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Abstract. In the usual linear regression model we investigate the geometric structure of a class of minimax optimality criteria containing Elfving's minimax and Kiefer's ϕ_p-criterion as special cases. It is shown that the optimal designs with respect to these criteria are also optimal for $A'\theta$ where A is any inball vector (in an appropriate norm) of a generalized Elfving set. The results explain the particular role of the A- and E-optimality criterion and are applied determining the optimal design with respect to Efving's minimax criterion for polynomial regression up to degree 9.

1. Introduction. For a compact metric space $\mathcal{X}$ which contains at least k different points we consider the usual linear regression model $y = f(x)'\theta$, $x \in \mathcal{X}$. For each $x \in \mathcal{X}$ a random variable $Y(x)$ with mean $f(x)'\theta$ and variance $\sigma^2 > 0$ can be observed where different observations are assumed to be uncorrelated. The vector of continuous, real valued and linearly independent regression functions $f(x) = (f_1(x), \dots, f_k(x))'$ is known while $\theta \in \mathbb{R}^k$ is an unknown parameter vector. A design ξ is a probability measure on a sigma field on $\mathcal{X}$ which contains all one point sets. The performance of a given design is evaluated by its information matrix

$$M(\xi) = \int_{\mathcal{X}} f(x)f(x)'\,d\xi(x) \in \mathbb{R}^{k\times k}.$$

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If ξ is an exact design concentrating masses $\frac{n_i}{n}$ at the points x_i $(i = 1, \ldots, s)$ the information matrix $M(\xi)$ is proportional to the inverse of the covariance matrix of the least squares estimator calculated from n observations, n_i at each x_i $(i = 1, \ldots, s)$.

Almost all optimality criteria which can be used to discriminate between competing designs depend on the information matrix $M(\xi)$ or its inverse (see e.g. Silvey (1980) or Pukelsheim (1993)). In this paper we will consider the geometric structure of two generalizations of the E-optimality criterion which minimizes the maximum eigenvalue of the inverse of the information matrix. The first extension of this criterion is due to Kiefer (1974), eq. (4.18), (see also Kiefer (1975), p. 337) who defines a design ξ_p to be ϕ_p-optimal if ξ_p minimizes

$$(1.1) \qquad \phi_p(M(\xi)) = \begin{cases} (\text{tr } (M(\xi)^{-p}))^{1/p} & \text{if } M(\xi) \text{ is positive definite} \\ \infty & \text{else} \end{cases}$$

Here $1 \leq p \leq \infty$ and the case $p = \infty$ gives the E-optimality criterion. Note that we have omitted the factor $1/k$ in our definition and that Kiefer's ϕ_p-criteria can be also considered for the case $-1 \leq p < 1$ (see Pukelsheim (1980)), but throughout this paper we will assume that $p \geq 1$. A generalization of the E-optimality criterion in a different direction results from the Courant Fischer characterization of the maximum eigenvalue of $M^{-1}(\xi)$

$$\lambda_{\max}(M^{-1}(\xi)) = \max\{c'M^{-1}(\xi)c | \ c \in \mathbb{R}^k, \ |c|_2 = 1\}$$

(here $|\cdot|_2$ denotes the euclidean norm on $\mathbb{R}^k$). Replacing the euclidean norm $|\cdot|_2$ by an arbitrary norm $|\cdot|$ on $\mathbb{R}^k$ we will call a design ξ minimax optimal with respect to the $|\cdot|$-norm if ξ minimizes

$$(1.2) \qquad \phi_{|\cdot|}(M(\xi)) = \max\{c'M^{-1}(\xi)c \mid c \in \mathbb{R}^k, \ |c| = 1\}.$$

We will omit the dependency on the norm in this definition whenever it is clear from the context which norm is used in the minimax optimality criterion (1.2).

In Section 2 we introduce a general minimax criterion which contains (1.1) and (1.2) as special cases. It is shown that the minimax optimal design with respect to this criterion is also optimal for $A'\theta$ where $A \in \mathbb{R}^{k \times k^2}$ is an inball vector of a k^3- dimensional Elfving set (in an appropriate norm). The criteria (1.1) and (1.2) are discussed as special cases in Section 3. Finally, the results are applied in Section 4 for the determination of the optimal design with respect to Elfving's minimax criterion (Elfving (1959)) in polynomial regression models up to degree 9.

2. Optimal Minimax Designs. Let $l \in \mathbb{N}$ and let $|\cdot|$ denote an arbitrary matrix norm on $\mathbb{R}^{k \times l}$ with dual or conjugate morm $|\cdot|_*$, i.e.

$$(2.1) \qquad |D|_* := \max\{tr(D'C) \mid C \in \mathbb{R}^{k \times l}, \ |C| = 1\}$$

(see e.g. von Neumann (1937), Rockafellar (1970) or Zietak (1988)). The unit spheres of $|\cdot|$ and $|\cdot|_*$ are denoted by $\mathcal{C}$ and $\mathcal{D}_*$, respectively, and we define a minimax criterion $\phi_{\mathcal{C}}$ and an information function $j_{\mathcal{D}_*}$ by

$$\text{(2.2)} \qquad \begin{aligned} \phi_{\mathcal{C}}(M(\xi)) &= \max\{tr(C'M^{-1}(\xi)C) \mid C \in \mathcal{C}\}, \quad M(\xi) > 0 \\ j_{\mathcal{D}_*}(M(\xi)) &= \min\{tr(D'M(\xi)D) \mid D \in \mathcal{D}_*\}, \quad M(\xi) \geq 0 \,. \end{aligned}$$

A design is called minimax optimal (with respect to the norm $|\cdot|$) if it minimizes $\phi_{\mathcal{C}}(M(\xi))$. In the following we will need an equivalence theorem for minimax optimal designs which can easily be obtained from general equivalence theorems for optimal designs (see e.g. Gaffke (1985, 1987), Pukelsheim (1993) or Hoang and Seeger (1991)).

Proposition 2.1. A design ξ_M is minimax optimal with respect to the $|\cdot|$-norm if and only if there exist an integer $1 \leq k_0 \leq k$, matrices $D_1, \ldots, D_{k_0} \in \mathcal{D}_*$ and positive numbers $\alpha_1, \ldots, \alpha_{k_0}$ with $\sum_{i=1}^{k_0} \alpha_i = 1$ such that $tr(D_i'M(\xi_M)D_i) = j_{\mathcal{D}_*}(M(\xi_M))$ $(i = 1, \ldots, k_0)$ and

$$\text{(2.3)} \qquad \sum_{i=1}^{k_0} \alpha_i tr(D_i' f(x) f(x)' D_i) \leq j_{\mathcal{D}_*}(M(\xi_M)) \qquad \text{for all } x \in \mathcal{X}.$$

Lemma 2.2 Let $M > 0$, then $\phi_{\mathcal{C}}(M) = [j_{\mathcal{D}_*}(M)]^{-1}$. Moreover $C_0 \in \mathcal{C}$ maximizes $tr(C'M^{-1}C)$ over $\mathcal{C}$ if and only if $D_0 = j_{\mathcal{D}_*}(M)M^{-1}C_0$ is an element of $\mathcal{D}_*$ and minimizes $tr(D'MD)$.

Proof: The relation $\phi_{\mathcal{C}}(M) = [j_{\mathcal{D}_*}(M)]^{-1}$ follows from Cauchy's inequality. If $C_0 \in \mathcal{C}$ maximizes $tr(C'M^{-1}C)$ we have for all $C \in \mathcal{C}$

$$tr^2(C'D_0) \leq (j_{\mathcal{D}_*}(M))^2 tr(C'M^{-1}C) tr(C_0'M^{-1}C_0) \leq 1$$

(with equality for $C = C_0$) which shows that $D_0 \in \mathcal{D}_*$. Conversely, if $D_0 \in \mathcal{D}_*$ minimizes $tr(D'MD)$ and $C_0 \in \mathcal{C}$ satisfies $tr(C_0'D_0) = |D_0|_* = 1$, then

$$1 = tr^2(C_0'D_0) \leq tr(C_0'M^{-1}C_0) tr(D_0'MD_0) \leq j_{\mathcal{D}_*}(M)\phi_{\mathcal{C}}(M) = 1$$

which shows that C_0 maximizes $tr(C'M^{-1}C)$ and $C_0 = [j_{\mathcal{D}_*}(M)]^{-1}MD_0$ ∎

Remark 2.3. Note that for all positive definite matrices M, all positive numbers α_i with $\sum_i \alpha_i = 1$ and all matrices $D_i \in \mathcal{D}_*$ with $tr(D_i'MD_i) = j_{\mathcal{D}_*}(M)$ the matrix

$$E = -\phi_{\mathcal{C}}(M) \sum_i \alpha_i D_i D_i'$$

is a subgradient of $\log(\phi_C)$ at M (see Gaffke (1985), Lemma 3).

Throughout this paper we will use the following matrix norm on $\mathbb{R}^{k\times lm}$ ($m \in \mathbb{N}$) induced by a given vector norm $|\cdot|$ on $\mathbb{R}^{k\times l}$. For a given matrix $\tilde{A} = (A_1, \ldots, A_m) \in \mathbb{R}^{k\times lm}$ ($A_i \in \mathbb{R}^{k\times l}$) define

$$||A|| = \left(\sum_{i=1}^{m} |A_i|^2\right)^{1/2},$$

then it is easy to see that the dual norm of $||\cdot||$ is given by

$$(2.4) \qquad ||\tilde{D}||_* = \left(\sum_{i=1}^{m} |D_i|_*^2\right)^{1/2}$$

where $|\cdot|_*$ is the dual of the given matrix norm $|\cdot|$ on $\mathbb{R}^{k\times l}$ ($\tilde{D} = (D_1, \ldots, D_m)$). We consider a generalized Eflving set

$$(2.5) \quad \mathcal{R}_m^{(l)} = co\left(\{(f(x)\varepsilon_1', \ldots, f(x)\varepsilon_m') \mid x \in \mathcal{X},\ \varepsilon_j \in \mathbb{R}^l,\ \sum_{j=1}^{m} |\varepsilon_j|_2^2 = 1\}\right) \subseteq \mathbb{R}^{k\times lm}$$

where $co(\mathcal{A})$ denotes the convex hull of the set $\mathcal{A} \subseteq \mathbb{R}^{k\times lm}$. Note that $\mathcal{R}_m^{(l)}$ is convex, compact, symmetric with respect to the origin and that for $l = m = 1$ this definition gives the set introduced by Elfving (1952) while for $l = 1$ or $m = 1$ the definition (2.5) yields the generalized Elfving set considered in Studden (1971). A more general version of this set and some examples illustrating its geometric structure are discussed in the context of model robust designs by Dette (1993). The minimum distance of all boundary points of $\mathcal{R}_m^{(l)}$ to the origin

$$r_m^{(l)} = \min\{||\tilde{A}|| \mid \tilde{A} \in \partial\mathcal{R}_m^{(l)}\}$$

is called inball radius of $\mathcal{R}_m^{(l)}$ and every matrix $\tilde{A}$ with $||\tilde{A}|| = r_m^{(l)}$ is called inball vector of $\mathcal{R}_m^{(l)}$. The following Theorem shows that inball radii and vectors of the Elfving set in (2.5) are intimitately related to the minimax optimal design problem.

Theorem 2.4 Let $m \geq k_0$ and $\alpha_1, \ldots, \alpha_{k_0}$, $D_1, \ldots, D_{k_0} \in \mathcal{D}_*$ denote the quantities of Proposition 2.1.

a) Let $\tilde{D} = (j_{\mathcal{D}_*}(M(\xi_M))^{-1/2}\ (\sqrt{\alpha_1}, D_1, \ldots,\ \sqrt{\alpha_{k_0}}\ D_{k_0}, 0, \ldots, 0) \in \mathbb{R}^{k\times lm}$ and define $\tilde{A} = M(\xi_M)\tilde{D}$, then $\tilde{A}$ is a $||\cdot||$-inball vector of $\mathcal{R}_m^{(l)}$ with supporting hyperplane $\tilde{D}$. The $||\cdot||$-inball radius is given by $r_m^{(l)} = (\phi_C(M(\xi_M)))^{-1/2}$.

b) The minimax optimal design ξ_M (with respect to the $|\cdot|$-norm) is optimal for $\tilde{A}'\theta$ where $\tilde{A} \in I\!R^{k\times lm}$ is any $||\cdot||$-inball vector of $\mathcal{R}_m^{(l)}$. If $\tilde{D} \in I\!R^{k\times lm}$ is a supporting hyperplane to $\mathcal{R}_m^{(l)}$ at the $||\cdot||$-inball vector $\tilde{A}$, we have $|\tilde{D}'f(x_i)|_2 = 1$ for all support points x_i of ξ_M.

Proof. Let $\tilde{N} = (N_1, \ldots, N_m) \in I\!R^{k\times lm}$ ($N_i \neq 0$), then we have for all $k \times k$ matrices $B \geq 0$

$$j_{\mathcal{D}_*}(B) \;=\; \min\left\{\frac{tr(N'BN)}{|N|_*^2} \mid N \in I\!R^{k\times l}\setminus\{0\}\right\} \;\leq\; \frac{tr(N_i'BN_i)}{|N_i|_*^2} \qquad (i = 1,\ldots,m)$$

which implies (using (2.4))

$$||\tilde{N}||_*^2 \;\leq\; \frac{\sum_{i=1}^m tr(N_iN_i'B)}{j_{\mathcal{D}_*}(B)} \;=\; \frac{tr(\tilde{N}\tilde{N}'B)}{j_{\mathcal{D}_*}(B)}.$$

Because $j_{\mathcal{D}_*}$ is an information function (see Pukelsheim (1980)) it thus follows for the polar function of $j_{\mathcal{D}_*}$

$$(2.6) \qquad j^o_{\mathcal{D}_*}(\tilde{N}\tilde{N}') \;=\; \inf\left\{\frac{tr(\tilde{N}\tilde{N}'B)}{j_{\mathcal{D}_*}(B)} \mid B \neq 0\right\} \;\geq\; ||\tilde{N}||_*^2\,.$$

From the definiton of $\tilde{D}$ and $\tilde{A}$ we have $tr(\tilde{D}'\tilde{A}) = 1$ and Proposition 2.1 implies that $\tilde{A} \in \partial\mathcal{R}_m^{(l)}$ with supporting hyperplane $\tilde{D}$. From Lemma 2.2, (2.6) and Pukelsheim's "Mutual Boundness" Theorem 3 (see Pukelsheim (1980)) that

$$(2.7) \quad [r_m^{(l)}]^2 \;\leq\; ||\tilde{A}||^2 \;=\; [\phi_C(M(\xi_M))]^{-1} \;=\; j_{\mathcal{D}_*}(M(\xi_M)) \;\leq\; \frac{1}{j^o_{\mathcal{D}_*}(\tilde{N}\tilde{N}')} \;\leq\; \frac{1}{||\tilde{N}||_*^2}$$

for all covering halfspaces $\tilde{N}$ of $\mathcal{R}_m^{(l)}$ (that is $|\tilde{N}f(x)|_2^2 = \sum_{i=1}^m f(x)'N_iN_i'f(x) \leq 1\ \forall x \in \mathcal{X}$) Using the representation

$$(2.8) \qquad r_m^{(l)} \;=\; \min\left\{\frac{1}{||\tilde{N}||_*} \mid \tilde{N} \in I\!R^{k\times lm},\ |\tilde{N}'f(x)|_2 \leq\ 1,\ \forall x \in \mathcal{X}\right\}$$

the assertion a) follows. Part b) is proved by exactly the same arguments as in Dette and Studden (1993) and therefore omitted. ∎

Remark 2.5. If $\tilde{D} = (D_1, \ldots, D_m) \in I\!R^{k\times lm}$ is a covering halfspace to $\mathcal{R}_m^{(l)}$ achieving the minimum in (2.8), then the matrix $\tilde{A} = (|D_1|_*A_1, \ldots, |D_m|_*A_m)/||D||_*^2$ defines a $||\cdot||$-inball vector of the Elfving set $\mathcal{R}_m^{(l)}$, where $A_j \in I\!R^{k\times l}$ is any matrix satisfying

$$|A_j| = 1\ , \qquad tr(D_j'A_j) = |D_j|_* \quad (j = 1,\ldots,m)$$

(the matrix A_j is called dual of D_j with respect to the $|\cdot|$-norm see Zietak (1988)). Even if the optimal covering halfspace cannot be determined the covering halfspaces of $\mathcal{R}_m^{(l)}$ provide lower bounds for the minimax efficiency

$$\mathrm{Eff}_{\mathcal{C}}(\xi) \ := \ \frac{\phi_{\mathcal{C}}(M(\xi_M))}{\phi_{\mathcal{C}}(M(\xi))}$$

of a given design ξ when the optimal minimax design ξ_M with respect to the $|\cdot|$-norm is unknown.

Corollary 2.6. Let $m \geq 1$ and $\tilde{D}$ denote a supporting hyperplane to $\mathcal{R}_m^{(l)}$, then the minimax efficiency (with respect to the $|\cdot|$-norm) of a given design ξ is bounded by

$$\mathrm{Eff}_{\mathcal{C}}(\xi) \ \geq \ \frac{j^o_{\mathcal{D}_*}(\tilde{D}\tilde{D}')}{\phi_{\mathcal{C}}(M(\xi))} \ \geq \ \frac{||\tilde{D}||_*^2}{\phi_{\mathcal{C}}(M(\xi))}.$$

If $\tilde{D}$ is an optimal supporting hyperplane (i.e. $\tilde{D}$ minimizes (2.8)) then the equality $j^o_{\mathcal{D}_*}(\tilde{D}\tilde{D}') = ||\tilde{D}||_*^2$ holds.

Proof. This is an immediate consequence of (2.7) and (2.8). ∎

Remark 2.7. The results of Theorem 2.4 can easily be generalized to minimax optimal design problems for parameter subsystems. For a given $k \times s$ matrix K of rank s a minimax optimal design for $K'\theta$ allows the estimability of $K'\theta$ (i.e. range $(K) \subseteq$ range $(M(\xi))$) and minimizes $\phi_{\mathcal{C}}((K'M(\xi)^-K)^{-1})$. According to Theorem 1 of Gaffke (1987) there exists a left inverse $L_0' \in \mathbb{R}^{s\times k}$ of K such that the minimax optimal design for $K'\theta$ is minimax optimal for the full parameter vector in the "new" regression setup $y = \tilde{\theta}'\tilde{f}(x)$ where $\tilde{f}(x) = L_0'f(x)$. Thus we obtain from Theorem 2.4 that the minimax optimal design for $K'\theta$ is optimal for $A'\tilde{\theta}$ for any $||\cdot||$-inball vector $A \in \mathbb{R}^{s\times sl}$ of the Elfving set $\tilde{\mathcal{R}}_s^{(l)}$ where $\mathcal{R}_m^{(l)}$ is defined as

$$\tilde{\mathcal{R}}_m^{(l)} = co\left(\left\{ \ L_0'f(x)(\varepsilon_1',\ldots,\varepsilon_m') \mid x \in \mathcal{X}, \varepsilon_j \in \mathbb{R}^l \ , \ \sum_{j=1}^m |\varepsilon_j|_2^2 = 1 \ \right\}\right) \subseteq \mathbb{R}^{s\times ml}.$$

$(m = 1,\ldots,s)$. The applications of this result are limited (except in the case $s = k$ where $L_0' = K^{-1}$) because in general L_0 is unknown and a $||\cdot||$-inball vector of $\tilde{\mathcal{R}}_s^{(l)}$ cannot be found.

3. Elfving's minimax and Kiefer's ϕ_p-criterion. In this section we will return to the criteria defined in (1.1) and (1.2) which now emerge as special cases from the general theory of Section 2 where the Elfving set (2.5) is the same for both criteria.

Firstly let $l = 1$, then the criterion (2.2) reduces to the minimax criterion (1.2). The geometric structure of the minimax problem is described in Theorem 2.4 ($l = 1$) where the generalized Elfving set in (2.5) reduces to the set

$$\mathcal{R}_m = co(\{f(x)\varepsilon' \mid x \in \mathcal{X}, \varepsilon \in I\!R^m, |\varepsilon|_2 = 1\}) \tag{3.1}$$

which was firstly introduced by Studden (1971) characterizing the optimal designs for $A'\theta$ (here $A \in I\!R^{k\times m}$ is a given matrix). Theorem 2.4 now generalizes the results of Dette and Studden (1993) ($|\cdot| = |\cdot|_2$) to arbitrary criteria of the form (1.2). The following important examples are mentioned as special cases.

1) Considering the l_2- norm we obtain the E-optimality criterion while the l_1- norm yields to Elfving's minimax criterion (Elfving (1959)), that is

$$\phi_{|\cdot|_1}(M(\xi)) = \max\left\{c'M^{-1}(\xi)c \mid |c|_1 = 1\right\} = \max_{i=1}^{k}\{M^{-1}(\xi)\}_{ii} \ . \tag{3.2}$$

2) If the regression norm (see Pukelsheim (1981))

$$|c|^R = \inf\{\alpha \geq 0 |\ c \in \alpha\mathcal{R}_1\}$$

on $I\!R^k$ is used in definition (1.2) then it is straightforward to see that the optimality criterion (1.2) gives the well known G-optimality criterion, i.e.

$$\phi_{|\cdot|^R}(M(\xi)) = \max\{c'M^{-1}(\xi)c|\ c \in \partial\mathcal{R}_1\} = \max_{x\in\mathcal{X}} f(x)'M^{-1}(\xi)f(x)$$

(note that $|\cdot|^R$ characterizes the Elfving set $\mathcal{R}_1$ as the unit ball). The dual norm of $|\cdot|^R$ is given by $|d|_*^R = \max_{x\in\mathcal{X}} |d'f(x)|$ (see e.g. Householder (1965)).

Secondly let $l = k$ and define a norm on $I\!R^{k\times k}$ by

$$||A||_{p'} = |\sigma(A)|_{p'} = (tr(AA')^{\frac{p'}{2}})^{\frac{1}{p'}} \qquad (1 \leq p' \leq \infty)$$

where $\sigma_1(A) \leq \ldots \leq \sigma_k(A)$ denote the singular values of a given matrx $A \in I\!R^{k\times k}$, $\sigma(A) = (\sigma_1(A), \ldots, \sigma_k(A))'$ and $|\cdot|_{p'}$ is the $l_{p'}$- norm on $I\!R^k$. Putting $p' = 2p/(p-1)$ we obtain for the optimality criterion (2.2).

$$\begin{aligned}\phi_{\mathcal{C}}(M(\xi)) &= \max\{tr(C'M^{-1}(\xi)C)|\ C \in I\!R^{k\times k}, ||C||_{p'} = 1\} \\ &= \max\{tr(B'M^{-1}(\xi)) \mid B \geq 0, ||B||_{p/(p-1)} = 1\} \\ &= ||M^{-1}(\xi)||_p \ = \ \phi_p(M(\xi))\end{aligned}$$

where the last line follows from Theorem 5.10 in Gaffke and Krafft (1982). Using Lemma 3 in Pukelsheim (1980) we obtain that for $1 \le p < \infty$ the quantities in Proposition 2.1 are given by

$$(3.3) \qquad k_0 = 1 \quad \text{and} \quad D_1 = (tr(M(\xi_p)^{-p}))^{-(p+1)/2p} M(\xi_p)^{-(p+1)/2} Q$$

where Q denotes an arbitrary orthogonal $k \times k$ matrix and ξ_p the ϕ_p- optimal design. If $p = \infty$ a possible choice for D_1 is the matrix

$$(3.4) \qquad D_1 = (\sqrt{\beta_1} z_1, \ldots, \sqrt{\beta_{k_1}} z_{k_1}, 0, \ldots 0) Q \in \mathbb{R}^{k \times k}$$

where $k_1 \le k$, $\beta_j > 0$, $\sum_{j=1}^{k_1} \beta_j = 1$ and $z_1, \ldots z_{k_1}$ are normalized eigenvectors of $M(\xi_\infty)$ corresponding to its minimum eigenvalue which satisfy the inequality (2.3) in Proposition 2.1 for the E-optimality criterion (see also Corollary 8.1 in Pukelsheim (1980)). By an application of Theorem 2.4 we thus obtain the following result.

Corollary 3.1. For $1 \le p \le \infty$ let ξ_p denote the ϕ_p-optimal design and let D_1 be defined by (3.3) if $1 \le p < \infty$ and by (3.4) if $p = \infty$. The matrix $A = \phi_p(M(\xi_p))^{1/2} M(\xi_p) D_1$ defines a $|| \cdot ||_{2q}$-inball vector of the Elfving set $\mathcal{R}_k$ with supporting hyperplane $\phi_p(M(\xi_p))^{1/2} D_1$ $(1/p + 1/q = 1)$. The $|| \cdot ||_{2q}$-inball radius of $\mathcal{R}_k$ is given by $(\phi_p(M(\xi_p)))^{-1/2}$.

b) If $1 \le p \le \infty$ and A is any $||\cdot||_{2q}$-inball vector of the Elfving set $\mathcal{R}_k$, then the ϕ_p-optimal design ξ_p is also optimal for $A'\theta$.

Remark 3.2. Let $\tilde{p} = 2p/(p+1)$ and let $D \in \mathbb{R}^{k \times k}$ denote an "optimal" covering halfspace, i.e. $||D||_{\tilde{p}} = 1/r_k$ with singular value decomposition $D = U$ diag$(\sigma(D))$ V' (here diag$(x_1, \ldots, x_k)$ means a diagonal matrix with diagonal elements $x_1, \ldots, x_k$), then a $||\cdot||_{2q}$ inball vector can be obtained as follows. Consider a dual vector $\sigma^*(D)$ of $\sigma(D) \in \mathbb{R}^k$ with respect to the ℓ_{2q}-norm (i.e. $\sigma^*(D)'\sigma(D) = |\sigma(D)|_{\tilde{p}}$, $|\sigma^*(D)|_{2q} = 1$) and define $A = U$ diag$(\sigma^*(D))V'/||D||_{\tilde{p}}$. Thus we obtain tr$(D'A) = |\sigma(D)|_{\tilde{p}}/||D||_{\tilde{p}} = 1$ and $||A||_{2q} = 1/||D||_{\tilde{p}}$ which shows that A defines an inball vector of $\mathcal{R}_k$. For $1 < p < \infty$ the strict convexity of the ℓ_{2q}-norm implies that A is the unique $|| \cdot ||_{2q}$-inball vector corresponding to D (Zietak (1988), Theorem 3.1, Corollary 4.2).

Remark 3.3. Recalling the discussion in Remark 2.7 we see that Theorem 3.1 gives new insight into the particular role of the ϕ_1-optimality criterion. Here $(q = \infty)$ any $s \times s$ orthogonal matrix Q (appropriately scaled) defines a $|| \cdot ||_\infty$-vector of $\tilde{\mathcal{R}}_1^{(s)}$ (this follows from Theorem 3.1 a)).

It should also be mentioned that the results of this section can easily be generalized for unitarily invariant norms on $\mathbb{R}^{k\times k}$. These norms are obtained by replacing the ℓ_p-norm in (3.1) by a so called symmetric gauge function $\psi(\cdot)$ on $\mathbb{R}^k$ which satisfies in addition to the norm properties the symmetry assumption

$$\psi((\varepsilon_1 a_{i_1},\dots,\varepsilon_k a_{i_k})') = \psi((a_1,\dots,a_k)')$$

for all permutations $a_{i_1},\dots,a_{i_k}$ of $a_1,\dots,a_k$ and for all $\varepsilon_j = \mp 1$ (see von Neumann (1937), Mudholkar (1966) or Zietak (1988) for more details).

4. Elfving's minimax criterion for polynomial regression. Let $l=1$, $\mathcal{X}=[-1,1]$, $f(x)=(1,x,\dots,x^d)'$, and $1\le p\le\infty$, thus we are faced with the minimax criterion (1.2) with respect to the ℓ_p-norm defined in (3.2). In contrary to an example for spring balance weighing designs ($p=2$) discussed in Dette and Studden (1993) the situation here is more complicated because we are not able to find the $||\cdot||_p$-inball radius of the Elfving set $\mathcal{R}_{d+1}$ defined in (3.1). However, if the (unknown) number k_0 in Proposition 2.1 is 1, Theorem 2.4b) shows that the minimax design $\xi_{|\cdot|}$ is already optimal for any $||\cdot||_p$-inball vector c of the first Elfving set $\mathcal{R}_1$. This fact was used by Pukelsheim and Studden (1993) to show that the E-optimal design (minimax with respect to the ℓ_2-norm) is supported at the Chebyshev points $s_j=\cos(\frac{d-j}{d}\pi)$ $(j=0,\dots,d)$. Observing these results and Corollary 2.6 it will therefore be useful to find (at least) the $||\cdot||_p$-inball vectors of $\mathcal{R}_1$ and the corresponding optimal designs. The optimal designs for these inball vectors seem to be good candidates for minimax optimality. Throughout this example let ξ_k denote the optimal design minimizing the variance of the least squares estimator for the individual coefficient θ_k in the polynomial regression $y=\theta_0+\theta_1 x+\dots+\theta_d x^d$ (see Studden (1968)) and define $t=(t_0,\dots,t_d)')$ as the vector of the coefficients of the Chebyshev polynomial of the first kind, i.e. $t'f(x)=T_d(x)=\cos(d \text{ arc } \cos x)$.

Theorem 4.1. a) If $1<p<\infty$, the $||\cdot||_p$-inball vector $c=(c_0,\dots,c_d)'$ of $\mathcal{R}_1$ has coordinates

$$c_i = \frac{\text{sign }(t_i)|t_i|^{q-1}}{|t|_q^q} \qquad i=0,\dots,d.$$

The c-optimal design for this inball vector is given by $\xi_c=\sum_{j=0}^d |t_j|^q/|t|_q^q\cdot\xi_j$ and the $||\cdot||_p$-inball radius is $1/|t|_q$

b) If $p=1$, the $||\cdot||_1$-inball vector has coordinates

$$c_i = \begin{cases} \text{sign }(t_i)\frac{g_i}{|t|_\infty} & \text{if } |t_i|=|t|_\infty \\ 0 & \text{if } |t_i|<|t|_\infty \end{cases}$$

where $g_i \geq 0$ and $\Sigma g_i = 1$. The c-optimal design is given by $\xi_c = \sum_j g_j \xi_j$ and the $||\cdot||_1$ inball radius is $1/|t|_\infty$.

Proof. Using (2.8) (for $m = l = 1$) we have to maximize $|a|_q$ subject to the restriction $|a'f(x)| \leq 1$ for all $x \in [-1, 1]$ ($a = (a_0, \ldots, a_d)' \in \mathbb{R}^{d+1}$). Using a result of Cantor (1977) we obtain for the coefficients of the vector a

$$|a_{d-2m}| + |a_{d-2m-1}| \leq |t_{d-2m}| \qquad (m = 0, \ldots, \lfloor \frac{d-1}{2} \rfloor)$$

with equality if and only if $a = \mp t$. This implies $|a|_q \leq |t|_q$ and (2.8) shows that the $||\cdot||_p$-inball radius of $\mathcal{R}_1$ is given by $|t|_q^{-1}$. By the discussion in Remark 2.5 we have to find a dual vector of t (with respect to the ℓ_p-norm) which can easily be obtained considering equality in the Hölder inequality (see e.g. Zietak p. 60). Thus the assertion about the inball vectors follows directly from Remark 2.5. Let $L_v(x) = \ell_{vo} + \ell_{v1}x + \ldots + \ell_{vd}x^d$ denote the v-th Lagrange interpolation polynomial at the points $s_0, \ldots, s_d$, then it follows from the results of Studden (1968) that the optimal design ξ_{d-2j} for estimating θ_{d-2j} puts masses $|\ell_{vd-2j}|/|t_{d-2j}|$ at the points $s_v (v = 0, \ldots, d)$ and Elfving's theorem (Elfving (1952)) yields

$$\frac{1}{|t_{d-2j}|} e_{d-2j} = \sum_{v=0}^{d} (-1)^{d-v+j} \frac{|\ell_{vd-2j}|}{|t_{d-2j}|} f(s_v) \qquad j = 0, \ldots, \lfloor \frac{d-1}{2} \rfloor.$$

Expressing the inball vector c as a linear combination of the unit vectors e_{d-2j} the assertion now follows directly by a further application of Elfving's theorem. ■

To be more explicit consider the case $d = 2$, then it is straightforward to show that the ℓ_p-optimal design ξ_c puts masses $2^{q-2}/(1 + 2^q)$ at the points -1 and 1 and mass $(1 + 2^{q-1})/(1 + 2^q)$ at the point 0. Using Lagrangian multipliers and Proposition 2.1 it can be shown by tedious computations that ξ_c is in fact the minimax design with respect to the ℓ_p-norm for all $1 \leq p < \infty$.

Recently Pukelsheim and Studden (1993) showed that ξ_c is E-optimal for all $d \in \mathbb{N}$ (that is minimax with respect to the ℓ_2-norm). We will conclude with an example demonstrating that this might not be true for arbitrary $p \geq 1$. To this end consider Elfving's minimax criterion (that is $p = 1$, $q = \infty$). Using a table of the Chebyshev polynomials of the first kind (see e.g. Davis (1963) p. 369) and Theorem 4.1b) we see that for $d = 1, 2, 3$ the design $\xi_c = \xi_d$ can be considered as a candidate for minimax optimality. For $d = 5, 6, 7, 8, 9$ we get ξ_{d-2} as a minimax candidate while in the case $d = 4$ (note that $T_4(x) = 8x^4 - 8x^2 + 1$) every convex combination $\alpha\xi_2 + (1-\alpha)\xi_4$ ($\alpha \in [0, 1]$) seems to be a good choice. Tedious algebra and Proposition 2.1 show that for $d = 1, 2, 3$ the design ξ_d is in fact minimax optimal with respect to Elfving's criterion. In the case $d = 5, 6, 7, 8, 9$ the design ξ_{d-2} can be shown to be minimax, while for $d = 4$ every convex combination of ξ_2 and ξ_4 fails to be minimax (with respect to Elfving's criterion). In this case the number k_0 in Proposition 2.1 is 2 the $||\cdot||_1$-inball radius of $\mathcal{R}_2$ can only be determined numerically and is smaller than $\frac{1}{|t|_\infty} = \frac{1}{8}$. However we can use Corollary 2.6 to obtain a lower bound for the minimax efficiency, i.e.

$$\mathrm{Eff}_{|\cdot|_1}(\xi) \geq |t|_\infty^2 \cdot \left[\max_{i=0}^{d}(e_i' M^{-1}(\xi) e_i)\right]^{-1}.$$

The average of the optimal designs for the coefficients θ_2 and θ_4 $\xi^* = \frac{1}{2}(\xi_2 + \xi_4)$ puts masses 3/32, 1/4, 5/16, 1/4, 3/32 and the points -1, $-1/\sqrt{2}$, 0, $1/\sqrt{2}$ and 1 and has at least minimax efficiency $\mathrm{Eff}_{|\cdot|_1}(\xi^*) \geq 30/31 \approx 0.9677$ which shows that ξ^* is a good choice with respect to Elfving's minimax criterion. Numerical calculations yield that for $d = 4$ the minimax design is not supported at the Chebyshev points and puts masses 0.0958, 0.246, 0.3164, 0.246, 0.0958 at the points -1, -0.7086, 0, 0.7086 and 1. Thus the exact minimax efficiency of the design ξ^* is 0.9997.

The results of the last paragraph suggest that for polynomial regression of degree d on the interval $[-1, 1]$ the minimax optimal design with respect to Elfving's minimax criterion is specified by the optimal design for the $|\cdot|_1$- inball vector of the first Elfving set $\mathcal{R}_1$ provided that $\#\{j \,|\, |t_j| = |t|_\infty\} = 1$. A partial proof of this conjecture and a more complete discussion of the problem including minimax optimal designs for parameter subsystems, different design spaces is given in a recent paper of Dette and Studden (1993).

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