establish in dependent situations. What complicates the matter is that there may be more than one natural choice of implementing the bootstrap.

Example 4. Suppose X_i's are observations from the linear model

$$X_i = \alpha + \beta Y_i + \epsilon_i$$

where the ϵ_i's are i.i.d. with mean zero and Y_i are known and nonrandom. Suppose it is desired to approximate the distribution of the least squares estimate β_n of β. Since the X_i's are not i.i.d. it is meaningless to do resampling from these. One could here calculate

$$\hat{\epsilon}_i = X_i - \beta_n Y_i$$

and proceed as if these are i.i.d. Let F_n be their *edf* (corrected to make it zero mean). Draw ϵ_i^* i.i.d. from F_n and generate X_i^* by the model using β_n as the "true" value of β. The least squares estimate β_n^* based on the bootstrap X_i^* is then a pseudovalue of β_n. This method does not preserve the pairing of Y_i with the ϵ_i. One could device a bootstrap which does preserve it by choosing $X_i^* = \beta_n Y_j + \epsilon_i^*$ if $\epsilon_i^* = \hat{\epsilon}_j$. See Freedman (1981) for general bootstrap consistency results in linear models of the above form. □

Example 5. A very common time series model is the autoregression model (of order one) given by

$$X_i = \beta X_{i-1} + \epsilon_i$$

where the ϵ_i's are i.i.d. with mean zero. The bootstrap here can be implemented as in the previous example by making obvious changes. See Freedman (1981) for consistency results in this model when $|\beta| < 1$ and Basawa *et al.* (1989) for the case $|\beta| \geq 1$. See Bose (1988) for higher order results in the former case.

It may be observed that the bootstrap in the above two examples were implemented by carefully preserving the structure of the model. Any naive application pretending that the observations are i.i.d. when in fact they are not, can be disastrous. □

Example 6. The moving average model (of order one) used in time series is given by

$$X_i = \epsilon_i + \beta \epsilon_{i-1}$$

where the ϵ_i's are i.i.d. with mean zero. The sample mean of $X_1, \dots X_n$ has an asymptotic normal distribution. If we take F_n as the empirical distribution of the X_i's and generate the bootstrap sample by drawing an i.i.d. re-sample from F_n and use the distribution of this sample mean to approximate the distribution of the original sample mean, it is not even asymptotically correct. (See Singh (1981) for the details of the asymptotics). See Bose (1990) for accuracy results for a properly implemented bootstrap procedure. This paper also has simulation results for this model and the autoregression model discussed earlier. □

2. Model-free dependent samples

Although Examples 4, 5, and 6 do not correspond to i.i.d. samples, they are nevertheless reduced to an i.i.d. setting by looking at the residuals. In the most interesting case of a model-free dependent sample such a device is not available. Consequently, modifications of Efron's bootstrap have been proposed to deal with this scenario, since direct application of the nonparametric bootstrap in dependent samples is inconsistent; see Singh (1981) for example.

For the rest of this paper suppose that $X_1, \ldots, X_N$ are observations from the (strictly) stationary multivariate time series $\{X_n, n \in \mathbf{Z}\}$, where X_1 takes values in $\mathbf{R}^d$. The time series $\{X_n, n \in \mathbf{Z}\}$ is assumed to have a weak dependence structure. Specifically, the α-mixing (also called strong mixing) condition will be assumed, i.e. that $\alpha_X(k) \to 0$, as $k \to \infty$, where $\alpha_X(k) = \sup_{A,B} |P(A \cap B) - P(A)P(B)|$, and $A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{k}^{\infty}$ are events in the σ-algebras generated by $\{X_n, n \leq 0\}$ and $\{X_n, n \geq k\}$ respectively. Recently Künsch(1989) and Liu and Singh(1992) have independently proposed a *block- resampling* bootstrap that is generally consistent; related procedures were also proposed in Politis and Romano (1991, 1992c). To fix ideas, suppose that we are interested in interval estimation of the univariate ($d = 1$) mean $\mu = EX_t$, based on the sample mean $\bar{X}_N = N^{-1} \sum_{i=1}^{N} X_i$, although the procedure applies for general statistics estimating a parameter of the first-marginal distribution of the $\{X_t\}$ sequence, i.e., the distribution of X_1. The Künsch–Liu–Singh *'moving blocks'* method, can be described as follows:

- Define $\mathcal{B}_i$ to be the block of b consecutive observations starting from X_i, that is $\mathcal{B}_i = (X_i, \ldots, X_{i+b-1})$, where $i = 1, \ldots, q$ and $q = N - b + 1$. Sampling with replacement from the set $\{\mathcal{B}_1, \ldots, \mathcal{B}_q\}$, defines a (conditional on the original data) probability measure P^* which is used in the 'moving blocks' bootstrap procedure. If k is an integer such that $kb \sim N$, then letting $\xi_1, \ldots, \xi_k$ be drawn i.i.d. from P^*, it is seen that each ξ_i is a block of b observations $(\xi_{i,1}, \ldots, \xi_{i,b})$. If all $l = kb$ of the $\xi_{i,j}$'s are concatenated in one long vector denoted by $Y_1, \ldots, Y_l$, then the 'moving blocks' bootstrap estimate of the variance of $\sqrt{N}\bar{X}_N$ is the variance of $\sqrt{l}\bar{Y}_l$ under P^*, and the 'moving blocks' bootstrap estimate of $P\{\sqrt{N}(\bar{X}_N - \mu) \leq x\}$ is $P^*\{\sqrt{l}(\bar{Y}_l - \bar{X}_N) \leq x\}$, where $\bar{Y}_l = \frac{1}{l}\sum_{i=1}^{l} Y_i$.

As a final step, confidence intervals for μ can be obtained either by means of the Central Limit Theorem using the 'moving blocks' bootstrap estimate of variance, or by approximating the quantiles of the distribution $P\{\sqrt{N}(\bar{X}_N - \mu) \leq x\}$ by the corresponding quantiles of $P^*\{\sqrt{l}(\bar{Y}_l - \bar{X}_N) \leq x\}$. If P^* probabilities turn out to be cumbersome to analytically calculate, one can always resort to Monte Carlo, i.e. drawing a large number of samples $\xi_1^{(j)}, \ldots, \xi_k^{(j)}$ i.i.d. from P^*, where $j = 1, \ldots, J$, and evaluating the required probabilities or quantiles empirically from the Monte Carlo set of the J *re-samples.* It is obvious that taking $b = 1$ makes the 'moving blocks' bootstrap coincide with the classical (i.i.d.) bootstrap of Efron(1979).

It can be shown (cf. Lahiri(1992)) that a slightly modified 'moving blocks' bootstrap estimate of sampling distribution turns out to be more accurate than the normal approximation, under some regularity conditions, resulting to more accurate confidence intervals for μ. The modification amounts to approximating the quantiles of $P\{\sqrt{N}(\bar{X}_N - \mu) \leq x\}$ by the corresponding quantiles of $P^*\{\sqrt{l}(\bar{Y}_l - E^*\bar{Y}_l) \leq x\}$, where $E^*\bar{Y}_l$ denotes the expected value of $\bar{Y}_l$ under the P^* probability (conditional on the original data).

In Politis and Romano(1992a,d), the *'blocks of blocks'* resampling scheme was introduced, in order to address the problem of setting confidence intervals for parameters associated with the whole (infinite-dimensional) distribution of the $X_1, X_2, \ldots$ observations, and not just a finite-dimensional marginal. A prime example of such a parameter is the spectral density function of the $\{X_n\}$ sequence, evaluated at a point. As a by-product, the 'blocks of blocks' method also provides more accurate confidence intervals for parameters associated with a finite-dimensional distribution of the observations, as compared to confidence intervals obtained by the normal approximation. Examples of such parameters include the autocovariance $Cov(X_0, X_s)$ and the autocorrelation $Cov(X_0, X_s)/Var(X_0)$ at lag s. The 'blocks of blocks' scheme is a generalization of the 'moving blocks' method, and the two coincide if the parameter under consideration is the mean EX_1.

To describe the 'blocks of blocks' method, first set up the estimation problem in the following manner. Suppose $\mu \in \mathbf{R}^D$ is a parameter of the m-dimensional joint distribution of sequence $\{X_n, n \in \mathbf{Z}\}$, where m could be infinite. For each $N = 1, 2, \ldots$ let $B_{i,M,L}$ be the block of M consecutive observations starting from $(i-1)L+1$, i.e., the subseries $X_{(i-1)L+1}, \ldots, X_{(i-1)L+M}$.

where M, L are integer functions of N. Define $T_{i,M,L} = \phi_M(B_{i,M,L})$, where $\phi_M : \mathbf{R}^{dM} \to \mathbf{R}^D$ is some function. So for fixed N, the $T_{i,M,L}$ for $i \in \mathbf{Z}$ constitute a strictly stationary sequence. In practice we would observe a segment $X_1, \ldots, X_N$ from the time series $\{X_n\}$, which would permit us to compute $T_{i,M,L}$ for $i = 1, \ldots, Q$ only, where $Q = [\frac{N-M}{L}] + 1$ and $[\cdot]$ is the integer part function. Also, define the general linear statistic:

$$\bar{T}_N = \frac{1}{Q}\sum_{i=1}^{Q} T_{i,M,L} \tag{1}$$

Under broad regularity conditions $\bar{T}_N$ is a consistent estimator of μ. Loosely stated, these regularity conditions consist of a weak dependence structure (allowing the variance of $\bar{T}_N$ to tend to zero as $N \to \infty$), and a condition of unbiasedness or asymptotic unbiasedness of $T_{1,M,L}$, i.e., $ET_{1,M,L} = \mu$, or $ET_{1,M,L} \to \mu$ as $M \to \infty$.

Some examples of time series statistics that can fit in this framework are the following. For the examples assume X_n is univariate, that is $d = 1$.

Example 7. The sample mean : $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$. Just take $M = L = 1$ and ϕ_M to be the identity function. □

Example 8. The sample autocovariance at lag s: $\frac{1}{N-s}\sum_{i=1}^{N-s} X_i X_{i+s}$. Take $L = 1$, $M = s + 1$ and $\phi_M(x_1, \ldots, x_M) = x_1 x_M$. □

Example 9. The lag-window spectral density estimator, where we take

$$\phi_M(B_{i,M,L}) = \frac{1}{2\pi M}\left|\sum_{t=L(i-1)+1}^{L(i-1)+M} W_t^{(M)} X_t e^{-jtw}\right|^2 \tag{2}$$

i.e., $T_{i,M,L}(w)$ is the periodogram of block $B_{i,M,L}$ of data 'tapered' by the function $W_t^{(M)}$, and evaluated at the point $w \in [0, 2\pi]$. (Note that the symbol j denotes the unit of imaginary numbers $\sqrt{-1}$, in order to avoid confusion with i, the block count.) □

Note that in example 7, μ is just EX_1, i.e. it is a parameter of the m-dimensional marginal distribution of sequence $\{X_n, n \in \mathbf{Z}\}$, with m=1. Similarly, in example 8, $\mu = EX_0X_s$ is a parameter of the m-dimensional marginal, with m=$s+1$, and in example 9, μ is the spectral density evaluated at the point w, i.e. a parameter of the whole (infinite-dimensional) joint distribution of $\{X_n, n \in \mathbf{Z}\}$.

With the objective of setting confidence intervals for μ, the 'blocks of blocks' bootstrap procedure goes as follows:

- Define $\mathcal{B}_{j,b}$ to be the block of b consecutive $T_{i,M,L}$'s starting from $T_{j,M,L}$; that is, let $\mathcal{B}_{j,b} = (T_{j,M,L}, \ldots, T_{j-1+b,M,L})$. Note that there are $q = Q - b + 1$ such $\mathcal{B}_{j,b}, j = 1, \ldots, q$. Sampling with replacement from the set $\{\mathcal{B}_{1,b}, \ldots, \mathcal{B}_{q,b}\}$ defines (conditionally on the original observations $X_1, \ldots, X_N$) a probability measure denoted by P^*, which is used in the 'blocks of blocks' bootstrap procedure. Let $Y_1, \ldots, Y_k$ be i.i.d. samples from P^*, where k is of the same asymptotic order as Q/b, (for instance, let $k = [Q/b] + 1$). Obviously, each Y_i is a block of size b which we denote as $Y_i = (y_{i1}, \ldots, y_{ib})$. Let us concatenate the y_{ij} in one long vector of size $l = kb$ denoted by $T_1^*, \ldots, T_l^*$, where $T_i^* = y_{rv}$, for $r = [i/b], v = i - br$. Now both $P^*\{\sqrt{l}(\bar{T}_l^* - \bar{T}_N) \leq x\}$ and $P^*\{\sqrt{l}(\bar{T}_l^* - E^*\bar{T}_l^*) \leq x\}$ constitute 'blocks of blocks' bootstrap estimates of $P\{\sqrt{Q}(\bar{T}_N - \mu) \leq x\}$, where $\bar{T}_l^* = \frac{1}{l}\sum_{i=1}^{l} T_i^*$, and the variance matrix of $\sqrt{l}\bar{T}_l^*$ under the P^* probability constitutes the 'blocks of blocks' bootstrap estimate of the variance matrix of $\sqrt{Q}\bar{T}_N$.

Under mixing and moment conditions, consistency of the 'blocks of blocks' bootstrap estimate of sampling distribution was proved in Politis and Romano(1992a,d) in the general case (where m might be infinite). It is interesting to note that if μ is a parameter of the m-dimensional marginal distribution of sequence $\{X_n\}$, with m finite, then M could be taken to be a *fixed* constant equal to m, and L can be taken equal to one in the 'blocks of blocks' procedure. In this case, and under some additional regularity conditions (including that $ET_{1,M,L} = \mu$, and that $\alpha(k)$ has an exponential decay), it has been proved (Lahiri(1992), Politis and Romano(1992a,d)) that the approximation provided by equation

$$P^*\{\sqrt{l}(\bar{T}_l^* - E^*\bar{T}_l^*) \leq x\} \simeq P\{\sqrt{Q}(\bar{T}_N - \mu) \leq x\} \tag{3}$$

that holds for any real x, is more than first-order accurate. This fact establishes that the'blocks of blocks' bootstrap approximation is preferrable to the normal approximation to the sampling distribution of $\bar{T}_N$, especially if there is significant skewness in the distribution of the $T_{i,M,L}$'s.

An important implication of the multivariate approximation (3) is that we can get asymptotically correct approximations to the sampling distributions of continuous functions of $\bar{T}_N$. For example, approximations to the distribution of $\max_{n=1,2,\ldots,D}\{\sqrt{Q}|\bar{T}_N^{(n)} - \mu^{(n)}|\}$ can be obtained with no extra effort; this allows for the possibility of constructing simultaneous confidence intervals for all coordinates of μ (cf. Politis and Romano (1990d)). In particular, μ can be taken to be the spectral or cross-spectral density function sampled at a grid of points and a confidence band using the 'blocks of blocks' bootstrap can be constructed.

Example 10. As our final example, consider the important case where the parameter of interest is the autocorrelation coefficient at lag s, i.e. the parameter $\rho(s) = R(s)/R(0)$, where $R(s) = EX_0X_s$ and for simplicity it is assumed that $EX_0 = 0$. In that case, the linear statistic $\bar{T}_N$ is $(s+1)$-dimensional, with $\bar{T}_N^{(n)} = \frac{1}{N-s}\sum_{i=1}^{N-s} X_iX_{i+n-1}$, and $L = 1$, $M = s+1$ and $\phi_M^{(n)}(x_1, \ldots, x_M) = x_1x_n$, for $n = 1, \ldots, s+1$. It is easy to see that $\bar{T}_N^{(n)}$, for $n = s+1$, is just the sample autocovariance $\hat{R}(s)$ at lag s. By the 'blocks of blocks' resampling scheme applied to the linear statistic $\bar{T}_N$, accurate confidence intervals for the autocovariances can be obtained, as well as variance estimates for the sample autocovariances. Considering the complicated form of the asymptotic variance of the sample autocovariances (that involves estimates of the fourth order cumulants, cf. Priestley(1981), the advantage of using an automatic procedure like the bootstrap is apparent.

Now the estimator $\hat{\rho}(s) = \bar{T}_N^{(s+1)}/\bar{T}_N^{(1)}$ is a smooth function of the linear statistic $\bar{T}_N$, and its statistical properties can be analyzed via the 'blocks of blocks' bootstrap. Of course, if we are only interested in $\rho(s)$, a 2-dimensional linear statistic, consisting of just $\bar{T}_N^{(1)} = \hat{R}(0)$ and $\bar{T}_N^{(s+1)} = \hat{R}(s)$, would suffice. The usefulness of considering the $(s+1)$-dimensional statistic $\bar{T}_N$ lies in that we can instantly obtain *simultaneous* confidence intervals (confidence band) for $\rho(k), k = 1, \ldots, s$ (and for $R(0)$), that are *not* available by classical methods (cf. Priestley(1981)). An obvious use of such confidence bands is in testing hypotheses regarding the covariance structure.

The way this can be done is as follows. For concreteness, assume that we are looking for a 95% confidence band for $\rho(k), k = 1, \ldots, s$. That is, we are looking for two sequences $c_1(k), c_2(k)$ such that $P\{\forall k \in \{1, \ldots, s\} : \hat{\rho}(k) - c_1(k) \le \rho(k) \le \hat{\rho}(k) + c_2(k)\} = 0.95$. To start with, apply Fisher's z transformation to approximately stabilize the variance of the estimates at different lags, i.e. let $\zeta(k) = \frac{1}{2}\log\frac{1+\rho(k)}{1-\rho(k)}$, and $\hat{\zeta}(k) = \frac{1}{2}\log\frac{1+\hat{\rho}(k)}{1-\hat{\rho}(k)}$, for $k = 1, \ldots, s$. Then, by the 'blocks of blocks' bootstrap, obtain an approximation to the distribution of the 'maximum modulus' $\sqrt{N}\max_{k=1,\ldots,s}|\hat{\zeta}(k) - \zeta(k)|$. This immediately leads to a uniform width (i.e. $c_1(k) = c_1$ and $c_2(k) = c_2, k = 1, \ldots, s$) and symmetric (i.e. $c_1(k) = c_2(k)$) confidence band for

$\zeta(k), k = 1, \ldots, s$, and can be translated to a confidence band (of non-uniform width) for $\rho(k), k = 1, \ldots, s$. Alternatively, we can get a (non-symmetric in general) equal-tailed uniform width confidence band for $\zeta(k), k = 1, \ldots, s$, by finding bootstrap approximations to x and y such that $P\{\sqrt{N} \max_{k=1,\ldots,s}(\hat{\zeta}(k) - \zeta(k)) \leq x\} = 0.975$, and $P\{\sqrt{N} \min_{k=1,\ldots,s}(\hat{\zeta}(k) - \zeta(k)) \leq y\} = 0.025$. □

In this paper, a brief review of the bootstrap methodology as applies to dependent data was provided, along with illustrative examples and references to the bibliography. Some recent research problems related to the bootstrap can also be found in the compilation of research papers by LePage and Billard (1992), the book by Hall (1992), and the forthcoming textbook by Efron and Tibshirani.

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