SOME FORMULAS FOR ANTICIPATIVE GIRSANOV TRANSFORMATIONS

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ABSTRACT

We study anticipative transformations of Ω to Ω of the form

$$\varphi_{s,t}(\cdot,\omega) = \omega. + \int_{s\wedge.}^{t\wedge.} \sigma_r(\varphi_{s,r}(\omega))dr$$

where σ is anticipating, and ω is a Wiener process. Under special assumptions on σ, we re-interpret results of Buckdahn using the semimartingale integral (via an expansion of the filtration) instead of the Skorohod integral, and we show the same formulas hold more generally.

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1. INTRODUCTION

Recently there has been interest in extending our understanding of Girsanov transformation type theorems to anticipating situations. This began with Ramer **[13]** and Kusuoka **[9]**, and has been continued with Nualart-Zakai **[11]**, Föllmer-Protter [4], Ustunel-Zakai **[14]**, and Buckdahn **[1-3]**, among others. Here we examine some of Buckdahn's results, and we show they have an interpretation using the classical Itô stochastic integral (interpreted as a semimartingale integral via an expansion of the filtration), rather than with the Skorohod integral that Buckdahn - and most other researchers - has been using. The use of the semimartingale integral allows us to extend Buckdahn's results in some special cases and further gives an idea of just how far such extensions can go before "trouble" occurs.

We are interested in studying "anticipative Girsanov transformations" of the form

$$\varphi_{s,t}(\cdot,\omega) = \omega_{\cdot} + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\varphi_{s,r}(\omega))dr.$$

Thus φ is the solution of an ordinary differential equation for each ω. For each fixed (s,t) with $0 \leq s \leq t \leq 1$, we have $\varphi_{s,t}$ maps Ω to Ω and therefore if $(\Omega, \mathcal{F}, P)$ is Wiener space, we can ask whether or not $P \circ \varphi_{s,t}^{-1}$ is absolutely continuous with respect to the Wiener measure P, and if so, then what is a formula for the Radon-Nikodym derivative (or "likelihood ratio")

$$\frac{dP \circ \varphi_{s,t}^{-1}}{dP} \quad ?$$

Such a question is classical when σ is deterministic or even non-anticipating of a reasonable form. Buckdahn [**1**], [**2**] has considered the case where σ is anticipating of a special form and he obtained explicit formulas for the derivative using the Skorohod integral. We reinterpret one of Buckdahn's results using the semimartingale integral, and we then are able to extend it, always using the semimartingale integral.

In Section Two we give a preliminary result on the expansion of filtrations which is a slight extension of a result of Itô [**6**]. For details on the semimartingale integral the reader can consult, for example, Protter [**12**]; for more on the expansion of filtrations the reader can consult either Jeulin [7] or the Springer Lecture Notes volume edited by Jeulin and Yor [8].

In Section Three we give an equivalent formula to a fundamental result of Buckdahn [1]. Buckdahn's formula, which uses the Skorohod integral, is given in Theorem (3.3), and our formula - which uses the semimartingale integral- is given in Theorem (3.11). In Section Four we relax Buckdahn's hypotheses, and we are able to show that the same semimartingale formula holds more generally (Theorem (4.15)).

In Section Five we relax the important boundedness assumptions. The measure $P\circ\varphi_{s,t}^{-1}$ is still absolutely continuous with respect to P, but the two measures no longer need be equivalent. Nevertheless we are still able to give a formula for the Radon-Nikodym density (Theorem (5.19)), which remains as unchanged as possible.

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2. Preliminaries

Throughout we let $W = (W_t)_{0\le t\le 1}$ be Brownian motion (the Wiener process) on the path space of continuous functions Ω, equipped with the Borel σ-field $\mathcal{F}^0$, and Wiener measure P. The completion of $\mathcal{F}^0$ under P is $\mathcal{F}$; we let $\mathcal{F}_t^0 = \sigma\{W_s; 0 \le s \le t\}$, and $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where $\mathcal{N}$ are the P-null sets of $\mathcal{F}$. Let $\underset{\sim}{\mathcal{F}}$ denote the filtration $(\mathcal{F}_t)_{0\le t\le 1}$.

We will need an elementary result from the theory of "expansion of filtrations." Let $(\alpha_1, \dots, \alpha_n)$ be an n-tuple of distinct points in $(0,1]$, let $\alpha_0 = 0$, and let us define a new filtration $\underset{\sim}{\mathcal{G}} = (\mathcal{G}_t)_{0\le t\le 1}$ by

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma\{W_{\alpha_1}, \dots, W_{\alpha_n}\}.$$

Then clearly $\mathcal{F}_t \subset \mathcal{G}_t$, $0 \le t \le 1$, that is, $\underset{\sim}{\mathcal{G}}$ is an "expansion" of $\underset{\sim}{\mathcal{F}}$. Moreover $\underset{\sim}{\mathcal{G}}$ is a sight continuous filtration, and $\mathcal{G}_t$ contains all the null sets of $\mathcal{F}$. The following theorem is a simple extension of a now classical result due to Itô [6], and we omit the proof.

THEOREM 2.1. *The Wiener process W is a $\underset{\sim}{\mathcal{G}}$ semimartingale, and the process*

$$B_t = W_t - \sum_{i=1}^{n} \int_{\alpha_{i-1}\wedge t}^{\alpha_i \wedge t} \frac{W_{\alpha_i} - W_s}{\alpha_i - s} ds \tag{2.2}$$

is a $\underset{\sim}{\mathcal{G}}$-Brownian motion.

We assume the reader is familiar with both the theory of stochastic integration for semimartingales (as given for example in Protter [**12**]), as well as the theory of the Skorohod integral and elementary Malliavin calculus as given for example in Nualart-Pardoux [**10**].

To distinguish between the semimartingale (or Itô) integral and the Skorohod integral, we denote the semimartingale integral of a predictable process u, for a filtration relative to which W is a semimartingale, by:

$$\int_0^t u_s dW_s, \quad 0 \leq t \leq 1;$$

and for a process $u \in L^2([0,1] \times \Omega)$, and $t \in (0,1]$ such that $1_{[0,t]}u$ is Skorohod integrable, we denote the Skorohod integral of $u1_{[0,t]}$ by

$$\int_0^t u_s \partial W_s.$$

3. A result of Buckdahn interpreted through semimartingales

For this section we let $\underset{\sim}{W_\alpha} = (W_{\alpha_1}, \ldots, W_{\alpha_n})$, and we define the simple process

$$\sigma_s = \sum_{j=1}^{m} 1_{(a_j,b_j]}(s) f_j(\underset{\sim}{W_\alpha}) \tag{3.1}$$

where $f_j \in \mathcal{C}_b^3$, where $\mathcal{C}_b^3$ denotes the space of three times continuously differentiable functions mapping $\mathbf{R}^n$ to $\mathbf{R}$ which are bounded and such that their first three partial derivatives are bounded. Here also $0 \leq a_j \leq b_j \leq 1$, all j. Let us now consider the following equation on Ω, where the dot, "·", denotes the "time" parameter:

$$\varphi_{s,t}(\cdot,\omega) = \omega. + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\varphi_{s,r}(\omega))dr \tag{3.2}$$

for $0 \leq s \leq t \leq 1$ and $\omega \in \Omega$.

By an inverse ψ to φ we mean a function $\psi_{t,s}(\cdot, \omega)$ such that

$$\varphi_{s,t}(\psi_{t,s}(\omega)) = \psi_{t,s}(\varphi_{s,t}(\omega)) = \omega$$

for all $\omega \in \Omega$. The following theorem is due to Buckdahn [**1**]. (Buckdahn assumes that the functions f_j are in $\mathcal{C}_b^\infty$, but an analysis of his proof shows that $f_j \in \mathcal{C}_b^3$ suffices.)

THEOREM 3.3. *Equation (3.2) has a unique strong solution. It is bijective. Its inverse is the unique strong solution of the equation*

$$\psi_{t,s}(\cdot, \omega) = \omega_\cdot - \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\psi_{t,r}(\omega))dr,$$

for $0 \leq s \leq t \leq 1$ *and* $\omega \in \Omega$. *Moreover the law of* $\varphi_{s,t}$ *and Wiener measure are mutually absolutely continuous and*

$$\frac{dP \circ \varphi_{s,t}^{-1}}{dP} = \exp\{\int_s^t \sigma_r(\psi_{t,r})\partial W_r - \frac{1}{2}\int_s^t (\sigma_r(\psi_{t,r}))^2 dr - \int_s^t \int_s^t (D_r\sigma_u)(\psi_{t,u})D_u(\sigma_r(\psi_{t,r}))dudr\}$$

where D *is the Malliavin derivative operator.*

In this section we give an alternative representation of $\frac{dP\circ\varphi_{s,t}^{-1}}{dP}$ where we use the semimartingale integral in place of the Skorohod integral. We need five technical lemmas, which will be followed by our main result, Theorem (3.11).

LEMMA 3.4. *Fix* t, $0 \leq t \leq 1$. *The process* $1_{[0,t]}(\cdot)\sigma_\cdot(\psi(t, \cdot))$ *is integrable with respect to the* $\mathcal{G}$*-semimartingale* W.

Proof: Fix t. Buckdahn [**1**, Section 3] has shown there is a function $\Lambda(t, s, x)$ mapping $[0,1]^2 \times \mathbf{R}^n$ to $\mathbf{R}^n$ which is continuous in s, is $\mathcal{C}^1$ in x, and its partials in x are bounded on $[0,1]^2 \times \mathbf{R}^n$, and finally

$$\sigma_\cdot(\psi_{t,\cdot}(\omega)) = \sum_{j=1}^m 1_{(a_j,b_j]}(\cdot)f_j(\Lambda(t, \cdot, \underset{\sim}{W_\alpha})). \tag{3.5}$$

This implies the result. ∎

Note that the function Λ in the proof of Lemma (3.4) is the unique solution of the equation

$$\Lambda^i(t,s,x) = x^i - \sum_{j=1}^{m} \int_{\alpha_i \wedge t \wedge s}^{\alpha_i \wedge t} 1_{(a_j,b_j]}(r) f_j(\Lambda(t,r,x))dr, \tag{3.6}$$

for $t, s \in [0,1]$, $1 \le i \le n$, and $x \in \mathbb{R}^n$, where x^i is the ith component of x.

In the next lemma we use the now somewhat standard notation $\mathbb{D}^{1,2}$ to denote the domain of the closed extension of the unbounded but closeable linear operator D (the "Malliavin derivative"); see, for example, Nualart-Pardoux [**10**].

LEMMA 3.7. *Let $\Lambda(t,s,x)$ be the unique solution of equation (3.6). For $1 \le j \le m$ and $0 \le t \le 1$ the function $f_j(\Lambda(t,\cdot,x))$ is absolutely continuous on $[0,t]$. Let $\Phi(t,\cdot,x)$ denote its Radon-Nikodym derivative. Then the process $1_{[0,t]}(\cdot)\Phi(t,\cdot,\underset{\sim}{W}_\alpha) \in L^2([0,1],\mathbb{D}^{1,2})$.*

Proof: The Lemma follows by combining Proposition 4.8(i) of Nualart-Pardoux [**10**] with the fact that $f_1,\dots,f_m \in \mathcal{C}_b^3(\mathbb{R}^n)$, and equation (3.6). ∎

LEMMA 3.8. *Let $1 \le j \le m$, and let $0 < t \le 1$, $0 \le s \le t$ be fixed. Let $\Lambda(t,s,x)$ be the unique solution of equation (3.6). Then*

$$\begin{aligned} f_j(\Lambda(t,s,\underset{\sim}{W}_\alpha))W_s = &\int_0^s W_r d_r(f_j(\Lambda(t,r,\underset{\sim}{W}_\alpha)) \\ &+ \int_0^s f_j(\Lambda(t,r,\underset{\sim}{W}_\alpha))\partial W_r + \int_0^s D_r(f_j(\Lambda(t,r,\underset{\sim}{W}_\alpha)))dr. \end{aligned}$$

Proof: This is simply an application of the integration by parts formula for the Skorohod integral (see Corollary 6.2 of Nualart-Pardoux [**10**]). ∎

LEMMA 3.9. *With the hypotheses of Lemma 3.8,*

$$f_j(\Lambda(t,s,\underset{\sim}{W}_\alpha))W_s = \int_0^s f_j(\Lambda(t,r,\underset{\sim}{W}_\alpha))dW_r + \int_0^s W_r d_r(f_j(\Lambda(t,r,\underset{\sim}{W}_\alpha))).$$

Proof: Since $\underset{\sim}{W}$ is a $\mathcal{G}$-semimartingale by Theorem (2.1), this is just the semimartingale integration by parts formula, since $f_j(\Lambda(t,\cdot,\underset{\sim}{W}_\alpha))$ is continuous and of finite variation. ∎

LEMMA 3.10. *Fix t, $0 \leq t \leq 1$. Then*

$$\int_0^t \sigma_s(\psi_{t,s})dW_s = \int_0^t \sigma_s(\psi_{t,s})\partial W_s + \int_0^t D_s(\sigma_s(\psi_{t,s}))ds.$$

Proof: This results from combining Lemmas (3.8) and (3.9) with equation (3.5) in the proof of Lemma (3.4). ∎

The preceding lemmas permit us to re-express Buckdahn's formula in Theorem (3.3) using the semimartingale integral in place of the Skorohod integral. It is our belief this gives an intrinsically simpler formula.

THEOREM 3.11. *Let $\varphi_{s,t}$ satisfy equation (3.2). Then*

$$(3.12) \qquad \frac{dP \circ \varphi_{s,t}^{-1}}{dP} = exp\{\int_s^t \sigma_r(\psi_{t,r})dW_r - \frac{1}{2}\int_s^t (\sigma_r(\psi_{t,r}))^2 dr - \int_s^t (D_r\sigma_r)(\psi_{t,r})dr\}.$$

Proof: In Lemma (3.9) of Buckdahn [**1**], the following equality for Malliavin derivaties in this framework is established:

$$D_s(\sigma_s(\psi_{t,s})) = (D_s\sigma_s)(\psi_{t,s}) - \int_s^t (D_r\sigma_s)(\psi_{t,s})D_s(\sigma_r(\psi_{t,r}))dr$$

for $0 \leq s \leq t \leq 1$. Combining this with Lemma (3.10) and the formula of Theorem (3.3) gives the result. ∎

At this point it may be appropriate to give an intuitive understanding of formula (3.12) for those people familiar only with the traditional formula. Let us recall a typical derivation of the classical, non-anticipating formula. In this case, for a sufficiently large class of random variables F, if we can show that

$$(3.13) \qquad X_t F(\psi_{t,s}) = F + \int_s^t \sigma_r X_r(\psi_{r,s})dW_r,$$

then because the Itô integral has zero expectation, we have

$$E\{X_t F(\psi_{t,r})\} = E\{F\},$$

so that $X_t = \frac{dP\circ\varphi_{s,t}^{-1}}{dP}$. Then taking F to be identically equal to 1 yields the classical formula

$$X_t = \exp(\int_s^t \sigma_r(\psi_{t,r})dW_r - \frac{1}{2}\int_s^t (\sigma_r(\psi_{t,r}))^2 dr).$$

In the anticipating case where W remains a semimartingale, the stochastic integral no longer need have expectation zero; indeed if H is an anticipating process we have the following general relationship, where $D_r^-(H_r) = \lim_{u\uparrow r} D_r(H_u)$:

$$E\left\{\int_s^t H_r dW_r\right\} = E\left\{\int_s^t D_r^-(H_r)dr\right\};$$

which follows trivially from the identity

$$\int_s^t H_r \partial W_r = \int_s^t H_r dW_r - \int_s^t D_r^-(H_r)dr$$

and the fact that the Skorohod integral has zero expectation. This leads us to a modification of equation (3.13), to consider instead:

$$(3.14) \qquad X_t F(\psi_{t,s}) = F + \int_s^t \sigma_r X_r F(\psi_{r,s})dW_r - \int_s^t D_r^-\{\sigma_r X_r F(\psi_{r,s})\}dr.$$

(Note that in the non-anticipating case $D_r^-\{\sigma_r X_r F(\psi_{t,r})\} = 0$, hence equation (3.14) reduces to (3.13).) If we take expectations in (3.14), we now have the desired

$$E\{X_t F(\psi_{t,s})\} = E\{F\},$$

and hence $X_t = \frac{dP\circ\varphi_{s,t}^{-1}}{dP}$. This perhaps helps to explain the presence of the extra term in formula (3.12) which contains a Malliavin-type derivative. This also outlines an approach to prove Buckdahn's Theorem (3.3) using only semimartingale techniques (that is, avoiding the Skorohod integral).

4. An extension of Buckdahn's result

In Section Three we gave an equivalent formula to Buckdahn's formula using the semimartingale integral in place of the Skorohod integral. In this section we show that the same formula remains valid with weaker hypotheses. These hypotheses do not satisfy the conditions given in Buckdahn [**1-3**], nor do they satisfy those given in Ustunel-Zakai [**14**]. The main result is Theorem (4.15).

We again let $\underset{\sim}{W_\alpha} = (W_{\alpha_1}, \ldots, W_{\alpha_n})$, and we define the process

$$\sigma_s = \sum_{j=1}^{m} f_j^1(s) f_j^2(\underset{\sim}{W_\alpha}), \tag{4.1}$$

where $f_j^1 \in L^p([0,1])$ for some $p > 2$, and $f_j^2 \in \mathcal{C}_b^3(\mathbf{R}^n)$ for $1 \le j \le m$. We denote by C a constant bounding all the first partials of f^2, as well as f^2 itself.

Let $f_{j,k}^1$, $1 \le j \le m$, denote a vector sequence of elementary functions such that

$$\lim_{k \to \infty} f_{j,k}^1 = f_j^1, \qquad 1 \le j \le m,$$

with convergence in $L^p([0,1])$, and moreover we can assume without loss

$$\sup_k \| f_{j,k}^1 \|_{L^p([0,1])} \le C.$$

For each k let φ^k, ψ^k denote the unique strong solutions of

$$\varphi_{s,t}^k(\cdot, \omega) = \omega_\cdot + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r^k(\varphi_{s,r}^k(\omega)) dr \tag{4.2}$$

$$\psi_{t,s}^k(\cdot, \omega) = \omega_\cdot - \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r^k(\psi_{t,r}^k(\omega)) dr \tag{4.3}$$

for $0 \le s \le t \le 1$ and $\omega \in \Omega$, where of course

$$\sigma_r^k = \sum_{j=1}^{m} f_{j,k}^1(r) f_j^2(\underset{\sim}{W_\alpha}).$$

Recall that Theorem (3.3) asserts that equations (4.2) and (4.3) do have unique, strong solutions, since $f_{j,k}^1$ are simple functions.

LEMMA 4.4. *With σ as given in (4.1), the equation*

$$\varphi_{s,t}(\cdot,\omega) = \omega_\cdot + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\varphi_{s,r}(\omega))dr \tag{4.5}$$

$0 \le s \le t \le 1$, $\omega \in \Omega$, *has a unique strong solution.*

Proof: Let $0 \le s \le t \le 1$ and $u \in [0,1]$. Then for $k < \ell$ we have

$$|\varphi^k_{s,t}(u) - \varphi^\ell_{s,t}(u)| \le \sum_{j=1}^m \int_s^t |f^1_{j,k}(r) f^2_j(W_{\underset{\sim}{\alpha}}(\varphi^k_{s,r})) - f^1_{j,\ell}(r) f^2_j(W_{\underset{\sim}{\alpha}}(\varphi^\ell_{s,r}))|dr$$

$$\le C \sum_{j=1}^m \left\{ \|f^1_{j,k} - f^1_{j,\ell}\|_{L^p([0,1])} + (\int_s^t |f^2_j(W_{\underset{\sim}{\alpha}}(\varphi^k_{s,r})) - f^2_j(W_{\underset{\sim}{\alpha}}(\varphi^\ell_{s,r}))|^2 dr)^{1/2} \right\}$$

and since f^2_j are Lipschitz, there exists a constant C_1 such that the preceding is

$$\le C_1 \left\{ \sum_{j=1}^m \|f^1_{j,k} - f^1_{j,\ell}\|_{L^p([0,1])} + (\int_s^t \sum_{i=1}^n |\varphi^k_{s,r}(\alpha_i) - \varphi^\ell_{s,r}(\alpha_i)|^2 dr)^{1/2} \right\}$$

and therefore there exists a constant C_2 such that.

$$\sup_{u\in[0,1]} |\varphi^k_{s,t}(u) - \varphi^\ell_{s,t}(u)|^2$$

$$\le C_2 \{ \sum_{j=1}^m \|f^1_{j,k} - f^1_{j,\ell}\|^2_{L^p([0,1])} + \int_s^t \sup_{1\le i\le n} |\varphi^k_{s,r}(\alpha_i) - \varphi^\ell_{s,r}(\alpha_i)|^2 dr \}$$

and applying Gronwall's inequality yields

$$\le C_2 \sum_{j=1}^m \|f^1_{j,k} - f^1_{j,\ell}\|^2_{L^p([0,1])} \exp\{C_2(t-s)\}.$$

From this we deduce that $(\varphi^k_{s,t})$ is Cauchy in $\mathcal{C}[0,1]$ with the sup norm. Denote $\varphi_{s,t}$ as the limit of $\varphi^k_{s,t}$. Next, use the inequality:

$$\int_s^t |\sigma^k_r(\varphi^k_{s,r}(\omega)) - \sigma_r(\varphi_{s,r}(\omega))|dr$$

$$\le C \sum_{j=1}^m \{ \|f^1_{j,k} - f^1_j\|_{L^p([0,1])} + (\int_s^t |f^2_j(W_{\underset{\sim}{\alpha}}(\varphi_{s,r}) - f^2_j(W_{\underset{\sim}{\alpha}}(\varphi^k_{s,r}))|^2 dr)^{1/2} \},$$

and the dominated convergence theorem to deduce that $\varphi_{s,t}$ is a solution of equation (4.5).

The uniqueness of the solution $\varphi_{s,t}$ follows from Gronwall's inequality and the fact that f^2_j is in $\mathcal{C}^3_b(\mathbb{R}^n)$. ∎

LEMMA 4.6. *With σ as given in (4.1), the equation*

$$\psi_{t,s}(\cdot,\omega) = \omega. - \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\psi_{t,r})dr, \tag{4.7}$$

$0 \leq s \leq t \leq 1$, $\omega \in \Omega$, *has a unique strong solution.*

Proof: The proof is analogous to the proof of Lemma (4.4), with ψ^k of (4.3) replacing φ^k of (4.2). ∎

Remark 4.8: It follows as in Lemma (3.4) (since $f_j^1 \in L^p([0,1])$ for $p > 2$, $1 \leq j \leq m$) that the process $1_{[0,t]}(\cdot)\sigma.(\psi_{t,\cdot})$ is integrable with respect to the $\underset{\sim}{\mathcal{G}}$-semimartingale W.

We also note, as was established in the proofs of Lemmas (4.4) and (4.6) that $\varphi_{s,t}^k$ and $\psi_{t,s}^k$ converge uniformly in $\mathcal{C}[0,1]$ to $\varphi_{s,t}$ and $\psi_{t,s}$, respectively.

Before stating and proving our main result (Theorem (4.15)), we establish four lemmas.

LEMMA 4.9. *Let φ^k and φ be the solutions of equations (4.2) and (4.5) respectively. Then for $0 \leq s \leq t \leq 1$:*

$$\lim_{k\to\infty} 1_{(s,t]}(\cdot)\sigma_\cdot^k(\varphi_{s,\cdot}^k) = 1_{(s,t]}(\cdot)\sigma.(\varphi_{s,\cdot})$$

with convergence in $L^2([0,1]\times\Omega)$.

Proof: Using the convergence of φ^k to φ in the sup norm, the hypothesis that $f_{j,k}^1$ converges boundedly in $L^p([0,1])$ to f_j^1, and the dominated convergence theorem, we have the result. ∎

LEMMA 4.10. *Let β be a positive real number. Then*

$$\sup_k E\{exp(-\frac{\beta}{2}\int_s^t (\sigma_r^k(\psi_{t,r}^k))^2 dr - \beta\int_s^t (D_r\sigma_r^k)(\psi_{t,r}^k)dr\} < \infty$$

for $0 \le s \le t \le 1$, *where* D *is the "Malliavin derivative" operator.*

Proof: Fix $0 \le s \le t \le 1$ and a positive integer k. Since f^2 and its first partials are all bounded by hypothesis we have:

$$
\begin{aligned}
|\frac{\beta}{2}\int_s^t (\sigma_r^k(\psi_{t,r}^k))^2 dr + \beta \int_s^t (D_r\sigma_r^k)(\psi_{t,r}^k) dr| \\
&\le \beta \sum_{j=1}^m \{2^m \int_s^t |f_{j,k}^1(r) f_j^2(W_{\underset{\sim}{\alpha}}(\psi_{t,r}^k))|^2 dr \\
&\quad + \int_s^t |f_{j,k}^1(r) \sum_{i=1}^n \frac{\partial f_j^2}{\partial x_i}(W_{\underset{\sim}{\alpha}}(\psi_{t,r}^k)) 1_{[0,\alpha_i]}(r)| dr\} \\
&\le \beta \sum_{j=1}^m \{C 2^m \int_s^t |f_{j,k}^1(r)|^2 dr + nC \int_s^t |f_{j,k}^1(r)| dr\}.
\end{aligned}
$$

Since $\sup_k \|f_{k,j}^1\|_{L^p([0,1])} \le C$ by hypothesis as well, the result follows. ∎

LEMMA 4.11. *Let* β *be a positive real number. Then*

$$\sup_k E\{\exp(\beta \int_s^t \sigma_r^k(\psi_{t,s}^k) dW_r)\} < \infty$$

for $0 \le s \le t \le 1$.

Proof: Fix $0 \le s \le t \le 1$ and let k be a positive integer. Using Lemma (3.7); that $f_{j,k}$ is in $L^p([0,1])$ for $p > 2$; the integration by parts formula for the semimartingale integral; and the identities (3.5) and (3.6) yield:

$$
\begin{aligned}
&\int_s^t \sigma_r^k(\psi_{t,r}^k) dW_r \\
&= \sum_{j=1}^m \{(\int_s^t f_{j,k}^1(r) dW_r) f_j^2(W_{\underset{\sim}{\alpha}}) - \int_s^t (\int_s^r f_{j,k}^1(u) dW_u) d_r f_j^2(\psi^k(t,r,W_{\underset{\sim}{\alpha}}))\} \\
&= \sum_{j=1}^m \{(\int_s^t f_{j,k}^1(r) dW_r) f_j^2(W_{\underset{\sim}{\alpha}}) \\
&\quad - \int_s^t (\int_s^r f_{j,k}^1(u) dW_u) \sum_{j=1}^n \frac{\partial f_j^2(\psi(t,r,W_{\underset{\sim}{\alpha}}))}{\partial x^i} 1_{[0,\alpha_i]}(r) \sigma_r^k(\psi_{t,r}^k) dr\},
\end{aligned}
$$

and the hypotheses that f^2_j and its first partials are bounded, and that $f^1_{j,k}$ is uniformly bounded in L^p further yield:

$$
\begin{aligned}
&|\int_s^t \sigma_r^k(\psi_{t,r}^k)dW_r| \\
&\leq C\sum_{j=1}^m\{|\int_s^t f^1_{j,k}(r)dW_r| + n\int_s^t |\int_s^r f^1_{j,k}(u)dW_u|\sum_{\ell=1}^m |f^1_{\ell,k}(r)f^2_\ell(\psi^k_{t,r})|dr\} \\
&\leq (C+1)^2 n\sum_{j=1}^m\{|\int_s^t f_{j,k}(r)dW_r|+ \\
&\sum_{\ell=1}^m\int_s^t|\int_s^r f^1_{j,k}(u)dW_u||f^1_{\ell,k}(r)|dr\} \\
&\leq (C+1)^3 mn\sum_{j=1}^m\{|\int_s^t f^1_{j,k}(r)dW_r| + (\int_s^t(\int_s^r f^1_{j,k}(u)dW_u)^2dr)^{1/2}\}.
\end{aligned}
\tag{4.12}
$$

Inequality (4.12) inspires us to consider the following (where we use Fubini's theorem and the Cauchy-Schwarz inequality):

$$
\begin{aligned}
&E\{\exp(\beta(\int_s^t(\int_s^r f^1_{j,k}(u)dW_u)^2dr)^{1/2})\} \\
&\leq 1+\beta(E\int_s^t(\int_s^r f^1_{j,k}(u)dW_u)^2dr)^{1/2} \\
&\quad+\frac{\beta^2}{2}(\int_s^t E(\int_s^r f^1_{j,k}(u)dW_u)^2dr) \\
&\quad+\sum_{\ell=3}^\infty\frac{\beta^\ell}{\ell!}(\int_s^t E(\int_s^r f^1_{j,k}(u)dW_u)^2dr)^{\ell/2} \\
&\leq 1+\beta(\int_s^t E(\int_s^r f^1_{j,k}(u)dW_u)^2dr)^{1/2} \\
&\quad+\frac{\beta^2}{2}\int_s^t E(\int_s^r f^1_{j,k}(u)dW_u)^2dr \\
&\quad+\int_s^t E\{\sum_{\ell=3}^\infty\frac{\beta^\ell|\int_s^r f^1_{j,k}(u)dW_u|^\ell}{\ell!}\}dr \\
&\leq \beta(\int_s^t E(\int_s^r f^1_{j,k}(u)dW_u)^2dr)^{1/2} \\
&\quad+\int_s^t E\{\exp\beta|\int_s^r f^1_{j,k}(u)dW_u|\}dr,
\end{aligned}
\tag{4.13}
$$

where of course we have been interpreting dW_u as the semimartingale integral for the $\underset{\sim}{\mathcal{G}}$-semimartingale W. However since $f^1_{j,k}$ are non-random, the $\underset{\sim}{\mathcal{G}}$-semimartingale integral of $f^1_{j,k}$ is the same as the $\underset{\sim}{\mathcal{F}}$-semimartingale integral, which is the Wiener integral. Using standard inequalities (Burkholder-Davis-Gundy) for stochastic integrals and the hypothesis that $f^1_{j,k}$ are uniformly bounded in L^p for $p > 2$, and combining (4.12) with (4.13) gives the result. ∎

LEMMA 4.14. *Let $0 \le s \le t \le 1$. The sequence of densities*

$$\left(\frac{dP \circ (\varphi^k_{s,t})^{-1}}{dP}\right)_{k \ge 1}$$

is uniformly integrable.

Proof: The lemma follows from Theorem (3.11) and Lemmas (4.10) and (4.11). ∎

THEOREM 4.15. *With σ as given in (4.1), let $\varphi_{s,t}$ be the unique strong solution of*

$$\varphi_{s,t}(\cdot,\omega) = \omega_\cdot + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\varphi_{s,r}(\omega))dr. \tag{4.16}$$

For every $0 \le s \le t \le 1$ the measures $P \circ \varphi^{-1}_{s,t}$ and P are mutually absolutely continuous and the density

$$\frac{dP \circ \varphi^{-1}_{s,t}}{dP}$$

is given by equation (3.12). That is,

$$\frac{dP \circ \varphi^{-1}_{s,t}}{dP} = exp\{\int_s^t \sigma_r(\psi_{t,r})dW_r - \frac{1}{2}\int_s^t (\sigma_r(\psi_{t,r}))^2 dr - \int_s^t (D_r\sigma_r)(\psi_{t,r})dr\}. \tag{4.17}$$

Proof: Note that Lemma (4.4) assures that equation (4.16) - which is the same as equation (4.5) - has a unique, strong solution. For the proof we need a result of Gihman-Skorohod [**5**], which we found in Buckdahn [**2**, Proposition 2.9], and we state it here for the reader's convenience:

LEMMA 4.18 (BUCKDAHN). *Let T^p, given by $T^p\omega = \omega + \int_0^\cdot K_s^p(\omega)ds$, be a sequence of absolutely continuous transformations such that*

(i) the sequence of processes K^p converges to a process K in $L^2([0,1]\times\Omega)$;

(ii) the sequence of densities

$$L^p = \frac{dP\circ(T^p)^{-1}}{dP}$$

is uniformly integrable;

Then the transformation T given by

$$T\omega = \omega + \int_0^\cdot K_s(\omega)ds$$

is also absolutely continuous and the density $L = \frac{dP\circ T^{-1}}{dP}$ is the limit of L^p in the weak topology $\sigma(L^1, L^\infty)$.

Continuing with the proof of Theorem (4.15), let us fix $0 \le s \le t \le 1$. Using Lemmas (4.9), (4.14), and (4.18) above we conclude that $P\circ\varphi_{s,t}^{-1} \ll P$ and moreover

$$\frac{dP\circ\varphi_{s,t}^{-1}}{dP} = \lim_{k\to\infty}\frac{dP\circ(\varphi_{s,t}^k)^{-1}}{dP},$$

where convergence is in the weak topology $\sigma(L^1, L^\infty)$. Thus to finish the proof we need to show only that $\frac{dP\circ(\varphi_{s,t}^k)^{-1}}{dP}$ converges in probability to the right side of equation (4.17). From Theorem (3.11) we have that

$$\text{(4.19)}\quad \frac{dP\circ(\varphi_{s,t}^k)^{-1}}{dP} = \exp\{\int_s^t \sigma_r^k(\psi_{t,r}^k)dW_r - \frac{1}{2}\int_s^t(\sigma_r^k(\psi_{t,r}^k))^2dr - \int_s^t(D_r\sigma_r^k)(\psi_{t,r}^k)dr\}.$$

Here we use that W is a $\underset{\sim}{\mathcal{G}}$-semimartingale, with its decomposition given by

$$W_t = B_t + \sum_{k=1}^n\int_{\alpha_{i-1}\wedge t}^{\alpha_i\wedge t}\frac{W_{\alpha_i}-W_s}{\alpha_i - s}ds,$$

where B is a $\underset{\sim}{\mathcal{G}}$-Brownian motion (or "Wiener process"). (See Theorem (2.1).) Therefore, in view of Lemma (4.9), in order to show that:

$$\text{(4.20)}\qquad \lim_{k\to\infty}\exp\{\int_s^t\sigma_r^k(\psi_{t,r}^k)dW_r\} = \exp\{\int_s^t\sigma_r(\psi_{t,r})dW_r\}$$

in probability; it suffices to show that

$$\lim_{k\to\infty} E\{\int_0^{\alpha_i\wedge t} |\sigma_r^k(\psi_{t,r}^k) - \sigma_r(\psi_{t,r})||\frac{W_{\alpha_i} - W_r}{\alpha_i - r}|dr = 0. \tag{4.21}$$

Lemma (4.9) takes care of the convergence of the Brownian Itô integral term (dB). To that end, for an i, $1 \le i \le n$, we have

$$\begin{aligned} &E\{\int_0^{\alpha_i\wedge t} |\sigma_r^k(\psi_{t,r}^k) - \sigma_r(\psi_{t,r})||\frac{W_{\alpha_i} - W_r}{\alpha_i - r}|dr \\ &\le \sum_{j=1}^m E\left\{\int_0^{\alpha_i\wedge t} |f_j^1(r)||f_j^2(W_{\underset{\sim}{\alpha}}(\psi_{t,r})) - f_j^2(W_{\underset{\sim}{\alpha}}(\psi_{t,r}^k))|\frac{|W_{\alpha_i} - W_r|}{\alpha_i - r}dr \right. \\ &\left. + C\int_0^{\alpha_i} |f_j^1(r) - f_{j,k}^1(r)|\frac{|W_{\alpha_i} - W_r|}{\alpha_i - r}dr\right\} \\ &\le \sum_{j=1}^m \left\{\int_0^{\alpha_i\wedge t} \frac{|f_j^1(r)|}{(\alpha_i - r)^{1/2}}(E(f_j^2(W_{\underset{\sim}{\alpha}}(\psi_{t,r})) - f_j^2(W_{\underset{\sim}{\alpha}}(\psi_{t,r}^k)))^2)^{1/2}dr \right. \\ &\left. + C\int_0^{\alpha_i} \frac{|f_j^1(r) - f_{j,k}^1(r)|}{(\alpha_i - r)^{1/2}}dr\right\}, \end{aligned}$$

and (4.21) follows by the dominated convergence theory and the hypotheses that f_j^2 is bounded and that f_j^1 is uniformly bounded in L^p for $p > 2$, combined with Remark (4.8), that ψ^k converges to ψ uniformly in $\mathcal{C}[0,1]$. This then establishes (4.20) as well.

We have now established the convergence of the first term in the argument of the exponential on the right side of equation (4.19). For the second term, observe that

$$\begin{aligned} &|\int_s^t (\sigma_r^k(\psi_{t,r}^k))^2 - (\sigma_r(\psi_{t,r}))^2 dr| \\ &\le (\int_s^t |\sigma_r^k(\psi_{t,r}^k) + \sigma_r(\psi_{t,r})|^2 dr)^{1/2}(\int_s^t |\sigma_r^k(\psi_{t,r}^k) - \sigma_r(\psi_{t,r})|^2 dr)^{1/2}, \end{aligned}$$

and again since $\psi_{t,r}^k$ converges uniformly to $\psi_{t,r}$ in $\mathcal{C}[0,1]$, an argument similar to the previous one yields

$$\lim_{k\to\infty} \exp\{-\frac{1}{2}\int_s^t (\sigma_r^k(\psi_{t,r}^k))^2 dr\} = \exp\{-\frac{1}{2}\int_s^t (\sigma_r(\psi_{t,r}))^2 dr\} \tag{4.22}$$

with convergence in probability. Analogously, we can show

$$\lim_{k\to\infty} \exp\{\int_s^t (D_s\sigma_r^k)(\psi_{t,r}^k)dr\} = \exp\{\int_s^t (D_r\sigma_r)(\psi_{t,r})dr\}$$

with convergence in probability. (Note that the presence of the Malliavin derivative operator does not pose a problem, since we can use the identity

$$\begin{aligned}&\frac{1}{2}(\sigma_\cdot^k(\psi_{t,\cdot}^k))^2 + (D_\cdot\sigma_\cdot^k)(\psi_{t,\cdot}^k)\\ &\qquad = \frac{1}{2}(\sum_{j=1}^m f_{j,k}^1(\cdot)f_j^2(W_{\underset{\sim}{\alpha}}(\psi_{t,\cdot}^k)))^2 + \sum_{j=1}^m\sum_{i=1}^n f_{j,k}^1(\cdot)\frac{\partial f_j^2}{\partial x^i}(W_{\underset{\sim}{\alpha}}(\psi_{t,\cdot}^k))1_{[0,\alpha_i]}(\cdot)\end{aligned}$$

from which, together with (4.22), the result follows without having to use continuity properties of D.) This completes the proof. ∎

We end this section by recording a technical result that we will need in Section Five.

LEMMA 4.23. *Let σ be as given in (4.1), and let φ be as in (4.5) and ψ be as in (4.7). Let $0 \le s \le t \le 1$. Then*

$$\varphi_{s,t}(\psi_{t,s}(\omega)) = \psi_{t,s}(\varphi_{s,t}(\omega)) = \omega, \ \text{all } \omega \in \Omega.$$

Proof: We note that Buckdahn [**1**, Lemma 3.4] has already proved this lemma for "simple" φ and ψ: that is, for our φ^k and ψ^k. Using that the functions f_j^2 are in $\mathcal{C}_b^3(\mathbf{R}^n)$, $1 \le j \le m$, we can show, analogously to the proof of Lemma (4.4), that there exists a constant K such that

$$\sup_{u\in[0,1]} |\varphi_{s,t}^k(u,\omega) - \varphi_{s,t}^k(u,\omega')| \le K \sup_{u\in[0,1]} |\omega(u) - \omega'(u)|$$

for all k and ω, $\omega' \in \Omega$. Moreover:

$$\sup_{u\in[0,1]} |\psi_{t,s}^k(u,\omega) - \psi_{t,s}^k(u,\omega')| \le K \sup_{u\in[0,1]} |\omega(u) - \omega'(u)| \tag{4.24}$$

for all k and ω, $\omega' \in \Omega$. We now use the fact (see Buckdahn [**1**, Lemma 3.4]) mentioned above that Lemma (4.23) holds for φ^k and ψ^k to conclude:

$$\begin{aligned}
&\sup_{u\in[0,1]} |\varphi_{s,t}(u,\psi_{t,s}(\omega)) - \omega(u)| \\
&\qquad = \sup_{u\in[0,1]} |\varphi_{s,t}(u,\psi_{t,s}(\omega)) - \varphi^k_{s,t}(u,\psi^k_{t,s}(\omega))| \\
&\qquad \le \sup_{u\in[0,1]} |\varphi_{s,t}(u,\psi_{t,s}(\omega)) - \varphi^k_{s,t}(u,\psi_{t,s}(\omega))| \\
&\qquad\qquad + K \sup_{u\in[0,1]} |\psi_{t,s}(u,\omega) - \psi^k_{t,s}(u,\omega)|,
\end{aligned}$$

for all k. The uniform convergence of φ^k and ψ^k to φ and ψ respectively (see Remark (4.8)) yields the result. ∎

5. The unbounded case

In this section we again let $\underset{\sim}{W_\alpha} = (W_{\alpha_1}, \dots, W_{\alpha_n})$, and we define the process

$$\sigma_s = \sum_{j=1}^{m} f_j^1(s) f_j^2(\underset{\sim}{W_\alpha}) \tag{5.1}$$

where $f_j^1 \in L^p([0,1])$ for some $p > 2$, and $f_j^2 \in \mathcal{C}^3(\mathbf{R}^n)$. Note that the difference in σ here, as opposed to σ as defined in (4.1), is that f_j^2 are no longer assumed to be bounded, and their partials are no longer assumed to be bounded.

We also assume that the equation

$$\varphi_{s,t}(\cdot,\omega) = \omega. + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r(\varphi_{s,r}(\omega))dr, \tag{5.2}$$

$0 \le s \le t \le 1$, $\omega \in \Omega$, has a unique, strong solution. (Note that in Sections Three and Four we were able to prove the statement analogous to (5.2); here we must assume it, for it is not true in general; in Section Six we give a counterexample.)

Let $g \in \mathcal{C}_b^\infty(\mathbf{R}^n)$ such that $\|g\|_{L^\infty} \le 1$, and

$$g(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2 \end{cases}$$

and we define for each $k \in \mathbf{N}$:

$$\sigma_s^k = \sum_{j=1}^m f_j^1(s) f_{j,k}^2(\underset{\sim}{W_\alpha}), \tag{5.3}$$

where $f_{j,k}^2(x) = g(\frac{x}{k}) f_j^2(x)$. Then σ^k satisfies (4.1) and we can apply the results of Section Four. In particular the equation

$$\varphi_{s,t}^k(\omega) = \omega. + \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r^k(\varphi_{s,r}^k) dr, \tag{5.4}$$

$0 \le s \le t \le 1$, $\omega \in \Omega$, has a unique strong soltion.

LEMMA 5.5. *Let $k \in \mathbf{N}$, φ as in (5.2), φ^k as in (5.4), $s \in [0,1]$, and define $\Omega_k^s = \{ \sup_{s \le r \le 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r})| \le k\}$. Then for all $\ell \ge k$*

$$\sup_{\substack{s \le t \le 1 \\ u \in [0,1]}} |\varphi_{s,t}(u) - \varphi_{s,t}^\ell(u)| = 0 \text{ on } \Omega_k^s.$$

Proof: Fix $s \in [0,1]$ and $\omega \in \Omega_k^s$. Let $t \in [s,1]$, $u \in [0,1]$, and $\ell \ge k$. Then

$$\begin{aligned} &|\varphi_{s,t}(u,\omega) - \varphi_{s,t}^\ell(u,\omega)| \\ &\quad \le \sum_{j=1}^m \int_{s\wedge u}^{t\wedge u} |f_j^1(s)| |f_j^2(\underset{\sim}{W_\alpha}(\varphi_{s,r}(\omega)) - f_{j,\ell}^2(\underset{\sim}{W_\alpha}(\varphi_{s,r}^\ell(\omega)))| dr \\ &\quad = \sum_{j=1}^m \int_{s\wedge u}^{t\wedge u} |f_j^1(s)| |f_{j,\ell}^2(\underset{\sim}{W_\alpha}(\varphi_{s,r}(\omega))) - f_{j,\ell}^2(\underset{\sim}{W_\alpha}(\varphi_{s,r}^\ell(\omega)))| dr. \end{aligned}$$

Since $f_{j,\ell}^2 \in \mathcal{C}_b^3(\mathbf{R}^n)$ we have that there exists a constant K_ℓ such that

$$|\varphi_{s,t}(u,\omega) - \varphi_{s,t}^\ell(u,\omega)| \le \sum_{j=1}^m K_\ell \int_s^t |f_j^1(s)| |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\omega)) - \underset{\sim}{W_\alpha}(\varphi_{s,r}^\ell(\omega)))| dr.$$

An application of Gronwall's inequality now gives the result. ∎

Remarks 5.6:

(i) Let $s \in [0,1]$. The continuity of the function $r \to |\underset{\sim}{W_\alpha}(\varphi_{s,r})|$, $s \le r \le 1$, implies that $\Omega = \bigcup_{k=1}^\infty \Omega_k^s$; moreover $\Omega_k^s \subset \Omega_\ell^s$ if $k \le \ell$.

(ii) In the proof of Lemma (5.5) we used the hypothesis that equation (5.2) has a unique, strong solution. Indeed we cannot prove that $\{\varphi_{s,t}^k(\omega)\}_{k \ge 1}$ is Cauchy in $\mathcal{C}[0,1]$ without this assumption.

LEMMA 5.7. *Let $0 \le s \le t \le 1$. Then the mapping $\omega \to \varphi_{s,t}(\cdot,\omega)$, from Ω into Ω, is measurable.*

Proof: Clearly Ω_k^s is measurable, since $W_{\underset{\sim}{\alpha}}(\varphi_{s,r})$ is continuous in r. However we have seen $\varphi_{s,t}(u) = \varphi_{s,t}^k(u)$ on Ω_k^s, and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k^s$. Thus φ is measurable because φ^k is. ∎

THEOREM 5.8. *Let σ be as in (5.1) and let φ be as in (5.2), and let $0 \le s \le t \le 1$. Then $P \circ (\varphi_{s,t})^{-1}$ is absolutely continuous with respect to P.*

Proof: Let $P(A) = 0$, where A is a measurable subset of Ω. For each $\omega \in \Omega$ there exists $k_0 = k_0(\omega)$ such that $\varphi_{s,t}(\cdot,\omega) = \varphi_{s,t}^k(\cdot,\omega)$, for $k \ge k_0$ (see the proof of Lemma (5.7)). Therefore

$$\lim_{k\to\infty} 1_A(\varphi_{s,t}^k(\omega)) = 1_A(\varphi_{s,t}(\omega)),$$

all $\omega \in \Omega$. Taking expectations, by the monotone convergence theorem

$$\lim_{k\to\infty} P((\varphi_{s,t}^k)^{-1}(A)) = P(\varphi_{s,t}^{-1}(A)).$$

However $P((\varphi_{s,t}^k)^{-1}(A)) = 0$ for all k, since $P \circ (\varphi_{s,t}^k)^{-1} \ll P$ by Theorem (4.15). Therefore $P(\varphi_{s,t}^{-1}(A)) = 0$ and the proof is complete. ∎

We now wish to establish a formula revealing the Radon-Nikodym density of $\frac{dP\circ\varphi_{s,t}^{-1}}{dP}$ which we know exists by Theorem (5.8). We need eight small preliminary results.

For the remainder of Section Five we let ψ^k denote the unique, strong solution of the equation

$$\psi_{t,s}^k(\cdot,\omega) = \omega. - \int_{s\wedge\cdot}^{t\wedge\cdot} \sigma_r^k(\psi_{t,r}^k)dr. \tag{5.9}$$

Note that by Lemma (4.6) equation (5.9) does have a unique, strong solution.

LEMMA 5.10. *Let $0 \le s \le t \le 1$. Then $\omega \in \varphi_{s,t}(\Omega)$ if and only if there exists $k_0 \in \mathbb{N}$ such that $\psi_{t,s}^k(\omega) = \psi_{t,s}^\ell(\omega)$ for all $k, \ell \ge k_0$.*

Proof: Let $\omega \in \varphi_{s,t}(\Omega)$. Then there exists $\omega' \in \Omega$ such that $\omega = \varphi_{s,t}(\omega')$, and so by Lemma (5.5) there exists k_0 such that $\omega = \varphi_{s,t}(\omega') = \varphi_{s,t}^\ell(\omega')$ for $\ell \ge k_0$. Lemma (4.23) then yields $\varphi_{t,s}^\ell(\omega) = \omega'$ for $\ell \ge k_0$.

For the sufficiency let $\omega, \omega' \in \Omega$ and let k_0 be such that $\omega' = \psi^k_{t,s}(\omega)$ for every $k \geq k_0$. By Lemma (4.23) $\varphi^k_{s,t}(\omega') = \omega$ for $k \geq k_0$, and by Lemma (5.5) we have $\varphi^k_{s,t}(\omega')$ tends to $\varphi_{s,t}(\omega')$ in $\mathcal{C}([0,1])$. This completes the proof. ∎

LEMMA 5.11. *Let $0 \leq s \leq t \leq 1$. Then*

(i) $\bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} \{\psi^k_{t,s} = \psi^{k+1}_{t,s}\} = \varphi_{s,t}(\Omega)$

(ii) $\varphi_{s,t}(\Omega) \in \mathcal{F}$.

Proof: (i) follows from Lemma (5.10), and (ii) follows from (i) above and the fact that ψ^k and ψ^{k+1} are continuous as functions of ω by inequality (4.24).

LEMMA 5.12. *Let $k \in \mathbf{N}$ and $s \in [0,1]$. Then the function $\omega \to \sup_{s \leq t \leq 1} |\underset{\sim}{W}_\alpha(\varphi^k_{s,t}(\omega))|$, from Ω into $\mathbf{R}$, is continuous.*

Proof: (Here we consider Ω as path space topologized by uniform convergence.) Fix $s \in [0,1]$ and $k \in \mathbf{N}$. Let $\omega, \omega' \in \Omega$. Since $f^2_{j,k}$, $1 \leq j \leq m$, are Lipschitz, we have the existence of a constant C such that

$$|\underset{\sim}{W}_\alpha(\varphi^k_{s,t}(\omega)) - \underset{\sim}{W}_\alpha(\varphi^k_{s,t}(\omega'))|$$

$$\leq n\{\sup_{u \in [0,1]} |\omega(u) - \omega'(u)| + \int_s^t |\sigma^k_r(\varphi^k_{s,t}(\omega)) - \sigma^k_r(\varphi^k_{s,r}(\omega'))|dr\}$$

$$\leq C\{\sup_{u \in [0,1]} |\omega(u) - \omega'(u)| + \sum_{j=1}^m \int_s^t |f^1_j(r)||\underset{\sim}{W}_\alpha(\varphi^k_{s,t}(\omega)) - \underset{\sim}{W}_\alpha(\varphi^k_{s,r}(\omega'))|dr\}$$

for every $t \in [s,1]$; the result now follows by Gronwall's inequality. ∎

COROLLARY 5.13. *Let $s \in [0,1]$. The function $\omega \to \sup_{s \leq t \leq 1} |\underset{\sim}{W}_\alpha(\varphi_{s,t}(\omega))|$ from Ω into $\mathbf{R}$ is measurable.*

Proof: This is an immediate consequence of Lemmas (5.5) and (5.12). ∎

For $\omega \in \varphi_{s,t}(\Omega)$, by Lemma (5.10) we know that $\psi^k_{t,s}(\omega)$ converges as k tends to ∞. Hence we define:

$$\psi_{t,s}(\omega) = \lim_{k \to \infty} \psi^k_{t,s}(\omega), \text{ for } \omega \in \varphi_{s,t}(\Omega). \tag{5.14}$$

LEMMA 5.15. *Let $0 \le s \le t \le 1$. The function $\omega \to 1_{\varphi_{s,t}(\Omega)}(\omega)\psi_{t,s}(\omega)$ is measurable.*

Proof: Since $1_{\varphi_{s,t}(\Omega)}\psi^k_{t,s}$ are measurable by Section Four and Lemma (5.11), the function is a limit of measurable functions and hence measurable. ∎

COROLLARY 5.16. *Let $0 \le s \le t \le 1$. The function*

$$\omega \to 1_{\varphi_{s,t}(\Omega)}(\omega) \sup_{s \le r \le 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\psi_{t,s}(\omega)))|,$$

from Ω into $\mathbf{R}$, is measurable

Proof: Combine Corollary (5.13) and Lemma (5.15). ∎

LEMMA 5.17. *Let $0 \le s \le t \le 1$. Then*

$$P\left\{\frac{dP \circ (\varphi_{s,t})^{-1}}{dP} = 0 | \varphi_{s,t}(\Omega)^c\right\} = 1.$$

Proof: Let F be a bounded random variable. Then

$$\begin{aligned} E\{F 1_{\{\varphi_{s,t}(\Omega)^c\}} \frac{dP \circ (\varphi_{s,t})^{-1}}{dP}\} \\ &= E\{F(\varphi_{s,t}) 1_{\{\varphi_{s,t}(\Omega)^c\}}(\varphi_{s,t})\} \\ &= 0. \end{aligned}$$

Since F was arbitrary, the result follows. ∎

LEMMA 5.18. *Let $k \in \mathbf{N}$, and $0 \le s \le t \le 1$. Let*

$$\Lambda_{s,t,k} = \varphi_{s,t}(\Omega) \cap \{ \sup_{s \le r \le 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\psi^k_{t,s}))| \le k\}$$

and

$$\Gamma_{s,t,k} = \varphi_{s,t}(\Omega) \cap \{ \sup_{s \le r \le 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\psi_{t,s}))| \le k\}.$$

Then $\Lambda_{s,t,k} = \Gamma_{s,t,k}$.

Proof: Fix k and $0 \leq s \leq t \leq 1$. Let $\omega \in \Lambda_{s,t,k}$. Then $\psi^k_{t,s}(\omega) \in \Omega^s_k$, where Ω^s_k is defined in Lemma (5.5). Therefore by Lemma (5.5),

$$\varphi_{s,t}(\psi^k_{t,s}(\omega)) = \varphi^\ell_{s,t}(\psi^k_{t,s}(\omega))$$

for $\ell \geq k$. Lemma (4.23) now yields

$$\psi^k_{t,s}(\omega) = \psi^\ell_{t,s}(\omega) \text{ for } \ell \geq k,$$

whence $\psi_{t,s}(\omega) = \psi^k_{t,s}(\omega)$, and hence $\omega \in \Gamma_{s,t,k}$.

Next suppose $\omega \in \Gamma_{s,t,k}$. Then $\psi_{t,s}(\omega) \in \Omega^s_k$ and therefore

$$\varphi_{s,t}(\psi_{t,s}(\omega)) = \varphi^\ell_{s,t}(\psi_{t,s}(\omega)) \text{ for } \ell \geq k;$$

since there is an ℓ large enough such that $\psi_{t,s}(\omega) = \psi^\ell_{t,s}(\omega)$, then $\varphi^k_{s,t}(\psi_{t,s}(\omega)) = \omega$, and hence again by Lemma (4.23) we conclude $\omega \in \Lambda_{s,t,k}$. ∎

THEOREM 5.19. *Let* $0 \leq s \leq t \leq 1$ *and* $\omega \in \Omega$. *Let* σ *be as in (5.1), and let* φ *be as in (5.2). Then*

$$\frac{dP \circ (\varphi_{s,t})^{-1}}{dP} = \begin{cases} \frac{dP \circ (\varphi^k_{s,t})^{-1}}{dP} & \text{on } \Lambda_{s,t,k} \\ 0 & \text{if } \omega \notin \varphi_{s,t}(\Omega) \end{cases}$$

where $\Lambda_{s,t,k} = \varphi_{s,t}(\Omega) \cap \{ \sup_{s \leq r \leq 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\psi_{t,s}))| \leq k \}$.

Proof: Let F be a bounded random variable and $k \in \mathbb{N}$. Then Lemmas (5.5) and (5.18), together with Corollary (5.16), imply:

$$\begin{aligned} &E\left\{ F 1_{\Lambda_{s,t,k}} \frac{dP \circ (\varphi_{s,t})^{-1}}{dP} \right\} \\ &= E\left\{ F(\varphi_{s,t}) 1_{\{\sup_{s\leq r\leq 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r})| \leq k\}} \right\} \\ &= E\left\{ F(\varphi^k_{s,t}) 1_{\{\varphi_{s,t}(\Omega)\}}(\varphi^k_{s,t}) 1_{\{\sup_{s\leq r\leq 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\psi^k_{t,s}))| \leq k\}}(\varphi^k_{s,t}) \right\} \\ &= E\left\{ F 1_{\{\varphi_{s,t}(\Omega)\}} 1_{\{\sup_{s\leq r\leq 1} |\underset{\sim}{W_\alpha}(\varphi_{s,r}(\psi^k_{t,s}))| \leq k\}} \frac{dP \circ (\varphi^k_{s,t})^{-1}}{dP} \right\} \\ &= E\left\{ F 1_{\Lambda_{s,t,k}} \frac{dP \circ (\varphi^k_{s,t})^{-1}}{dP} \right\}, \end{aligned}$$

and then Lemma (5.17) completes the proof. ∎

Remark 5.20: If for $0 \leq s \leq t \leq 1$ we have that $P(\varphi_{s,t}(\Omega)) = 1$, then Lemmas (4.23), (5.5), and (5.10) imply that

$$\omega = \varphi_{s,t}(\psi_{t,s}(\omega)) = \psi_{t,s}(\varphi_{s,t}(\omega))$$

almost surely, and the measures P, $P \circ \varphi_{s,t}^{-1}$ and $P \circ \psi_{t,s}^{-1}$ are all mutually absolutely continuous (i.e., equivalent).

6. A Counterexample

In this section we present an example which shows that, with the assumptions of Section Five, equation (5.2) need not have a strong solution.

Let $f \in L^p([0,1])$ for $p > 2$, and define

$$\sigma_r = f(r)\exp(cW_1).$$

For simplicity let us take $s = 0$ in equation (5.2). We then have

$$\varphi_t(\cdot, \omega) = \omega_\cdot + \int_0^{t\wedge\cdot} f(r)\exp(c\varphi_r(1,\omega))dr, \tag{6.1}$$

$0 \leq t \leq 1$ and $\omega \in \Omega$. Simplify further by taking $f \equiv c = 1$.

Suppose that equation (6.1) has a strong solution; then

$$\varphi_t(1,\omega) = \omega_1 + \int_0^t \exp(\varphi_r(1,\omega))dr, \tag{6.2}$$

$0 \leq t \leq 1$, has a strong solution. Hence

$$\frac{d\,\exp\{-\varphi_t(1,\omega)\}}{dt} = -1$$

for almost all $t \in [0,1]$, and hence from (6.2) we have

$$\exp\{-\varphi_t(1,\omega)\} = -t + \exp(-\omega_1),$$

for all $t \in [0,1]$, which is impossible if $\omega \in \{\omega_1 > 0\}$. Therefore equation (6.1) cannot have a strong solution in this case.

Note that if $f(s) > 0$ for $s \in [0,1]$, and if $c < 0$, then we are in the framework of Section Five, and equation (6.1) does have a unique strong solution. In this case the equation

$$\varphi_t(1,\omega) = \omega_1 + \int_0^t f(r)\exp(c\varphi_r(1,\omega))dr,$$

$0 \leq t \leq 1$, has the (unique) solution:

$$\varphi_t(1,\omega) = \frac{-1}{c}\, \ell n\{-c \int_0^t f(r)dr + \exp(-c\omega_1)\},$$

for $0 \leq t \leq 1$. Therefore:

$$\varphi_t(u,\omega) = \omega_u - \omega_1 + \varphi_{t\wedge u}(1,\omega),$$

which is a closed form formula for the solution of equation (6.1). Note that in this case

$$\varphi_t(\Omega) \subset \{\omega_1 > -\frac{1}{c}\ell n\{-c \int_0^t f(r)dr\}\}$$

and therefore $P(\varphi_t(\Omega)) < 1$, which is interesting in view of Remark (5.20).

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