REFLECTING STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

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Abstract

Lions and Sznitman (1984) studied diffusions reflected at the boundary of a domain in R^d. Saisho (1987) extended their results by weakening the conditions on the boundary of the domain. Menaldi and Robin (1985) published results on stochastic differential equations driven by a Lévy process with reflection at the boundary of a domain. The main condition Menaldi and Robin imposed on the Lévy process is that the jumps of the process have to put the solution process inside the domain.

We study a different model for stochastic differential equations driven by general semimartingales with reflection. This model introduced by Marcus (1978, 1981), later called Stratonovich type stochastic differential equation, imposes weaker conditions for existence and uniqueness than those in Menaldi and Robin (1985) when reflection is considered.

We also study stability results and the time reversal of the solutions of Stratonovich type stochastic differential equations with reflection.

1 Introduction

We are interested in addressing the following problem:

Given a bounded domain D in R^d, we want to find a "generalized" diffusion with given drift and diffusion coefficients driven by a general semimartingale (possibly with jumps) instead of the classical Brownian motion and Lebesgue measure. We also want the diffusion to live in D. Every time this generalized diffusion reaches the boundary it is contained by the boundary or it bounces back into the domain by the effect of any of the two coefficients.

First, start with the simpler Skorohod problem (originally formulated in 1961 for $D = R_+$); for a given continuous non-random $w(t)$, a domain D in R^d, find $(x(t), \phi(t))$ such that:

$$x(t) = w(t) - \phi(t), \ \ x(t) \in \bar{D}, \tag{1}$$

$$\phi(t) = \int_0^{t+} \theta(x(s)) d \mid \phi \mid (s), \quad \theta(x(s)) \in \Theta(x(s)), \mid \theta(s) \mid = 1, \ \ d \mid \phi \mid a.e., \tag{2}$$

$$\int_0^t 1(\{s \in R^+ / x(s) \in D\}) d \mid \phi \mid (s) = 0. \tag{3}$$

The equation (2) refers to the fact that the "reflection" function $\phi(t)$ has its differential in the direction $\theta(s)$. $\Theta(x)$ is a set of directions at the point $x \in \partial D$.

The condition (3) establishes the fact that $\phi(t)$ is only going to change (or "act") when $x(t) \in \partial D$. This also assures that $\phi(t)$ cannot kick the function $x(t)$ to the interior of the domain, but can only keep $x(t)$ within the closure of the domain.

The solution to this problem is a pair $(x(t), \phi(t))$, where $\phi(t)$ is sometimes called the reflection function or the regulator function in queueing theory contexts (see Williams and Harrison (1990)). If $\Theta(x)$ is a set of normal vectors, then the reflection is called normal. If not, then the reflection is called oblique.

For example, when $d = 1$ and $D = R_+$, it is known that

$$\phi(t) = sup_{s \leq t}((-w(s)) \vee 0),$$

where $\Theta(0) = -1$ (see Karatzas and Shreve (1988)).

The normal reflection problem has been solved under certain set of conditions by Lions and Sznitman (1984). Their results were subsequently refined by Saisho (1987).

Costantini (1991) worked with càdlàg functions and oblique reflection, while the previous two works deal exclusively with the continous case. Uniqueness has been proved in the continuous case with normal reflection but it is still open for the càdlàg case and for the oblique reflection case, as pointed out by Costantini (1991). One of the main problems related to the uniqueness is the non-uniqueness of the projection to the boundary mapping, which does hold for most non-convex domains .

Each of these studies approach the stochastic problem by first solving the deterministic problem, which is then applied to solve a stochastic differential equation with reflection of the following type:

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt - d\Phi(t), \ \ X(0) \in \bar{D},$$

where $B(t)$ is a Brownian motion, $X(t)$ stays in $\bar{D}$, $\Phi(t)$ only changes when $X(t) \in \partial D$ and its differential has the normal direction $n(X_t)$. Here σ and b are Lipschitz and bounded. In the stochastic case $\Phi(t)$ is called the reflection process.

For example, in the case $D = R_+, \sigma = 1,\ b = 0$, it is known that $X(t) = \mid B(t) \mid$ and by Lévy's Theorem we have:

$$L_t^0 = \Phi(t) = \max_{s \leq t}((-B(s)) \vee 0).$$

Here L_t^0 stands for the local time for Brownian motion at 0, and the above equality is in law.

Recently, Kurtz, Pardoux and Protter (1991) have reexamined the formulation of the Stratonovich differential equation driven by a general semimartingale with jumps. They proposed the following stochastic differential equation:

$$\begin{aligned} X(t) &= X_0 + \int_0^t f(X(s)_-)dZ_s + \frac{1}{2}\int_0^t f'f(X(s)_-)d[Z,Z]_s^c \\ &\quad + \sum_{0<s\leq t} \varphi(\Delta Z_s f, X(s)_-) - X(s)_- - \Delta Z_s f(X(s)_-). \end{aligned} \tag{4}$$

Here, $\varphi(g,x) = y(1)$ where y is the solution of the following ODE:

$$y(t) = x + \int_0^t g(y(s))ds.$$

Note that:

$$\Delta X(t) = \varphi(\Delta Z_t f, X(t)_-) - X(t)_-.$$

Here, f is a C_b^2-matrix function, whose ij element is denoted by f_j^i. The formula (4) expresses the intuition of its formulation. The idea is to open an interval of size 1 every time there is a jump and let the system be driven by $\Delta Z_s f$ for one unit of surreal time, then shrink this interval to zero and thereby obtain the jump. This could also be seen as a modelization of the fact that any process with jumps should be a continuous process that at certain times undergoes sudden and sharp changes (e.g. when we turn on a light the amount of current suddenly "jumps" up). Through a "time change" everything could be understood as a sort of continuous process. This idea was first used by S. Marcus (1978, 1981), his main interest being the stability of the solutions of stochastic differential equation driven by processes with jumps. Other types of Wong-Zakai results were obtained by Kushner (1979).

In order to justify this definition through a mathematical viewpoint, we have the following theorem:

Theorem 1 *(Kurtz, Pardoux and Protter) Let Z be a semimartingale (possibly with jumps) and $f \in C_b^1$. Define Z^h by*

$$Z_t^h = \frac{1}{h}\int_{t-h}^t Z_s ds. \tag{5}$$

Let X^h be a solution of the following stochastic differential equation:

$$dX_t^h = f(X_t^h)dZ_t^h,\ X_0^h = X_0. \tag{6}$$

Then there exists a sequence of random time changes $T_h(t)$ which converge uniformly on bounded intervals to t, such that $X_{T_h(t)}^h$ converges weakly to a process X_t which is the unique solution of the stochastic differential equation

$$X(t) = X_0 + \int_0^t f(X(s)_-)dZ_s + \int_0^t \frac{1}{2} f'f(X(s)_-)d[Z,Z]_s^c + \sum_{0<s\le t} \varphi(\Delta Z_s f, X(s)_-) - X(s)_- - \Delta Z_s f(X(s)_-). \tag{7}$$

Here, $\varphi(g,x) = y(1)$ *where* y *is the solution of the following ODE:*

$$y(t) = x + \int_0^t g(y(s))ds.$$

Moreover, $\int_0^t f'f(X(s)_-)d[Z,Z]_s^c$ *is understood as (note that this is not a matrix notation)*

$$\sum_{j,l,m} \int_0^t \frac{\partial f_j}{\partial x^l} f_m^l (X(s)_-)d[Z^j, Z^m]_s.$$

An extension of this theorem will be proved in Section 3.1. Our goal is to study stochastic differential equations (SDE's) with reflection driven by semimartingales with jumps. We will also show that this new type of SDE have desirable properties that are not known for the classical models. We will study the SDE for $0 \le t \le 1$, although 1 could obviously be replaced by any other constant $c < \infty$. Consider the following equation:

$$X(t) = X_0 + \int_0^t f(X(s)_-)dZ_s + \frac{1}{2}\int_0^t f'f(X(s)_-)d[Z,Z]_s^c - \Phi(t) + \sum_{0<s\le t} \varphi(\Delta Z_s f, X(s)_-) - X(s)_- - \Delta Z_s f(X(s)_-). \tag{8}$$

$X(t) \in \bar{D}$, $0 \le t \le 1$, $\Phi(t)$ is continuous and $\varphi(g,x) = y(1)$, where y is the solution of the following deterministic differential equation with normal reflection:

$$y(t) = x + \int_0^t g(y(s))ds - \kappa(t), \ y(t) \in \bar{D}, \tag{9}$$

$$\kappa(t) = \int_0^t \theta(y(s))d \mid \kappa \mid_s, \ \ \theta(y(s)) \in \Theta(y(s)), \ |\theta(y(s))| = 1, \ d|\kappa| a.e., \tag{10}$$

$$\kappa(t) = \int_0^t 1(y(s) \in \partial D)d\kappa(s). \tag{11}$$

The interpretation of the last two equations is the same as in (1). $\varphi(g,x)$ can be interpreted as the fact that every jump is to be the effect of the field $\Delta Z_s f$ reflected on the boundary of D, which is so sudden that it can not be appropriately seen unless we do a time change and rescale everything to an interval of meta-time size equal to 1 (the size of the interval does not matter as long as the rescaling is appropiate). As usual, we assume the following two conditions on Φ:

$$\Phi(t) = \int_0^t \theta(X(s))d \mid \Phi \mid (s), \tag{12}$$

$$\Phi(t) = \int_0^t 1(X(s) \in \partial D)d\Phi(s). \tag{13}$$

This stochastic differential equation will henceforth be called the Stratonovich type stochastic differential equation (SDE) with reflection driven by Z. We adopt this name because in the case that Z is continuous we obtain the classical Stratonovich integral on the right side of (8). In the following sections we will sometimes fail to recall some or all of these five conditions ((9)- (13)), assuming that they are understood.

In Section 3.1 we consider general semimartingales in smooth domains. We prove existence and uniqueness of solutions of 8 as well as adaptedness and Markov properties of the solution process. The existence theorem will justify our approach to the jumps as sudden discontinuities.

Using the idea that this new SDE potentially has an infinite number of continuous parts, we will show in Section 4 how to obtain inequalities that give (in the normal reflection case) existence and uniqueness of solutions for general driving semimartingales with summable jumps on domains that are not smooth but hold some weaker conditions.

Our original interest was to prove some stability results with respect to the solutions of stochastic differential equations with reflection. In Section 5, we divided this study in two cases. First, when some sort of uniformity condition holds for the approximating sequence; and second, when the Wong-Zakai effect takes place. The study of the stability problem could lead to some results on the computer simulation of the solution processes.

In Section 6 we study the time reversal of solutions. The proofs turn to be very simple because the jumps are generated by the solution of an ordinary differential equation. These results also follow in the oblique reflection case, with conditions similar to the ones achieved by Lions and Sznitman.

We start with a simple case; where Z is continuous and the reflection is normal. We use the notation and definitions for semimartingale theory as found in, e.g., Protter's "Stochastic Integration and Differential Equations: A new approach." (1990). For stochastic processes we will use X_t or $X(t)$ interchangeably. $D[0,1]$ denotes the space of càdlàg functions on $[0,1]$ with the Skorohod topology. The arrow $\Rightarrow$ denotes weak convergence on $D[0,1]$. Although it might seem that we are only discussing the generalized diffusion case, it is not difficult to obtain extensions in which the coefficients f_i^j depend on (t,ω,x) instead of only x as we are doing here. The techniques to obtain such results are discussed in Doléans-Dade and Meyer (1977) as well as in Protter's book.

In what follows $C([0,T],D)$ denotes the space of continuous paths from $[0,T]$ into D, a subscript b stands for bounded and a numerical superscript stands for the number of times a function is continuously differentiable. Sometimes we write $C(D)$ for $C(D;R^k)$ when the situation is clear. Define $\widehat{C}(D) := \{w \in C(R_+;D)/w(0) \in \bar{D}\}$. Also we denote by $BV(0,T)$ the set of paths of bounded variation in R^d. All vectors are considered as row vectors and A^T denotes the transpose of A.

2 Preliminaries

2.1 Reflected SDE's driven by a continuous semimartingale

In this section we recall some results when the driving semimartingale is continuous, for proofs we refer the reader to Kohatsu-Higa (1992). Consider the following stochastic differential equation:

$$X(t) \;=\; X_0 + \int_0^t f(X(s))dZ_s + \int_0^t \frac{1}{2} f'f(X(s))d[Z,Z]_s - \Phi(t), \qquad (14)$$

where $X(t) \in \bar{D}$ and

$$\Phi(t) = \int_0^t n(X(s))d \mid \Phi \mid (s), \tag{15}$$

$$\Phi(t) = \int_0^t 1(X(s) \in \partial D)d\Phi(s). \tag{16}$$

Z is a continuous semimartingale on R^k, and D is a domain holding the following conditions:

Define the set N_x of inward normal unit vectors at $x \in \partial D$ by

$$N_x = \bigcup_{r>0} N_{x,r}, \tag{17}$$

$$N_{x,r} = \{n \in R^d :\mid n \mid= 1, \; B(x - rn, r) \cap D = \emptyset\}, \tag{18}$$

where $B(z,r) = \{y \in R^d :\mid y - z \mid< r\}$, $z \in R^d$, $r > 0$.

Condition (A)(uniform exterior sphere condition) There exists a constant $r_0 > 0$ such that

$$N_x = N_{x,r_0} \neq \emptyset \text{ for any } x \in \partial D.$$

Condition (B). There exist constants $\delta > 0$ and $\beta \in [1, \infty)$ with the following property: For any $x \in \partial D$ there exists a unit vector l_x such that
$< l_x, n >\geq \frac{1}{\beta}$ for any $n \in \bigcup_{y \in B(x,\delta)\cap\partial D} N_y$.

For example, when the boundary of D is of class C^2, it is known that *Condition (A)* holds (see Gilbarg and Trudinger (1977)) and *Condition (B)* is a consequence of the continuity of the normal to the boundary function. Cases in which these two conditions hold but they are not C^2 abound. For example, any domain which is C^2 except a finite number of obtuse corners hold these conditions. We denote the finite variation part of the semimartingale Z by $FV(Z)$ and

$$\| [Z, Z] \|_s := \sum_{i=1}^{k} [Z^i, Z^i]_s.$$

From now on we will assume that $f_j^i \in C_b^1(\bar{D}; R)$, $i = 1, ..., d$ and $j = 1, ..., k$.

Theorem 2 *Let the domain D satisfy Condition (A) and (B). Then the system (14) has a unique solution.*

Corollary 3 *Let the domain D have a smooth boundary, then the system (14) has a unique solution.*

A weaker version of this result was mentioned by Lions and Sznitman (1984) but not proven.

2.2 Weak convergence of stochastic integrals

In the next sections we will extensively use results of weak convergence and tightness of stochastic integrals due to Kurtz and Protter (1991), we state them for completeness.

They studied the following problem:

Let $Z_n = M_n + A_n$ where M_n is a local martingale and A_n is a process of bounded variation paths (i.e. $\{Z_n\}$ is a sequence of semimartingales). Given (X_n, Z_n) converging weakly to (X, Z) in the Skorohod topology with Z_n semimartingales, under which conditions does $\int X_n dZ_n$ converge weakly to $\int XdZ$?

The following condition (some sort of uniform integrability condition) is the key condition on Z_n. Here, $T_t(A_n)$ denotes the total variation of A_n up to time t.

(*)Goodness condition.

A sequence of semimartingales $\{Z_n\}$ is said to hold the goodness condition if for each $\alpha > 0$, there exist stopping times $\{\tau_n^\alpha\}$ with $P\{\tau_n^\alpha \leq \alpha\} \leq \frac{1}{\alpha}$ such that for each $t \geq 0$,

$$sup_n E[[M_n, M_n]_{\tau_n^\alpha \wedge t} + T_{t \wedge \tau_n^\alpha}(A_n))] < \infty.$$

We have preferred to use a stronger condition than the goodness condition of Kurtz and Protter, because it is easier to understand and will suffice for our needs.

Theorem 4 *Suppose (*) holds and that X_n is bounded in probability (under the sup norm). If $\{Z_n\}$ is tight , then there exists a sequence of continuous, increasing stopping times $a_n(t)$ such that $(Z_n(a_n), \int_0^{a_n} X_n dZ_n, a_n)$ is tight.*

If $(Z_n(a_n), X_n(a_n), \int_0^{a_n} X_n dZ_n, a_n)$ is tight then $(Z_n, X_n, \int X_n dZ_n)$ is tight .

Also, if $(X_n, Z_n) \Rightarrow (X, Z)$ in the Skorohod topology, then Z is a semimartingale with respect to a filtration to which X and Z are adapted and $(X_n, Z_n, \int X_n dZ_n) \Rightarrow (X, Z, \int XdZ)$.

The above result holds if weak convergence is replaced by convergence in probability.

Although the tightness results might not seem to be satisfactory, there is an example in Kurtz and Protter (1991) that shows that better results fail without some extra conditions (see Remark 4.2(d) in Kurtz and Protter (1991)).

We will need the tightness of $(Z_n, \int X_n dZ_n)$ in a situation in which Theorem 4 only gives the tightness of $(Z_n(a_n), \int_0^{a_n} X_n dZ_n)$. In such a situation we refer to the proof of Proposition 4.3 in Kurtz and Protter where it can be seen that the key of the proof is to prove that on any interval on which a_n is constant $(\hat{Z}, \hat{A})$ is constant except for at most one jump ($(\hat{Z}, \hat{A})$ is any weak limit of $(Z_n(a_n), \int_0^{a_n} X_n dZ_n)$.

Kurtz and Protter have also obtained a Wong-Zakai type theorem, i.e., they characterized the weak limit for solutions of stochastic differential equations in which the integrator does not satisfy (*), but some other property close to (*) can be obtained.

Theorem 5 *Suppose $Z_n = Y_n + W_n$, where Y_n and W_n are also semimartingales adapted to the same filtration to which Z_n is adapted (these could change with n). Also, let X_n be the unique solution to the following stochastic differential equation*

$$X_n = X_n(0) + \int_0^t F(X_n(s-))dZ_n(s),$$

where $F \in C_b^2$. Define H_n and K_n by

$$H_n^{\beta\gamma}(t) = \int_0^t W_n^\beta(s-)dW_n^\gamma(s),$$

and

$$K_n^{\beta\gamma}(t) = [Y_n^\beta, W_n^\gamma]_t.$$

Suppose that $\{Y_n\}$ *and* $\{H_n\}$ *satisfy (*) and that*

$$(X_n(0), Y_n, W_n, H_n, K_n) \Rightarrow (X(0), Y, 0, H, K),$$

Then$(X_n(0), Y_n, W_n, H_n, K_n)$ *is relatively compact, and any limit point* $(X(0), Y, 0, H, K)$ *satisfies*

$$X(t) = X(0) + \int_0^t F(X(s-))dY(s) + \sum_{\alpha,\beta,\gamma} \int_0^t \partial_\alpha F_\beta(X(s-)) F_{\alpha\gamma}(X(s-)) d(H^{\gamma\beta}(s) - K^{\gamma\beta}(s)),$$

where ∂_α *denotes the partial derivative with respect to the* α*th variable and* F_β *denotes the* β*th column of* F. *This result holds true if weak convergence is replaced by convergence in probability.*

For proofs and exact statements, we refer the reader to Kurtz and Protter (1991).

2.3 Tightness results on the Skorohod problem

Costantini (1991) studied the relative compactness properties of solutions of the Skorohod problem. Here, we state weaker versions of these results which will be used later. In the next theorem we assume that D holds *Condition* (A) *and* (B).

Theorem 6 *Let* $w \in \widehat{C}(D)$, *then there exist positive constants* $K(w)$ *and* $K'(w)$ *such that, for any solution* (x, ϕ) *to the Skorohod problem (1):*

$$\sup_{s \le t_1 \le t_2 \le t} | x(t_1) - x(t_2) | \le K(w) \sup_{s \le t_1 \le t_2 \le t} | w(t_1) - w(t_2) | , \forall 0 \le s \le t \le 1, \quad (19)$$

$$| \phi | (t) - | \phi | (s) \le K'(w) \sup_{s \le t_1 \le t_2 \le t} | w(t_1) - w(t_2) | , \forall 0 \le s \le t \le 1. \quad (20)$$

If $W \subseteq \widehat{C}(D)$ *is relatively compact in the uniform topology, then:*

$$\sup_{w \in W} K(w) = K < \infty, \ \sup_{w \in W} K'(w) = K' < \infty. \quad (21)$$

The next theorem states the continuity property of the solution of the Skorohod problem.

Theorem 7 *Let* (x^n, ϕ^n) *be the solution to the Skorohod problem for* $w^n \in \widehat{C}$. *If* w^n *converges to a function* w *in the uniform topology, then the limit of* $\{(x^n, \phi^n)\}$ *exists and this limit is the solution of the Skorohod problem for* w.

For the proofs of the above two theorems see Costantini (1991), Theorems 2.4 and 3.2. It is not difficult to obtain the stochastic process version of the above two theorems . Just replace every ocurrence of a function with the corresponding continuous stochastic process.

3 Stratonovich type SDE's with reflection on a smooth domain

3.1 Existence of solutions

In this section, we will prove existence and uniqueness of Stratonovich type equations with reflection on smooth domains. The method will use ideas from Kurtz, Pardoux and Protter (1992). This entails the use of an approximating sequence for Z and a sequence of "random time changes" under which the approximating semimartingale does not jump. The results of the previous section can be applied to the approximations. The theorems will follow through the use of appropriate limits theorems from Kurtz and Protter (1991).

We will prove not only existence and uniqueness, but also give a scheme to approximate the solution for the general Stratonovich type stochastic differential equation with reflection. This argument also proves Theorem 1 in the Introduction.

Let Z be a semimartingale (possibly with jumps) that belongs to $\mathcal{H}^2$, i.e., there exists a local martingale M and a bounded variation process A such that $Z = M + A$ and

$$\|[M,M]_1^{\frac{1}{2}} + \int_0^1 |dA_s|\|_{L^2} < \infty.$$

This is not restrictive as we will note at the end of the section (see Remark 29).

We are looking for solutions of the following Stratonovich type stochastic differential equation with reflection when $f \in C_b^2(\bar{D})$:

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-})dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-})d[Z,Z]_s^c - \Phi(t) \\ &\quad + \sum_{0<s\leq t} \{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\}, \end{aligned} \tag{22}$$

$$\Phi(t) = \int_0^t n(X_s)d \mid \Phi \mid (s),\ X_t \in \bar{D},$$

and $\varphi(g,x) = y(1)$, where $y(t)$ is the solution of

$$y(t) = x + \int_0^t g(y(s))ds - \kappa(t),\ y(t) \in \bar{D}\ ,$$

$$\kappa(t) = \int_0^t n(y(s))d \mid \kappa \mid (s).$$

One of the conditions for X to be a solution for (22) is that the sum:

$$\sum_{0<s\leq t} \{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\} < \infty \text{ a.s.}$$

We start by defining the random time change that will let us consider approximating continous semimartingales to Z. It is known that

$$\Gamma_t^d = \sum_{0<s\leq t} |\Delta Z_s|^2 < \infty \text{ a.s.}$$

(d stands for discontinuous). The random time change that we want to introduce is

$$\gamma_h(t) = \frac{1}{h}\int_{t-h}^t (\Gamma_s^d + s)ds. \tag{23}$$

γ_h is continuous, adapted, strictly increasing and of trivially bounded variation on bounded intervals. We also have that $\gamma_h \to \gamma_0$ pointwise, where $\gamma_0(t) = \Gamma_t^d + t$. The approximating semimartingale is:

$$Z_t^h = \frac{1}{h}\int_{t-h}^{t} Z_s ds. \tag{24}$$

From the above definition it is easy to see that Z_t^h is continuous, adapted, and of bounded variation (therefore, a semimartingale). Define X_t^h as the unique continuous solution of the following stochastic differential equation with reflection:

$$X_t^h = X_0 + \int_0^t f(X_s^h)dZ_s^h - \Phi_t^h. \tag{25}$$

This solution exists and is unique because Z^h is a continuous semimartingale. To be able to achieve the convergence in $D[0,1]$, we need further to change the time frame. Accordingly, we define:

$$V_t^h = Z_{\gamma_h^{-1}(t)}^h. \tag{26}$$

Then V_t^h is continuous, adapted to the filtration $\mathcal{F}_{\gamma_h^{-1}(t)}$, and it is of bounded variation because γ_h is continuous and strictly increasing. Also note that $\gamma_h^{-1}(t)$ is an increasing sequence of stopping times.

Lemma 8 *Let Y_t^h be the unique solution of:*

$$Y_t^h = X_0 + \int_0^t f(Y_s^h)dV_s^h - \Psi_t^h. \tag{27}$$

Then $Y_t^h = X_{\gamma_h^{-1}(t)}^h$ and $\Psi_t^h = \Phi_{\gamma_h^{-1}(t)}^h$.

Proof.
The proof is standard and uses a change of variables formula for integrals with respect to bounded variation functions (see Dellacherie and Meyer p.153). □

Although γ_0 is not an onto function in general, it is possible to define a weaker inverse by

$$\gamma_0^{-1}(t) = inf\{s : \gamma_0(s) \geq t\}.$$

This inverse is a stopping time, nondecreasing in t, and constant every time γ_0 jumps.

Lemma 9 $\gamma_h^{-1}(t) \to \gamma_0^{-1}(t)$ *uniformly on bounded intervals.*

For the proof see Kurtz, Pardoux and Protter (1992).
Remark 10. Actually, as $\gamma_h(t) \leq \gamma_0(t)$, we have that $\gamma_h^{-1}(t) \geq \gamma_0^{-1}(t)$ for all t.

To characterize the limit process of the sequence V^h, define:

$$\eta_1(t) = sup\{s : \gamma_0^{-1}(s) < \gamma_0^{-1}(t)\},$$

$$\eta_2(t) = inf\{u : \gamma_0^{-1}(u) > \gamma_0^{-1}(t)\},$$

$$V_t^0 = \begin{cases} Z_{\gamma_0^{-1}(t)} & \text{if } \eta_1(t) = \eta_2(t); \\ Z_{\gamma_0^{-1}(t)}\dfrac{t-\eta_1(t)}{\eta_2(t)-\eta_1(t)} + Z_{\gamma_0^{-1}(t)-}\dfrac{\eta_2(t)-t}{\eta_2(t)-\eta_1(t)} & \text{if } \eta_2(t) \neq \eta_1(t). \end{cases}$$

Therefore, $\eta_1(t) \leq \eta_2(t)$ and $\gamma_0^{-1}(s)$ is constant for $[\eta_1(t), \eta_2(t)]$. V^0 is essentially Z time changed according to γ_0^{-1}, but we open a fictitious time interval $[\eta_1(t), \eta_2(t)]$ every time Z jumps. In this interval the value before the jump is joined linearly with the value after the jump, making V^0 continuous. V^0 is adapted to $\mathcal{F}_{\gamma_0^{-1}(t)}$, and is not necessarily a semimartingale because the interpolated pieces may add to infinity. In the rest of the discussion we will use the Skorohod topology for cádlág functions extensively. Theorems, results and notations can be found in Jacod and Shiryaev (1987). The proof of the following Lemma can be found in Kurtz, Pardoux and Protter (1992).

Lemma 11 *$V^h \to V^0$ uniformly on bounded intervals.*

We are interested in applying limit theorems to obtain the limit of (27) . For this we will use results from Kurtz and Protter (1991). Define $U_t^h = V_t^h - Z_{\gamma_0^{-1}(t)}$. Therefore, $U^h \Rightarrow U := V^0 - Z_{\gamma_0^{-1}(.)}$. The following sequence of lemmas show that the conditions of Theorem 5 hold in this case, and therefore we can take limits in (27). We also have to be careful with the manipulation of V^0 because as mentioned earlier V^0 is not necesarily a semimartingale and therefore an integral with respect to it might not exist.

Remark 12.

In the following lemmas we will use the following two facts extensively.

1. Using (26) and (24) we have:

$$\frac{dV_t^h}{dt} = \frac{dZ^h_{\gamma_h^{-1}(t)}}{dt} = \frac{1}{h}(Z_{\gamma_h^{-1}(t)} - Z_{\gamma_h^{-1}(t)-h})\frac{d\gamma_h^{-1}(t)}{dt}.$$

2. $\frac{d\gamma_h^{-1}(t)}{dt}$ can be obtained by an implicit differentiation of the following equation that can be obtained from (23),

$$t = \frac{1}{h}\int_{\gamma_h^{-1}(t)-h}^{\gamma_h^{-1}(t)} (\Gamma_s^d + s)ds.$$

One then obtains:

$$\frac{d\gamma_h^{-1}(t)}{dt} = \frac{h}{\Gamma^d_{\gamma_h^{-1}(t)} - \Gamma^d_{\gamma_h^{-1}(t)-h} + h}.$$

Lemma 13 *The sequence $\int_0^t f(Y_s^h)dV_s^h$ is tight.*

To prove Lemma 13, we use the following result:

Lemma 14 *(Lions and Sznitman) Let (x, κ) be the unique solution to the Skorohod problem associated with $w \in C^1(D)$. Then:*

$$\kappa_t = \int_0^t 1(x(s) \in \partial D)n(x(s)) < n(x(s)), \dot{w}(s) > ds. \tag{28}$$

Proof of Lemma 13.
As:

$$\int_0^t f(Y_s^h)dV_s^h = \int_0^t f(Y_s^h)dZ_{\gamma_0^{-1}(s)} + \int_0^t f(Y_s^h)dU_s^h,$$

it is enough to prove that $\int_0^t f(Y_s^h)dU_s^h$ is tight. To make things easier, we will suppose that $\{\int_0^t U_s^h dU_s^h\}$ is good (Condition (*) in subsection 2) in the sense of Kurtz and Protter (we will prove this later). Using the integration by parts formula we have:

$$\int_0^t f(Y_s^h)dU_s^h = f(Y_t^h)U_t^h - \int_0^t U_s^h f'(Y_s^h)dY_s^h - [f(Y_t^h), U_t^h]. \tag{29}$$

Here,

$$[f(Y_t^h), U_t^h] = \int_0^t f'(Y_s^h)f(Y_s^h)d[V^h, U^h]_s = 0. \tag{30}$$

Also (using equation (28)),

$$\begin{aligned}
\int_0^t U_s^h f'(Y_s^h)dY_s^h &= \int_0^t U_s^h f'f(Y_s^h)dZ_{\gamma_0^{-1}(s)} + \int_0^t U_s^h f'f(Y_s^h)dU_s^h + \\
&\quad - \int_0^t U_s^h f'(Y_s^h)d\Psi_s^h \\
&= \int_0^t U_s^h f'f(Y_s^h)dZ_{\gamma_0^{-1}(s)} + \int_0^t U_s^h f'f(Y_s^h)dU_s^h + \\
&\quad - \int_0^t U_s^h f'(Y_s^h)1(Y_s^h \in \partial D)n(Y_s^h)(n(Y_s^h), f(Y_s^h)dU_s^h) \\
&\quad - \int_0^t U_s^h f'(Y_s^h)1(Y_s^h \in \partial D)n(Y_s^h)(n(Y_s^h), f(Y_s^h)dZ_{\gamma_0^{-1}(s)}).
\end{aligned} \tag{31}$$

By looking at the effect of the calculations in (30) and (31) upon (29), it is not difficult to see (using Proposition 4.3 in Kurtz and Protter (1991) or Theorem (4)) that the tightness depends upon the goodness of $\int_0^t U_s^h dU_s^h$. We will prove the goodness in the next lemma. The tightness of

$$f(Y_t^h)U_t^h = f(Y_t^h)(V_t^h - V_t^0) + f(Y_t^h)U_t$$

can be obtained as follows. Take $\delta \in R$, and choose a partition $\{t_i\}$ so that $t_{i+1} - t_i > \delta$ and

$$\max_i w(Z, [t_i, t_{i+1}[) < \epsilon.$$

Here w denotes the modulus of continuity . This implies that all jumps of size bigger than ϵ are at some of the points t_i, and that any jump within $[t_i, t_{i+1}[$ has to be less than ϵ.

Obviously, $f(Y_t^h)(V_t^h - V_t^0)$ is tight because it converges to zero. If t, s are such that $0 < s - t < \delta$, then:

$$\begin{aligned}
&| f(Y_t^h)U_t - f(Y_s^h)U_s | \\
&\le \ |f(Y_t^h)\Delta Z_{\gamma_0^{-1}(t)} \frac{\eta_2(t) - t}{\eta_2(t) - \eta_1(t)} 1_{[\eta_1(t), \eta_2(t)[}(t) \\
&\quad - f(Y_t^h)\Delta Z_{\gamma_0^{-1}(t)} \frac{\eta_2(t) - s}{\eta_2(t) - \eta_1(t)} 1_{[\eta_1(t), \eta_2(t)[}(s) 1_{[\eta_1(t), \eta_2(t)[}(t) \ |
\end{aligned} \tag{32}$$

$$+ \mid f(Y_t^h)\Delta Z_{\gamma_0^{-1}(t)} \frac{\eta_2(t)-s}{\eta_2(t)-\eta_1(t)} 1_{[\eta_1(t),\eta_2(t)[}(t) 1_{[\eta_1(t),\eta_2(t)[}(s)$$
$$-f(Y_s^h)\Delta Z_{\gamma_0^{-1}(s)} \frac{\eta_2(s)-s}{\eta_2(s)-\eta_1(s)} 1_{[\eta_1(s),\eta_2(s)[}(s) \mid . \tag{33}$$

Now, (32) is less than:

$$\frac{|\Delta Z_{\gamma_0^{-1}(t)} f(Y_t^h)|}{\eta_2(t)-\eta_1(t)} 1_{[\eta_1(t),\eta_2(t)[}(t) |\eta_2(t) - t - (\eta_2(t)-s) 1_{[\eta_1(t),\eta_2(t)[}(s) \mid, \tag{34}$$

which if analyzed by cases (if s is in $[\eta_1(t), \eta_2(t)]$ or not) can be proved to be smaller than $C\epsilon$. For (33), assume first that $s \in [\eta_1(t), \eta_2(t)]$, then (33) is smaller than (see the Remark 11):

$$\frac{|\Delta Z_{\gamma_0^{-1}(t)}|}{\eta_2(t)-\eta_1(t)} (\eta_2(t)-s) C \Big| \int_s^t \frac{f(Y_t^h)(Z_{\gamma_h^{-1}(t)} - Z_{\gamma_h^{-1}(t)-h})}{\Gamma^d_{\gamma_h^{-1}(t)} - \Gamma^d_{\gamma_h^{-1}(t)-h} + h} dt$$
$$- \int_s^t U_s^h f'(Y_s^h) 1(Y_s^h \in \partial D) n(Y_s^h)(n(Y_s^h), \frac{f(Y_s^h)(Z_{\gamma_h^{-1}(s)} - Z_{\gamma_h^{-1}(s)-h})}{\Gamma^d_{\gamma_h^{-1}(s)} - \Gamma^d_{\gamma_h^{-1}(s)-h} + h}) ds \Big|, \tag{35}$$

which clearly can be made smaller that ϵ. In the second case where $\eta_1(s) \neq \eta_1(t)$, we have that (33) is equal to:

$$\Big| \frac{\Delta Z_{\gamma_0^{-1}(s)} f(Y_s^h)(\eta_2(s)-s)}{\eta_2(s)-\eta_1(s)} \Big| 1_{[\eta_1(s),\eta_2(s)[}(s), \tag{36}$$

this can be small if $\Delta Z_{\gamma_0^{-1}(s)}$ is small as $\eta_1(t) \neq \eta_1(s)$ and $|t-s| < \delta$. Using the sequence $\{t_i\}$ obtained from Z, the above inequalities show that it is possible to obtain a partition $\{s_i\}$ for which

$$\max_i w(f(Y_t^h)U_t^h, [s_i, s_{i+1}[) < C\epsilon + \delta \max_{0 \le s \le 1} |\Delta Z_s|.$$

Therefore the tightness of $f(Y_t^h)U_t^h$ follows. □

Remark 15.

The tightness of $\int_0^t f(Y_s^h) dZ_{\gamma_0^{-1}(s)}$ follows as in the Proof of Proposition 4.3 in Kurtz and Protter (1991), noting that the above integral jumps every time Z does and stays constant if Z stays constant (see comments after Theorem 4). For details we refer the reader to Kurtz and Protter (1991).

Lemma 16 *The sequence $\int_0^t (U_s^h)^T dU_s^h$, is good (Condition (*) in subsection 2) in the sense of Kurtz and Protter (1991). Also,*

$$\int_0^t (U_s^h)^T dU_s^h + (\int_0^t (U_s^h)^T dU_s^h)^T$$

converges a.s. to $((V_t^0 - Z_{\gamma_0^{-1}(t)})^T (V_t^0 - Z_{\gamma_0^{-1}(t)}) - [Z,Z]_{\gamma_0^{-1}(t)})$ in the Skorohod topology (A^T denotes the transpose of the matrix A).

For the proof see Kurtz, Pardoux and Protter (1992).

Let's review what we have done so far. We want to find a solution to (22) by finding a limit of Y^h which is a continuous process that solves (27). We know that $V^h \to V^0$, and that $\int_0^t f(Y_t^h)dV_t^h$ is tight.

Now, we only need to use a tightness result (Theorem 3.1) in Costantini (1991) or Theorem 6, to obtain the following lemma.

Lemma 17

$$(Y_\cdot^h, \int_0^\cdot f(Y_s^h)dV_s^h, \Psi_\cdot^h) \textit{ is tight.}$$

Also, Ψ^h is a good sequence.

Proof.

The only thing left to prove is the goodness of Ψ^h. By Theorem 6 , we have:

$$\mid \Psi^h \mid_t \leq K(w) \sup_{0\leq s\leq t} \mid X_0 + \int_0^s f(Y_u^h)dV_u^h \mid,$$

where the constant $K(w)$ is bounded when $\mid X_0 + \int_0^s f(Y_u^h)dV_u^h \mid$ is in a compact set of $D[0,T_0]$. Using this fact together with the tightness of $X_0 + \int_0^s f(Y_u^h)dV_u^h$, it follows that $\mid \Psi^h \mid_t$ is stochastically bounded. Therefore, by Remark 2.3 in Kurtz and Protter (1991), it follows that Ψ^h is good. □

As a consequence of Lemma 17, we know that there exists a limit point for a subsequence of $(Y_\cdot^h, \int_0^\cdot f(Y_s^h)dV_s^h, \Psi_\cdot^h)$. Denote the subsequence with the same superscript h and its limit point by $(Y_\cdot, S_\cdot, \Psi_\cdot)$. It is also known that this limit should solve the Skorohod problem for S. Now, we give the characterization for S.

Theorem 18

$$(Y_\cdot^h, \int_0^\cdot f(Y_s^h)dV_s^h, \Psi_\cdot^h) \to (Y_\cdot, S_\cdot, \Psi_\cdot) \textit{ a.s. in the Skorohod topology,}$$

where

$$\begin{aligned} S_t &= \int_0^t f(Y_s)dZ_{\gamma_0^{-1}(s)} + f(Y_t)U_t - \int_0^t U_s f'f(Y_s)dZ_{\gamma_0^{-1}(s)} \\ &- \int_0^t f'f(Y_s)d(\frac{U_s^2}{2} - \frac{[Z,Z]_{\gamma_0^{-1}(s)}}{2}) + \int_0^t U_s f'(Y_s)d\Psi_s. \end{aligned} \tag{37}$$

Also, Y_t, S_t and Ψ_t are continuous processes.

Proof.

In order to characterize S, we need to look at the proof of Lemma 13.

As:

$$\int_0^t f(Y_s^h)dV_s^h = \int_0^t f(Y_s^h)dZ_{\gamma_0^{-1}(s)} + \int_0^t f(Y_s^h)dU_s^h,$$

it is enough to find the weak limit of $\int_0^t f(Y_s^h)dU_s^h$. Although we can not apply a limit theorem for this integral, we have by integration by parts (as in Lemma 13):

$$\int_0^t f(Y_s^h)dU_s^h = f(Y_t^h)U_t^h - \int_0^t U_s^h f'f(Y_s^h)dZ_{\gamma_0^{-1}(s)} - \int_0^t U_s^h f'f(Y_s^h)dU_s^h + \int_0^t U_s^h f'(Y_s^h)d\Psi_s^h. \tag{38}$$

As Ψ^h and $\int_0^t U_s^h dU_s^h$ are good, it is enough to take limits in (38) as in Theorem 4. By performing this operation one obtains (37).

Here, notice that although U might not be a semimartingale, its square is a process of bounded variation (therefore a semimartingale). The continuity property of (Y, S, Ψ) follows because the sequence is tight and each element of the sequence is continuous (for details, see Proposition 3.26 in Jacod and Shiryaev (1987)). □

Remark 19.

1. Heuristically the limit of (27) is a limit process Y, the unique solution to the following Stratonovich type stochastic differential equation with reflection:

$$Y_t = X_0 + \int_0^t f(Y_s)dV_s^0 + \frac{1}{2}\int_0^t f'f(Y_s)d[V^0, V^0]_s - \Psi_t. \tag{39}$$

But in this case as V^0 and Y are not necesarily semimartingales we have to use another way to write this expression. This is provided by the integration by parts formula used in (38).

2. In the vector case $\int_0^t f'f(Y_s)d(\frac{(U_s)^2}{2})$ stands for

$$\sum_{j,k,l}\int_0^t U_s^j \frac{\partial f_{.j}}{\partial x_k} f_{kl} dU_s^l.$$

As a consequence of the above theorem, we have

Corollary 20 *There is at least one solution to the Stratonovich type stochastic differential equation with reflection (22).*

Proof.

S_t can be simplified as follows:

$$\begin{aligned} S_t &= \int_0^t f(Y_s)dZ_{\gamma_0^{-1}(s)} + f(Y_t)U_t - \int_0^t U_s f'f(Y_s)dZ_{\gamma_0^{-1}(s)} \\ &\quad - \int_0^t f'f(Y_s)d(\frac{U_s^2}{2} - \frac{[Z,Z]_{\gamma_0^{-1}(s)}}{2}) + \int_0^t U_s f'(Y_s)d\Psi_s \\ &= \int_0^t f(Y_s)dZ_{\gamma_0^{-1}(s)} + f(Y_t)U_t - \int_0^t f'f(Y_s)d(\frac{U_s^2}{2}) \\ &\quad + \frac{1}{2}\int_0^t f'f(Y_s)d[Z,Z]^c_{\gamma_0^{-1}(s)} + \int_0^t U_s f'(Y_s)d\Psi_s \end{aligned} \tag{40}$$

Here, we have used some facts about U_t:

1. U_t is continuous everywhere except at $\eta_1(t)$ when $\eta_1(t) \neq \eta_2(t)$, and its jump at that point has size $-\Delta Z_{\gamma_0^{-1}(t)}$.

2. U_t is zero everywhere except in the intervals of the form $[\eta_1(t), \eta_2(t)[$ when $\eta_1(t) \neq \eta_2(t)$.

Now, we study the behaviour of Y_t when $t \in [\eta_1(t), \eta_2(t)[$ for $\eta_1(t) \neq \eta_2(t)$. For this we will need the following formula:

$$U_t = -\Delta Z_{\gamma_0^{-1}(t)} \frac{\eta_2(t) - t}{\eta_2(t) - \eta_1(t)} 1_{[\eta_1(t),\eta_2(t)[}(t). \tag{41}$$

In such a case we have:

$$
\begin{aligned}
Y_t &= Y_{\eta_1(t)} - f(Y_t)\Delta Z_{\gamma_0^{-1}(t)} \frac{\eta_2(t)-t}{\eta_2(t)-\eta_1(t)} 1_{[\eta_1(t),\eta_2(t)[}(t) + f(Y_{\eta_1(t)})\Delta Z_{\gamma_0^{-1}(t)} \\
&\quad - \int_{(\eta_1(t),t]} f'f(Y_s)(\Delta Z_{\gamma_0^{-1}(t)})^2 \frac{\eta_2(t)-s}{(\eta_2(t)-\eta_1(t))^2} ds \\
&\quad + \int_{(\eta_1(t),t]} \Delta Z_{\gamma_0^{-1}(t)} \frac{\eta_2(t)-s}{\eta_2(t)-\eta_1(t)} f'(Y_s) d\Psi_s - (\Psi_t - \Psi_{\eta_1(t)}). \qquad (42)
\end{aligned}
$$

We also know that $Y_t^h - Y_{\eta_1(t)}^h \to Y_t - Y_{\eta_1(t)}$, in the uniform topology. Therefore:

$$
\begin{aligned}
Y_t^h - Y_{\eta_1(t)}^h &= \int_{\eta_1(t)}^t f(Y_s^h) dV_s^h - (\Psi_t^h - \Psi_{\eta_1(t)}^h) \\
&= \int_{\eta_1(t)}^t f(Y_s^h) \frac{(Z_{\gamma_h^{-1}(s)} - Z_{\gamma_h^{-1}(s)-h})}{\Gamma^d_{\gamma_h^{-1}(s)} - \Gamma^d_{\gamma_h^{-1}(s)-h} + h} ds - (\Psi_t^h - \Psi_{\eta_1(t)}^h). \qquad (43)
\end{aligned}
$$

Applying convergence theorems to the above integral we get:

$$
Y_t = Y_{\eta_1(t)} + \int_{\eta_1(t)}^t f(Y_s) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t)-\eta_1(t)} ds - (\Psi_t - \Psi_{\eta_1(t)}), \qquad (44)
$$

when $t \in [\eta_1(t), \eta_2(t)[$. Therefore by equating (44) with (42), and applying that result to (40) gives:

$$
\begin{aligned}
Y_t &= X_0 + \int_0^t f(Y_s) dZ_{\gamma_0^{-1}(s)} + \frac{1}{2}\int_0^t f'f(Y_s) d[Z,Z]^c_{\gamma_0^{-1}(s)} \\
&\quad + \sum_{0<u\le(\gamma_0^{-1}(\eta_2(t)))} \left(\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{f(Y_s)\Delta Z_u}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))} ds - f(Y_{\eta_1(\gamma_0(u))})\Delta Z_u\right) \\
&\quad - \int_t^{\eta_2(t)} f(Y_s) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t)-\eta_1(t)} ds - \Psi_t. \qquad (45)
\end{aligned}
$$

This representation is possible because , it is not difficult to show that each term in the sum above is of order $(\Delta Z_s)^2$. That is, by Taylor's theorem we have:

$$
\begin{aligned}
&\int_{\eta_1(t)}^{\eta_2(t)} \frac{f(Y_s)\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t)-\eta_1(t)} ds - f(Y_{\eta_1(t)})\Delta Z_{\gamma_0^{-1}(t)} \\
&= \left. \left(f'(Y_t)\, \frac{dY_t}{dt} \right) \right|_{(t=\xi)} \Delta Z_{\gamma_0^{-1}(t)} (\eta_2(t) - \eta_1(t)), \qquad \xi \in [\eta_1(t), \eta_2(t)] \\
&= f'(Y_t) \left. \left(f(Y_t) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t)-\eta_1(t)} - \frac{d\Psi_t}{dt} \right) \right|_{(t=\xi)} \Delta Z_{\gamma_0^{-1}(t)} (\eta_2(t) - \eta_1(t)) \\
&= O((\Delta Z_{\gamma_0^{-1}(t)})^2).
\end{aligned}
$$

The last equality follows because (see equation (28))

$$
\frac{d\Psi_t}{dt} = 1(Y_t \in \partial D) n(Y_t) < n(Y_t), f(Y_t) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t)-\eta_1(t)} > . \qquad (46)
$$

Now, define $X_t = Y_{\gamma_0(t)}$, then using (45) one obtains:

$$\begin{aligned} Y_{\gamma_0(t)} &= X_0 + \int_0^{\gamma_0(t)} f(Y_s) dZ_{\gamma_0^{-1}(s)} + \frac{1}{2}\int_0^{\gamma_0(t)} f'f(Y_s) d[Z,Z]^c_{\gamma_0^{-1}(s)} \\ &+ \sum_{0<u\le(\gamma_0^{-1}(\eta_2(\gamma_0(t))))} \left(\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{f(Y_s)\Delta Z_u}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))} ds - f(Y_{\eta_1(\gamma_0(u))})\Delta Z_u\right) \\ &- \Psi_{\gamma_0(t)}. \end{aligned} \tag{47}$$

Applying change of variables formulas we have that X_t solves (22). □

Remark 21.

1. This proof also shows that the nature of the summability of

$$\sum_{0<s\le t} \{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\}$$

is twofold. First,

$$\Delta Z_s \int_0^1 f(y(u))du - f(X_{s-})\Delta Z_s = O((\Delta Z_s)^2);$$

and secondly $\sum_{0<s\le t} \kappa^{\Delta Z_s}(1) = (\Psi(\gamma_0(t)))^d$.

Theorem 22 *There exists a version of the solution to (22) which is adapted to the filtration to which Z is adapted.*

Proof.

Since $Y^h \Rightarrow Y$ and Y is a continuous process adapted to $\mathcal{F}_{\gamma_0^{-1}(t)}$, we have that the convergence is in the uniform topology. If we look carefully at the previous proofs, is not difficult to see that the convergence of Y^h to Y is *a.s.* and in the uniform topology.

Since $Y^h_{\gamma_0(t)} \to Y_{\gamma_0(t)}$ *a.s.* uniformly in t, we obtain the measurability of X_t, by proving the "approximate" measurability of $Y^h_{\gamma_0(t)}$. That is,

$$Y^h_{\gamma_0(t)} = X^h_{\gamma_h^{-1}(\gamma_0(t))},$$

therefore by using a classical argument by simple processes we obtain the measurability of $Y^h_{\gamma_0(t)}$ with respect to $\mathcal{F}_{t+2h}$, and using that the filtration is right continuous we get the adaptedness of X. □

3.2 Uniqueness

The following inequality will serve to prove uniqueness of solutions. Here, we will use strongly the smoothness of the boundary of D, which will permit the use of Lipschitz properties of the reflection process on half space. This localization argument has been previously used by Anderson and Orey (1976) and it can also be found in Ikeda and Watanabe (1981). The idea is to think of $\bar{D}$ as a C^2 manifold with coordinate neighborhoods $\{(U_i, g_i); i = 1, ..., n\}$ for some finite n. Let Y and Y^* be two solutions of (45) for two initial points x_0 and x_0^*. Under these conditions we will prove the following result:

Lemma 23 *Define:*

$$\tau_1 = inf\{t > 0 : Y_t \notin \bar{U}_i\},$$
$$\tau_2 = inf\{t > 0 : Y_t^* \notin \bar{U}_i\}.$$

Then,

$$E \sup_{0<t\leq(\tau_1\wedge\tau_2)} \mid Y_t - Y_t^* \mid^2 \leq C_1 \mid x_0 - x_0^* \mid^2 + C_2 E(\int_0^{\tau_1\wedge\tau_2} \sup_{0<s\leq t} \mid Y_s - Y_s^* \mid^2 dA_s), \quad (48)$$

where A is an increasing stochastic process of bounded variation. Also, if $x_0 = x_0^*$ then $\tau_1 = \tau_2$.

Remark 24.
As we have noted before (see Remark 18), the main problem here is to be able to obtain results withouth integrating with respect to V^0 or Y which are not necesarily semimartingales.

Proof of Lemma 23.
The idea of the proof is to map Y and Y^* to the half plane, obtain the equivalent inequality there and transform back to U_i; by using the metric equivalence property. For this, if $U_i \cap \partial D \neq \emptyset$ let $g_i : U_i \to R_+^d$ be twice continuously differentiable, such that:
1. There exists a constant m such that:

$$\frac{1}{m} \mid g_i(x) - g_i(y) \mid \leq \mid x - y \mid \leq m \mid g_i(x) - g_i(y) \mid. \quad (49)$$

2. $g_i^1(x) = 0$ if $x \in \partial D$.
If $U_i \cap \partial D = \emptyset$ then $g_i : U_i \to R^d$ with property 1 above.
The method used in the previous subsection can be used to obtain a version of Itô's Lemma for $g(Y_t)$. That is, apply Itô's Lemma and limit theorems to $g(Y_t^h)$ to obtain:

$$\begin{aligned} g(Y_t) &= g(x_0) + \int_0^t g'f(Y_s)dZ_{\gamma_0^{-1}(s)} + g'f(Y_t)U_t \\ &- \int_0^t U_s(g'f)'f(Y_s)dZ_{\gamma_0^{-1}(s)} - \int_0^t (g'f)'f(Y_s)d(\frac{U_s^2}{2}) \\ &+ \int_0^t U_s(g'f)'f(Y_s)d\Psi_s - \int_0^t g'(Y_s)d\Psi_s \\ &+ \frac{1}{2}\int_0^t (g'f')'f(Y_s)d[Z,Z]^c_{\gamma_0^{-1}(s)}. \end{aligned} \quad (50)$$

Which by a similar argument as in the proof of Corollary 20 gives:

$$\begin{aligned} g(Y_t) &= g(x_0) + \int_0^t g'f(Y_s)dZ_{\gamma_0^{-1}(s)} + \frac{1}{2}\int_0^t (g'f)'f(Y_s)d[Z,Z]^c_{\gamma_0^{-1}(s)} \\ &+ \sum_{0<u\leq(\gamma_0^{-1}(\eta_2(t)))} (\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{g'f(Y_s)\Delta Z_u}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))} ds - g'f(Y_{\eta_1(\gamma_0(u))})\Delta Z_u)) \\ &- \int_t^{\eta_2(t)} g'f(Y_s)\frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t) - \eta_1(t)} ds - \int_0^t g'(Y_s)d\Psi_s. \end{aligned} \quad (51)$$

We have obtained a stochastic differential equation with oblique reflection in the half space R^d_+. In this case it is known that the reflection process is Lipschitz (see Dupuis and Ishii (1991)). Using (51) we obtain:

$$
\begin{aligned}
\mid g(Y_t) - g(Y_t^*) \mid \leq 2 \mid g(x_0) - g(x_0^*) \mid +2 \mid & \int_0^t (g'f(Y_s) - g'f(Y_s^*)) dZ_{\gamma_0^{-1}(s)} \\
& + \frac{1}{2} \int_0^t ((g'f)'f(Y_s) - (g'f)'f(Y_s^*)) d[Z,Z]^c_{\gamma_0^{-1}(s)} \\
& + \sum_{0<u\leq(\gamma_0^{-1}(\eta_2(t)))} \Big(\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{(g'f(Y_s) - g'f(Y_s^*))\Delta Z_u}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))} ds \\
& - (g'f(Y_{\eta_1(\gamma_0(s))}) - g'f(Y^*_{\eta_1(\gamma_0(s))}))\Delta Z_u \Big) \\
& - \int_t^{\eta_2(t)} (g'f(Y_s) - g'f(Y_s^*)) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t) - \eta_1(t)} ds \mid . \qquad (52)
\end{aligned}
$$

This gives (48) , if we use (49).

To show that $\tau_1 = \tau_2$ when $x_0 = x_0^*$ is enough to follow the same ideas as in the proof of Theorem 2. □

Let X be the solution we obtained in the previous subsection, and let X' be another solution of (22). The idea of the proof of the uniqueness is to generate another solution of (45). This raises a contradiction because of the local uniqueness provided by Lemma 23.

Let X^* be a solution for

$$
\begin{aligned}
X_t^* &= X_0 + \int_0^t f(X_{s-}^*) dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-}^*) d[Z,Z]^c_s - \Phi^*(t) \\
&\quad + \sum_{o<s\leq t} \{\varphi(\Delta Z_s f, X_{s-}^*) - X_{s-}^* - f(X_{s-}^*)\Delta Z_s\}. \qquad (53)
\end{aligned}
$$

Define,

$$
Y_t^* = \begin{cases} X^*_{\gamma_0^{-1}(t)} & \text{if } \eta_1(t) = \eta_2(t); \\[2ex] X^*_{\gamma_0^{-1}(t)-} + \displaystyle\int_{\eta_1(t)}^t f(Y^*(u)) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t) - \eta_1(t)} du - \kappa\left(\frac{\eta_2(t) - t}{\eta_2(t) - \eta_1(t)}\right) & \begin{array}{l}\text{if } \eta_1(t) \leq t \leq \eta_2(t) \\ \text{and } \eta_2(t) \neq \eta_1(t),\end{array} \end{cases} \qquad (54)
$$

Then, by using a similar argument as in the proof of the Corollary 20, we have that Y^* solves the stochastic equation (45). By uniqueness of solutions we have that $Y^* = Y$ *a.e.*, and therefore $Y^*_{\gamma_0(t)} = Y_{\gamma_0(t)}$. From there, we get that $X_t^* = X_t$ *a.e.*

To make the argument more explicit, we give the following Lemma.

Lemma 25

$$
\begin{aligned}
Y_t^* &= X_0 + \int_0^t f(Y_s^*) dZ_{\gamma_0^{-1}(s)} + \frac{1}{2}\int_0^t f'f(Y_s^*) d[Z,Z]^c_s \\
&\quad + \sum_{0<u\leq(\gamma_0^{-1}(\eta_2(t)))} \Big(\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{f(Y_s^*)\Delta Z_s}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))} ds - f(Y^*_{\eta_1(\gamma_0(s))})\Delta Z_s \Big) \\
&\quad - \int_t^{\eta_2(t)} f(Y_s^*) \frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t) - \eta_1(t)} ds - \Psi_t^*. \qquad (55)
\end{aligned}
$$

Here,

$$\Psi_t^* = \Phi^*(\gamma_0^{-1}(t)) + (\sum_{0<s\leq t} \kappa^{\Delta Z_s}(1)) - (\kappa^{\Delta Z_t}(1) - \kappa^{\Delta Z_t}(\frac{t-\eta_1(t)}{\eta_2(t)-\eta_1(t)})).$$

Now by uniqueness of solutions for Stratonovich type stochastic differential equations driven by continuous semimartingales, we know that $Y^* = Y$ *a.e.* and therefore, $X^* = X$ *a.e.*

In order to prove that this solution X is a Markov process when the driving semimartingale Z is a Lévy process, we will need the continuity of the flows defined by solutions of Stratonovich type stochastic differential equations with reflection. This continuity is with respect to the initial point. Using Lemma 23 is not difficult to prove this assertion.

Theorem 26 *If the driving semimartingale Z is a Lévy process then any solution of the Stratonovich SDE with reflection is a strong Markov process.*

Proof.

First, we note that the flow defined by any solution is continuous with respect to the initial point. To prove this it is enough to use Kolmogorov's lemma locally in Lemma 23. Therefore we obtain the continuity of Y^a with respect to the initial condition $Y_0^a = a$.

From here, is not difficult to follow the same steps as in Theorem 32 of Chapter V in Protter (1990). Let $[Z,Z]_t = \alpha t$ for some constant α. Let T be a stopping time and $\mathcal{G}^T = \sigma\{Z_{T+u} - Z_T : u \geq 0\}$. Then $\mathcal{G}^T$ is independent of $\mathcal{F}_T$.

Let $X(x,t,s)$ be the unique solution to the Stratonovich type stochastic differential equation with reflection driven by the semimartingale $Z_u - Z_t$, that is:

$$\begin{aligned} X(x,t,s) &= x + \int_t^s f(X(x,t,u-))dZ_u + \frac{1}{2}\int_0^t f'f(X(x,t,u-))\alpha du - \Phi(x,t,s) \\ &+ \sum_{0<s\leq t}\{\varphi(\Delta Z_s f, X(x,t,s-) - X(x,t,s-) - f(X(x,t,s-))\Delta Z_s\}. \quad (56) \end{aligned}$$

As in the proof of Theorem 22 we obtain that $X(x,t,s) \in \mathcal{G}^t$. Also, as mentioned before, one uses the continuity of the flows to show that

$$X_{T+s}^x = X(X(x,0,T),T,T+s) \ a.s.$$

for a stopping time T. Let h be a bounded measurable function from R^d to R, then

$$\begin{aligned} E[h(X_{T+s}^x) \mid \mathcal{F}_T] &= E[h(X(X_T^x,T,T+s)) \mid \mathcal{F}_T] \\ &= j(X_T^x) \\ \text{where } j(y) &= E[h(X(y,T,T+s))] \quad (57) \end{aligned}$$

As we have proved strong uniqueness and Z is a Lévy process it is easy to see that

$$j(y) = E[h(X_s^y)].$$

□

3.3 Oblique reflection case

Lions and Sznitman obtained the weakest known conditions to obtain uniqueness for the solution of the oblique Skorohod problem for diffusions. These are:

$$\begin{aligned} &1. D \text{ is a bounded smooth open set in } R^d. \\ &2.\ \exists \nu > 0,\ \forall x \in \partial D,\ (\Theta(x), n(x)) \geq \nu, \text{ where } \Theta \in C_b(\bar{D}). \end{aligned} \tag{58}$$

We consider the general case here, because we have already assumed that D is a smooth domain. The procedures to obtain the lemmas in this subsection are similar to those previously proved, therefore we will only note the significant changes that one has to introduce, leaving the details to the reader.

We start with the continuous case as before. The equation that we will focus on is:

$$X(t) = X_0 + \int_0^t f(X(t))dZ_t + \int_0^t \frac{1}{2} f'f(X(t))d[Z,Z]_t - \Phi(t), \tag{59}$$

where $X(t) \in \bar{D}$ and

$$\Phi(t) = \int_0^t \theta(X(s))d \mid \Phi \mid (s), \tag{60}$$

$$\Phi(t) = \int_0^t I(X(s) \in \partial D)d\Phi(s). \tag{61}$$

Since Z is a continuous semimartingale on R^k, we also assume the three conditions above with the extra condition that $\Theta \in C_b^2(\bar{D})$. The following Lemma extends a result obtained by Lions and Sznitman (1984). The proof is similar as theirs.

Lemma 27 *Let*

$$F(X)(t) = X_0 + \int_0^t f(X(t))dZ_t + \int_0^t \frac{1}{2} f'f(X(t))d[Z,Z]_t - \Phi(t), \tag{62}$$

where $\Phi(t)$ is the reflection process for the oblique reflection problem, so that $F(X)(t) \in D$ for all $t \leq 1$. Then for X, X' such that

$$E \sup_{0\leq t\leq 1} \mid X(t) \mid^4 < \infty, \text{ and } E \sup_{0\leq t\leq 1} \mid X'(t) \mid^4 < \infty$$

we have:

$$E[\sup_{0\leq t\leq u} \mid F(X)(t) - F(X')(t) \mid^4] \leq KE[\int_0^u \sup_{0\leq v\leq s} \mid X(v) - X'(v) \mid^4 dA_s], \tag{63}$$

where $A_s = \| [Z,Z] \|_s + FV(Z)_s$, and K is a non-random constant.

From here onwards, the procedure followed on the previous subsection can be applied similarly.

Theorem 28 *Let $f \in C_b^2(\bar{D})$ and $\Theta(x)$ be a field of directions for $x \in \partial D$ satisfying conditions (58). Then there exists a unique solution to the following Stratonovich type SDE with oblique reflection:*

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-})dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-})d[Z,Z]_s^c - \Phi(t) \\ &\quad + \sum_{o<s\leq t} \{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\}, \end{aligned} \tag{64}$$

$$\Phi(t) = \int_0^t \theta(X_s)d \mid \Phi \mid (s),\ X_t \in \bar{D},$$

and $\varphi(g,x) = y(1)$, where $y(t)$ is the solution of

$$y(t) = x + \int_0^t g(y(s))ds - \kappa(t),\ y(t) \in \bar{D}\ ,$$

$$\kappa(t) = \int_0^t \theta(y(s))d \mid \kappa \mid (s).$$

Also if Z is a Lévy process then the solution X is a strong Markov process.

Remark 29. The restriction that $Z \in \mathcal{H}^2$ is not an additional condition to impose, as this can be obtained from any semimartingale through an appropriate change of measures. For details, see Lenglart (1980).

The procedure is as follows, let X be a solution process on $(\Omega, \mathcal{F}_t, Q)$ where Q is a new probability measure under which $Z \in \mathcal{H}^2$. On the space $(\Omega, \mathcal{F}_t, Q)$ it is possible to find a solution X to the Stratonovich type stochastic differential equation with reflection (65). This solution has fourth bounded moments. By carrying this solution X to the space $(\Omega, \mathcal{F}_t, P)$ we have the existence of solutions because all integrals are uniquely defined for equivalent measures. Therefore the existence follows.

For uniqueness, consider first the case in which D is bounded. Let X, X' be 2 solutions on $(\Omega, \mathcal{F}_t, P)$ then obviously

$$E(\sup_{0<s\leq t} |X_s|^4) < \infty \text{ and } E(\sup_{0<s\leq t} |X'_s|^4) < \infty,$$

therefore $X = X'$. If D is unbounded, is enough to consider $X_{t\wedge\tau^r}$ where

$$\tau^r = inf\{t : X_t \in \partial D \cup \partial B(0,r)\}$$

then $X_{t\wedge\tau^r} = X'_{t\wedge\tau^r}$. As $\tau^r \to \infty$ as $r \to \infty$, we conclude that $X_t = X'_t$.

4 Stratonovich type SDE's with reflection driven by semimartingales with summable jumps

In Section 2.1 we dealt with smooth domains and general semimartingales. Here we will weaken conditions on the domain, but in exchange we will have to limit the class of semimartingales driving these reflected SDE's.

These conditions on the domain are the same as the ones used by Saisho (1987) to obtain the existence and uniqueness of Brownian diffussions.

Theorem 30 *(Saisho, normal reflection case) Suppose that a domain D in R^d satisfies conditions (A) and (B). Then for any $w \in \hat{C}(D)$ there exists a unique solution $\phi(t,w)$ of the equation (1.1) and $\phi(t,w)$ is continuous in (t,w).*

The case of continuous functions and oblique reflection is treated by Lions and Sznitman.

4.1 Existence of solutions

Now we state the problem we are going to solve. We will prove existence and uniqueness of solutions for the following system:

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-})dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-})d[Z,Z]_s^c - \Phi(t) \\ &+ \sum_{o<s\leq t} \{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\} \end{aligned} \qquad (65)$$

$$\Phi(t) = \int_0^t n(X_s)d \mid \Phi \mid (s), \; X_t \in \bar{D},$$

$$\Phi(t) = \int_0^t 1(X_s \in \partial D)d\Phi(s),$$

and $\varphi(g,x) = y(1)$, where $y(t)$ is the solution of

$$y(t) = x + \int_0^t g(y(s))ds - \kappa(t), \; y(t) \in \bar{D},$$

$$\kappa(t) = \int_0^t n(y(s))d \mid \kappa \mid (s),$$

$$\kappa(t) = \int_0^t 1(y(s) \in \partial D)d\kappa(s).$$

In order to prove the existence of solutions for the system (65), we consider only the jumps of Z that are bigger than $\epsilon > 0$ in absolute value, and then prove that this sequence of solutions X^ϵ is tight in $D[0,1]$ under the Skorohod topology. In order for this argument to work we need to assume that the sum of the jumps of Z converges absolutely, i.e.,

$$\sum_{0<s\leq t} \mid \Delta Z_s \mid < \infty \text{ a.s.}$$

Moreover we assume that $f \in C_b^1(\bar{D})$.

We define:

$$Z^\epsilon(t) = Z_t^c + \sum_{0<s\leq t} 1(\mid \Delta Z(s) \mid > \epsilon)\Delta Z(s),$$

where Z^c is the continous part of the semimartingale Z (see Protter (1990)), i.e.,

$$Z^c(t) = Z_t - \sum_{0<s\leq t} 1(|\Delta Z(s)| > 0)\Delta Z(s).$$

Lemma 31 *For the driving semimartingale Z^ϵ the system (65) has a unique solution X^ϵ.*

Proof.
We know that the number of jumps of Z^ϵ are finite, and therefore the problem can be solved sequentially. More explicitly, let $T_i^\epsilon = inf\{t > T_{i-1}^\epsilon / \mid \Delta Z_t^\epsilon \mid > 0\}$, where $T_0^\epsilon = 0$. As Z_t^ϵ is a càdlàg function with jump sizes bounded below, we know that there exists i such that $T_i^\epsilon = \infty$.

To define the solution X^ϵ, we proceed by induction. Suppose X_t^ϵ is defined for $t < T_j^\epsilon$. Define a new semimartingale $Z_t' = Z_{T_j+t}^c$ under a new family of σ-fields

$$\mathcal{F}_t' = \mathcal{F}_{T_j+t}$$

Then, there exists a unique solution for the system

$$Y_t = X_{T_j}^\epsilon + \int_0^t f(Y_{s-})dZ'^c_s + \frac{1}{2}\int_0^t f'f(Y_{s-})d[Z',Z']_s^c - \Phi_t^Y, \tag{66}$$

because Z' is a continuous semimartingale. Therefore, the existence and uniqueness are consequences of Theorem 4. We follow this solution up to time $T_{j+1} - T_j$ and obtain the unique solution $y(t)$ of

$$y(t) = Y_{T_{j+1}-T_j} + \Delta Z_{T_{j+1}}^\epsilon \int_0^t f(y(u))du - \kappa(t),$$

$$\text{where } \kappa(t) = \int_0^t n(y(s))d \mid \kappa \mid (s).$$

Now define

$$X_t^\epsilon = Y_{t-T_j} 1_{(T_j<t<T_{j+1})} + y(1)1_{(t=T_{j+1})}, \tag{67}$$

for $T_j < t \leq T_{j+1}$. X^ϵ holds equation (65) for $t \leq T_{j+1}$, where

$$\Phi_t^\epsilon = \begin{cases} \Phi_{T_j}^\epsilon + \Phi_{t-T_j}^Y & \text{if } T_j < t < T_{j+1}; \\ \Phi_{T_{j+1}-}^\epsilon & \text{if } t = T_{j+1}. \end{cases}$$

Note that Φ_t^ϵ is still continuous, of bounded variation, changes only when X_t^ϵ is at the boundary and $d\Phi_t^\epsilon$ has the normal direction. As this procedure need be done only a finite number of times, we can construct a solution X_t^ϵ for (65) driven by Z_t^ϵ. □

Theorem 32 *The sequence X^ϵ is tight in $D[0,1]$ under the Skorohod topology. Any limit point is a solution of the system driven by the semimartingale Z.*

Proof. We know by the previous lemma that

$$X_t^\epsilon = X_0 + \int_0^t f(X_{s-}^\epsilon)dZ_s^\epsilon + \frac{1}{2}\int_0^t f'f(X_{s-}^\epsilon)d[Z,Z]_s^c - \Phi^\epsilon(t) + \int_0^t h(\omega,s)dU_s^\epsilon, \tag{68}$$

where $U^\epsilon_s = \sum_{0<u\leq s} |\Delta Z^\epsilon_u|$, and

$$h(\omega, s) = \frac{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - \Delta Z_s f(X_{s-})}{|\Delta Z_s|} I(\Delta Z_s \neq 0).$$

The next step is to use some type of tightness result. But, as we stated before in Theorem 7, this result is not true in general for càdlàg functions.

The process

$$X_0 + \int_0^t f(X^\epsilon_{s-}) dZ^\epsilon_s + \frac{1}{2}\int_0^t f' f(X^\epsilon_{s-}) d[Z, Z]^c_s + \int_0^t h(\omega, s) dU^\epsilon_s, \tag{69}$$

has a finite number of jumps, and by Theorem 6 we have:

$$\begin{aligned}
&\sup_{s\leq t_1\leq t_2<t} | X^\epsilon_{t_1} - X^\epsilon_{t_2} | \\
&\leq K(w) \sup_{s\leq t_1\leq t_2<t} \Big| \int_{t_1}^{t_2} f(X^\epsilon_{s-}) dZ^\epsilon_s + \frac{1}{2}\int_{t_1}^{t_2} f' f(X^\epsilon_{s-}) d[Z, Z]^c_s \Big| + \int_{t_1}^{t_2} |h(\omega, s)| dU^\epsilon_s, \qquad (70) \\
&| \Phi^\epsilon | (t-) - | \Phi^\epsilon | (s) \\
&\leq K'(w) \sup_{s\leq t_1\leq t_2<t} \Big| \int_{t_1}^{t_2} f(X^\epsilon_{s-}) dZ^\epsilon_s + \frac{1}{2}\int_{t_1}^{t_2} f' f(X^\epsilon_{s-}) d[Z, Z]^c_s \Big| + \int_{t_1}^{t_2} |h(\omega, s)| dU^\epsilon_s, \qquad (71) \\
&\qquad \forall T^\epsilon_i \leq s \leq t \leq T^\epsilon_{i+1},
\end{aligned}$$

Here, T^ϵ_i are stopping times at which Z^ϵ jumps. From these inequalities it is possible to obtain the tightness of $(X^\epsilon, \Psi^\epsilon)$ if the tightness of the sum of the three integrals in (68) is proved. To show this last assertion, we apply Theorem 4. As f is supposed to be Lipschitz and bounded, we need to prove only that h is stochastically bounded and that condition (*) holds. The goodness condition for Z^ϵ and U^ϵ holds because

$$\begin{aligned}
[Z^\epsilon, Z^\epsilon]_s &= [Z, Z]_s - \sum_{0<u\leq s} 1(|\Delta Z_u| < \epsilon)(\Delta Z_u)^2, \\
FV(Z^\epsilon)_s &= FV(Z)_s - \sum_{0<u\leq s} 1(|\Delta Z_u| < \epsilon)\Delta Z_u, \\
FV(U^\epsilon) &= U^\epsilon.
\end{aligned}$$

To conclude the proof of the tightness is enough to show that h is bounded, we note that by using an estimate of Lions and Sznitman (1984), we can obtain:

$$| \kappa(1) | \leq | \Delta Z_s | \left(\int_0^1 f(y(u))^2 du\right)^{\frac{1}{2}}.$$

(This inequality can also be obtained through Theorem 4.2 in Saisho (1987)). As h can be written as:

$$h(\omega, s) = \frac{\Delta Z_s \int_0^1 (f(y(u)) - f(X_{s-})) du - \kappa(1)}{|\Delta Z_s|} 1(\Delta Z_s \neq 0),$$

then the the tightness of $\int_0^t h(\omega, s) dU^\epsilon_s$ follows because f is bounded.

We have just proved that

$$(Z^\epsilon, U^\epsilon, X_0 + \int_0^t f(X^\epsilon_{s-})dZ^\epsilon_s + \frac{1}{2}\int_0^t f'f(X^\epsilon_{s-})d[Z,Z]^c_s, \int_0^t h(\omega,s)dU^\epsilon_s)$$

is tight, therefore we obtain the tightness of

$$(Z^\epsilon, U^\epsilon, X_0 + \int_0^t f(X^\epsilon_{s-})dZ^\epsilon_s + \frac{1}{2}\int_0^t f'f(X^\epsilon_{s-})d[Z,Z]^c_s, \int_0^t h(\omega,s)dU^\epsilon_s, X^\epsilon, \Phi^\epsilon, \mid \Phi^\epsilon \mid).$$

Now, we only need to take any limit point of this sequence. This limit point will satisfy (65), and as Φ^ϵ is C-tight (see Theorem 3.26 in Jacod and Shiryaev (1989)) we also obtain that $\Phi(t)$ is continous. Furthermore, the bounded variation property and the normal direction property of the differential of Φ can be obtained by an argument similar to Theorem 3.1 in Costantini(1992). □

4.2 Uniqueness

In this section, we will use extensively one of Saisho's conditions that give most of the inequalities that follow. The following statement is equivalent to *Condition* (A) in Section 2.1 (see Remark 1.1 in Saisho),

$$\exists C_0 \geq 0,\ \forall x \in \partial D\ \forall x' \in \bar{D}\ \forall k \in n(x),$$
$$(x - x', k) + C_0 \mid x - x' \mid^2 \geq 0. \tag{72}$$

First, we give some lemmas that are variations of Lemma 3.1 of Lions and Sznitman.

Lemma 33 *If* y *is the unique solution of (continuous stochastic differential equation with reflection) :*

$$y(t) = X_{u-} + \Delta Z_u \int_0^t f(y(s))ds - \kappa(t), \tag{73}$$

and $\tilde{y}$ *is the unique solution of:*

$$\tilde{y}(t) = Y_{u-} + \Delta Z_u \int_0^t f(\tilde{y}(s))ds - \tilde{\kappa}(t);$$

then,

$$\mid y(t) - \tilde{y}(t) \mid^2 \leq B \mid X_{s-} - Y_{s-} \mid^2 exp(C \mid \Delta Z(s) \mid t), \tag{74}$$

for some non-random constants B *and* C.

Proof.
The following condition used by Lions and Sznitman holds locally according to Lemma 5.3 of Saisho.

$$\exists \phi \in C^2_b(R^d) \text{ (bounded in their two derivatives), such that}$$
$$\exists \alpha > 0\ \forall x \in \partial D\ \forall \zeta \in n(x),\ \ \nabla\phi(x)\zeta \leq -\alpha C_0. \tag{75}$$

To simplify our argument, we will assume this condition holds for the whole boundary. An argument similar to Saisho's will give the result in the general case . Using the chain rule we have,

$$\begin{aligned}
&exp(\frac{-2}{\alpha}(\phi(y(t)) + \phi(\tilde{y}(t)))) \mid y(t) - \tilde{y}(t) \mid^2 \\
= \quad & F_0 + 2\int_0^t exp(\frac{-2}{\alpha}(\phi(y(s)) + \phi(\tilde{y}(s))))(y(s) - \tilde{y}(s)) \\
& \{(f(y(s)) - f(\tilde{y}(s)))\Delta Z_u ds - d\kappa(s) + d\tilde{\kappa}(s)\} \\
- \quad & \frac{2}{\alpha}\int_0^t exp(\frac{-2}{\alpha}(\phi(y(s)) + \phi(\tilde{y}(s)))) \mid y(s) - \tilde{y}(s) \mid^2 \\
& \{\nabla\phi(y(s))\{f(y(s)\Delta Z_u ds - d\kappa(s)\} + \nabla\phi(\tilde{y}(s))\{f(\tilde{y}(s))\Delta Z_u ds - d\tilde{\kappa}(s)\}\}. \quad (76)
\end{aligned}$$

$$\text{where } F_0 = exp(\frac{-2}{\alpha}(\phi(X_{u-}) + \phi(Y_{u-}))) \mid X_{u-} - Y_{u-} \mid^2 .$$

As:

$$\frac{1}{\alpha}(\nabla\phi(y(s))n(y(s))) \mid y(s) - \tilde{y}(s) \mid^2 - (y(s) - \tilde{y}(s))n(y(s)) \leq 0 \quad d \mid \kappa \mid (s) \ a.s..$$

Analogously for $\tilde{\kappa}(s)$, we obtain a similar inequality. Combining these two inequalities in (76) results in

$$L(t) \mid y(t) - \tilde{y}(t) \mid^2 \leq L(0) \mid X_{s-} - Y_{s-} \mid^2 + C \mid \Delta Z_u \mid \int_0^t L(s) \mid y(s) - \tilde{y}(s) \mid^2 ds, \quad (77)$$

where $L(t) = exp(\frac{-2}{\alpha}(\phi(y(t)) + \phi(\tilde{y}(t))))$, and C is a constant. Then, by Gronwall's lemma,

$$\mid (y(t) - \tilde{y}(t))L(t) \mid^2 \leq L(0) \mid X_{u-} - Y_{u-} \mid^2 exp(C \mid \Delta Z_u \mid t).$$

The proof can be completed since $\phi(t)$ is bounded. □

As $\varphi(\Delta Z_u f, X_{u-}) = y(1)$ and $\varphi(\Delta Z_u f, Y_{u-}) = \tilde{y}(1)$, we also have the following result:

Corollary 34

$$\mid \varphi(\Delta Z_u f, X_{u-}) - \varphi(\Delta Z_u f, Y_{u-}) \mid^2 \leq B \mid X_{u-} - Y_{u-} \mid^2 exp(C \mid \Delta Z_u \mid).$$

Theorem 35 *If X and X' are any two solutions for the same Stratonovich SDE with reflection (65), we have:*

$$E[\sup_{s<t} \mid X_s - X'_s \mid^4] \leq CE\int_{[0,t)} \sup_{u<s} \mid X_u - X'_u \mid^4 dA_s, \quad (78)$$

where $A_s = \| [Z,Z]_s \| + FV(Z)_s$ (FV stands for the finite variation part).

Proof.

As in the proof of the previous lemma;

$$
\begin{aligned}
& exp(\frac{-2}{\alpha}(\phi(X_t)+\phi(X'_t))) \mid X_t - X'_t \mid^2 \\
= \;& 2\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-})))(X_{s-}-X'_{s-})\{(f(X_{s-})-f(X'_{s-}))dZ^c_s \\
& +\frac{1}{2}(f'f(X_{s-})-f'f(X'_{s-}))d[Z,Z]^c_s - d\Phi(s) + d\Phi'(s)\} \\
& -\frac{2}{\alpha}\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-}))) \mid X_{s-}-X'_{s-}\mid^2 \\
& \{\nabla\phi(X_{s-})\{f(X_{s-})dZ^c_s + \frac{1}{2}f'f(X_{s-})d[Z,Z]^c_s - d\Phi(s)\} \\
& +\nabla\phi(X'_{s-})\{f(X'_{s-})dZ^c_s + \frac{1}{2}f'f(X'_{s-})d[Z,Z]^c_s - d\Phi'(s)\}\} \\
& +\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-})))(f(X_{s-})-f(X'_{s-}))^2 d[Z,Z]^c_s \\
& +\frac{4}{\alpha^2}\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-}))) \mid X_{s-}-X'_{s-}\mid^2 \\
& \{\nabla\phi(X_{s-})f(X_{s-})+\nabla\phi(X'_{s-})f(X'_{s-})\}^2 d[Z,Z]^c_s \\
& -\frac{4}{\alpha}\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-})))(X_{s-}-X'_{s-})\{(f(X_{s-})-f(X'_{s-})) \\
& (\nabla\phi(X_{s-})f(X_{s-})+\nabla\phi(X'_{s-})f(X'_{s-})) \\
& +\{D^2\phi(X_{s-})f^2(X_{s-})+D^2\phi(X'_{s-})f^2(X'_{s-})\} \\
& +2\nabla\phi(X_{s-})\nabla\phi(X'_{s-})f(X_{s-})f(X'_{s-})\}d[Z,Z]^c_s \\
& -\frac{2}{\alpha}\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-})))(X_{s-}-X'_{s-}) \\
& (\nabla\phi(X_{s-})-\nabla\phi(X'_{s-}))f(X_{s-})f(X'_{s-})d[Z,Z]^c_s \\
& +\sum_{0<s\le t}(exp(\frac{-2}{\alpha}(\phi(X_s)+\phi(X'_s))) \mid X_s - X'_s\mid^2 \\
& -exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-}))) \mid X_{s-}-X'_{s-}\mid^2). \qquad (79)
\end{aligned}
$$

This equation can be analyzed in pieces:
(1) Consider first the terms that contain Φ and Φ'.

$$
\begin{aligned}
& 2\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-})))(X_{s-}-X'_{s-})\{-d\Phi(s)+d\Phi'(s)\} \\
- \;& \frac{2}{\alpha}\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-}))) \mid X_{s-}-X'_{s-}\mid^2 \\
& \{-\nabla\phi(X_{s-})d\Phi(s)-\nabla\phi(X'_{s-})d\Phi'(s)\}.
\end{aligned}
$$

The above expression can be rewritten as:

$$
2\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-})+\phi(X'_{s-})))(X_{s-}-X'_{s-})
$$

$$\{-n(X_{s-})d \mid \Phi \mid (s) + n(X'_{s-})d \mid \Phi' \mid (s)\}$$

$$- \quad \frac{2}{\alpha}\int_0^t exp(\frac{-2}{\alpha}(\phi(X_{s-}) + \phi(X'_{s-}))) \mid X_{s-} - X'_{s-} \mid^2$$

$$\{-\nabla\phi(X_{s-})n(X_{s-})d \mid \Phi \mid (s) - \nabla\phi(X'_{s-})n(X'_{s-})d \mid \Phi' \mid (s)\}. \qquad (80)$$

Using (75) and (72) we can build the following inequality:

$$\frac{1}{\alpha}(\nabla\phi(x), n(x)) \mid x - x' \mid^2 -(x - x', n(x)) \leq 0 \qquad (81)$$

which proves together with the fact $\Delta X_s = 0$ if $d \mid \Phi \mid_s \neq 0$ (recall that the jump times conform a set of measure zero), that (80) is negative.

(2) Next, consider the last term in (79) that contains all the jumps. This last term is bounded by (see (77)):

$$\sum_{0<s\leq t} C \mid \Delta Z_s \mid \int_0^1 L(u) \mid y(u) - y'(u) \mid^2 du. \qquad (82)$$

This equation can also be analyzed in pieces:
By the previous Lemma

$$\sum_{0<s\leq t} C \mid \Delta Z_s \mid\mid X_{s-} - X'_{s-} \mid^2 \qquad (83)$$

which is

$$\int_0^t \mid X_{s-} - X'_{s-} \mid^2 d(\sum_{0<u\leq s} C|\Delta Z_u|).$$

(3) Finally consider the remaining terms not covered in (1) and (2) above.

These terms are the ones that are obtained in the usual uniqueness proof for stochastic differential equations. If we take expectation of the supremum of the square of (79) using standard inequalities we obtain the result . □

From here, the classical argument of slicing Z proves the uniqueness of solutions (see Protter (1990)).

Theorem 36 *The solution of the Stratonovich SDE with reflection (65) is unique.*

Proof.

By Theorem 5, Chapter V in Protter (1990), we know that Z^{T-} is $\alpha - sliceable$ for any small α for a large stopping time T. Since we are studying the SDE with reflection in a bounded domain in time (say, $[0,1]$), we can assume that $T > 1$. We also let α be small compared to C in (78) (e.g. $\alpha < \frac{C}{3}$).

Let's suppose that the $\alpha - slice$ for Z is given by the increasing sequence of stopping times $0 = T_0 \leq T_1 \leq ... \leq T_k$, such that

$$\|(Z - Z^{T_i})^{T_{i+1}-}\|_{\mathcal{H}^\infty} \leq \alpha, \; 0 \leq i \leq k-1$$

by using (78) and the above inequality it is not difficult to obtain that $X_s = X'_s$ for $s < T_1$. As we know that the jumps of X or X' are the same as the jumps for Z, we also have that $X_{T_1} = X'_{T_1}$ if T_1 is a continuity point for Z. If T_1 is a discontinuity point, we still have that $X_{T_1} = X'_{T_1}$ given that $X_{T_1-} = X'_{T_1-}$. Using induction, one obtains uniqueness in (65). □

5 Stability results

In this section we analyze the weak convergence of solutions of reflected Stratonovich SDE's when coefficients f_n and integrators Z^n are used instead of f and Z. Let $Z^n = M^n + A^n$, where M^n is a local martingale and A^n is a process of bounded variation on compact intervals.

5.1 Continuous case

Let X^n be the unique solution for the following Stratonovich type stochastic differential equation with reflection driven by the continuous semimartingale Z^n:

$$X_t^n = X_0^n + \int_0^t f_n(X_{s-}^n)dZ_s^n + \frac{1}{2}\int_0^t f_n' f_n(X_{s-}^n)d[Z^n, Z^n]_s - \Phi^n(t) \qquad (84)$$

$$\Phi^n(t) = \int_0^t n(X_s^n)d \mid \Phi^n \mid (s), \; X_t^n \in \bar{D},$$

We further assume that $f_n, f \in C_b^1(\bar{D})$ and that the domain D satisfies *Conditions* (A) and (B) in Section 2.1.

Lemma 37 *If* $Z^n \Rightarrow Z$ *and* $\{Z^n\}$ *is good then*

$$[Z^n, Z^n] \Rightarrow [Z, Z].$$

Proof.
The proof is just an application of Theorem 4 (or Theorem 2.2 in Kurtz and Protter (1991)).
As

$$[Z^n, Z^n]_t = (Z_t^n)^2 - 2\int_0^t Z_{s-}^n dZ_s^n,$$

and Z^n is good, we have that the right side converges weakly to $Z_t^2 - 2\int_0^t Z_{s-}dZ_s$. This equals $[Z, Z]_t$. □

Theorem 38 *Let* $\{(X_0^n, Z^n)\}$ *be a good sequence of semimartingales converging weakly to* (X_0, Z) *in the Skorohod topology, then* (X^n, Φ^n) *converge weakly. If we denote the limit point by* (X, Φ)*, the pair solves the following Stratonovich type stochastic differential equation with reflection:*

$$X_t = X_0 + \int_0^t f(X_{s-})dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-})d[Z, Z]_s - \Phi(t). \qquad (85)$$

Proof.
The proof is a direct application of Theorem 4 as well as Lemma 37. To prove the tightness of (X^n, Φ^n, X_0^n) it is enough to use the Costantini's results on tightness of solutions of the Skorohod problem (see Theorem 7). □

5.2 Non-summable jumps case

The idea in this case is to construct a sequence Y^n corresponding to X^n through the formula (54) and take limits for Y^n. Once the limit Y is obtained we will prove that the process X obtained from Y is the limit for the sequence X^n.

Let X^n be a solution for:

$$\begin{aligned} X_t^n &= X_0^n + \int_0^t f_n(X_{s-}^n)dZ_s^n + \frac{1}{2}\int_0^t f_n'f_n(X_{s-}^n)d[Z^n,Z^n]_s^c - \Phi^n(t) \\ &\quad + \sum_{o<s\leq t}\{\varphi(\Delta Z_s^n f_n, X_{s-}^n) - X_{s-}^n - f_n(X_{s-}^n)\Delta Z_s^n\}, \end{aligned} \tag{86}$$

where Z^n is a good sequence, D is a smooth domain in R^d (C^2 will suffice), $f_n \in C_b^2(\bar{D})$ and $(f_n, f_n') \to (f, f')$ uniformly in $\bar{D}$. A consequence of these assumptions is that $\{f_n\}$ is uniformly bounded. We have assumed the necessary conditions for existence of solutions for (86), exposed in Section 3.

Now, let Y^n be defined through (54). Then Y^n is a solution of:

$$\begin{aligned} Y_t^n &= X_0^n + S_t^n - \Psi_t^n, \\ Y_t^n &= X_0^n + \int_0^t f_n(Y_s^n)dZ_{\gamma_n^{-1}(s)}^n + \frac{1}{2}\int_0^t f_n'f_n(Y_s^n)d[Z^n,Z^n]_{\gamma_n^{-1}(s)}^c \\ &\quad + \sum_{0<u\leq(\gamma_n^{-1}(\eta_2^n(t)))}\left(\int_{\eta_1^n(\gamma_n(u))}^{\eta_2^n(\gamma_n(u))}\frac{f_n(Y_s^n)\Delta Z_u^n}{\eta_2^n(\gamma_n(u))-\eta_1^n(\gamma_n(u))}ds - f_n(Y_{\eta_1^n(\gamma_n(u))}^n)\Delta Z_u^n\right) \\ &\quad - \int_t^{\eta_2^n(t)} f_n(Y_s^n)\frac{\Delta Z_{\gamma_n^{-1}(t)}^n}{\eta_2^n(t)-\eta_1^n(t)}ds - \Psi_t^n. \end{aligned} \tag{87}$$

Here $\gamma_n(t)$, $\eta_1^n(t)$, $\eta_2^n(t)$ and $V_n(t)$ are defined as in Section 3.

Lemma 39 *(Y^n, S^n, Ψ^n) is a tight sequence.*

Proof.

Recall that S^n can be written as:

$$\begin{aligned} S_t^n &= \int_0^t f_n(Y_s^n)dZ_{\gamma_n^{-1}(s)}^n + f_n(Y_t^n)U_t^n - \int_0^t U_s^n f_n'f_n(Y_s^n)dZ_{\gamma_n^{-1}(s)}^n \\ &\quad - \int_0^t f_n'f_n(Y_s^n)d\left(\frac{(U_s^n)^2}{2} - \frac{[Z^n,Z^n]_{\gamma_n^{-1}(s)}}{2}\right) + \int_0^t U_s^n f_n'(Y_s^n)d\Psi_s^n. \end{aligned} \tag{88}$$

Here, $U_t^n = Z_{\gamma_n^{-1}(t)}^n - V_t^n$. The goodness of $(U^n)^2$ is a consequence of Lemma 37 and the goodness of Z^n. That is,

$$T_t((U^n)^2) \leq C \sum_{0<s\leq\gamma_n^{-1}(t)} |\Delta Z_s^n|^2,$$

therefore it is obvious that

$$\sup_n E[T_t((U^n)^2)] < \infty$$

The tightness of $f_n(Y_t^n)U_t^n$ can be proved as in Lemma 13. This also proves the tightness of U^n. There is only one term left:

$$\int_0^t U_s^n f_n'(Y_s^n)d\Psi_s^n = \int_0^t U_s^n f_n'(Y_s^n)n(Y_s^n)1(Y_s^n \in \partial D) < n(Y_s^n), f(Y_s^n)dU_s^n > + \int_0^t U_s^n f_n'(Y_s^n)n(Y_s^n)1(Y_s^n \in \partial D) < n(Y_s^n), f(Y_s^n)dZ^n_{\gamma_n^{-1}(s)} > . \quad (89)$$

Therefore the tightness of this term follows the same pattern shown in Lemma 13. □

Also by Lemma 37 we have that $\gamma_n \Rightarrow \gamma_0$ in the Skorohod topology. Let $(Y, S^\star, \Psi)$ be a limit point of a subsequence. As before,

$$U_t^n = -\Delta Z^n_{\gamma_n^{-1}(t)} \frac{\eta_2^n(t) - t}{\eta_2^n(t) - \eta_1^n(t)} 1_{[\eta_1^n(t), \eta_2^n(t)[}(t). \quad (90)$$

Lemma 40 *$S^\star = S$. That is,*

$$S_t^\star = X_0 + \int_0^t f(Y_s)dZ_{\gamma_0^{-1}(s)} + \frac{1}{2}\int_0^t f'f(Y_s)d[Z,Z]^c_{\gamma_0^{-1}(s)} + \sum_{0<u\leq(\gamma_0^{-1}(\eta_2(t)))} \left(\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{f(Y_s)\Delta Z_u}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))}ds - f(Y_{\eta_1(\gamma_0(u))})\Delta Z_u\right) - \int_t^{\eta_2(t)} f(Y_s)\frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t) - \eta_1(t)}ds. \quad (91)$$

Proof.

Because of the goodness of Z^n and $(U^n)^2$, it is enough to prove that Ψ^n is good also. This is clear if we recall Lemma 17. 91 is obtained by taking limits for S_t^n:

$$S_t^n = \int_0^t f_n(Y_s^n)dZ^n_{\gamma_n^{-1}(s)} + f(Y_t^n)U_t^n - \int_0^t U_s^n f_n' f_n(Y_s^n)dZ^n_{\gamma_n^{-1}(s)} - \int_0^t f_n' f_n(Y_s^n)d\left(\frac{(U_s^n)^2}{2} - \frac{[Z^n, Z^n]_{\gamma_n^{-1}(s)}}{2}\right) + \int_0^t U_s^n f_n'(Y_s^n)d\Psi_s^n. \quad (92)$$

□

Theorem 41 *Let X_t^n be the solution to the Stratonovich type stochastic differential equation with reflection (86). Assume that ∂D is smooth, Z^n is a good sequence, $(X_0^n, Z^n) \Rightarrow (X_0, Z)$ and $(f_n, f_n') \to (f, f')$ uniformly in $C_b^2(\bar{D})$. Then $X_n \Rightarrow X$ where X solves (86) for (X_0, Z, f).*

The proof is an argument like the one used in equation (47), if we transform Y through $X_t = Y_{\gamma_0(t)}$.

In the oblique reflection case similar stability theorems as in the normal reflection case (Theorem 41) can be proved.

Theorem 42 *Let D be a bounded smooth open set, and let $\Theta(x)$ be an C_b^2 oblique field such that*

$$\exists \nu > 0,\ \forall x \in \partial D,\quad (\theta(x), n(x)) \geq \nu.$$

Also let $(X_0^n, Z^n) \Rightarrow (X_0, Z)$ in the Skorohod topology, and suppose that the sequence $\{Z^n\}$ is good. Let X^n be the unique solution to the Stratonovich type stochastic differential equation with oblique reflection driven by the semimartingale Z^n. Also, let X be analoguosly defined. Then $X^n \Rightarrow X$ in the Skorohod topology.

5.3 Summable jump case

In this case we study the following sequence X^n of solutions of the following Stratonovich type stochastic differential equations with reflection:

$$X_t^n = X_0^n + \int_0^t f_n(X_{s-}^n)dZ_s^n + \frac{1}{2}\int_0^t f_n' f_n(X_{s-}^n)d[Z^n, Z^n]_s^c - \Phi^n(t)$$
$$+ \sum_{0<s\leq t}\{\varphi(\Delta Z_s^n f, X_{s-}^n) - X_{s-}^n - f_n(X_{s-}^n)\Delta Z_s^n\}, \quad (93)$$

$$\Phi^n(t) = \int_0^t n(X_s^n)d \mid \Phi^n \mid (s),\ X_t^n \in \bar{D},$$

and $\varphi(g,x) = y(1)$, where $y(t)$ is the solution of

$$y(t) = x + \int_0^t g(y(s))ds - \kappa(t),\ y(t) \in \bar{D}\ ,$$

$$\kappa(t) = \int_0^t n(y(s))d \mid \kappa \mid (s).$$

The question we will answer in the next theorems is the weak convergence, as well as the tightness of the sequence X^n under certain conditions on f_n and Z^n; assuming that Z^n has summable jumps. As before, this subsection is based on results by Kurtz and Protter (1991). We assume that each $f_n \in C_b^1(\bar{D})$ and D holds *Conditions* (A) and (B).

Here, we will need to extend some results obtained previously in Lemma 37. Define A^d as the discontinuous part process of the bounded variated process A as:

$$A_t^d = \sum_{0<s\leq t}\Delta A_s.$$

The sum on the right side of the above inequality is absolutely convergent because A has bounded variation.

Lemma 43 $[Z^n, Z^n]^d \Rightarrow [Z, Z]^d$.

Proof.
As

$$[Z^n, Z^n]^d = \sum_{0<s\leq t}(\Delta Z_s^n)^2,$$

we have:

$$\mid \sum_{0<s\leq t}(\Delta Z_s^n)^2 - \sum_{0<s\leq t}(\Delta Z_s)^2 \mid \leq \mid \sum_J((\Delta Z_{\lambda_n(s)}^n)^2 - (\Delta Z_s)^2)\mid + \mid \sum_{\lambda_n(J)^c}(\Delta Z_s^n)^2 - \sum_{J^c}(\Delta Z_s)^2 \mid . \quad (94)$$

Here, λ_n are strictly increasing, continuous functions converging uniformly to t, such that,

$$\sup_{0\leq t\leq 1}|Z_{\lambda_n(t)}^n - Z_t| \to 0.$$

J is a finite set of indices so that

$$\sum_{J^c}(\Delta Z_s)^2 < \epsilon,$$

therefore the first term on the right side of (94) goes to zero. J has to be chosen so that the second sum is arbitrarily small. For that it is enough to notice that

1. $\mid \sum_{\lambda_n(J)^c}(\Delta Z_s^n)^2 \mid \leq \max_{s\in\lambda_n(J)^c} \mid \Delta Z_s^n \mid\mid \sum_{\lambda_n(J)^c} \mid(\Delta Z_s^n) \mid$,

2.The above sum is uniformly bounded in probability, because Z^n is good. Also $\max_{s\in J^c} \mid \Delta Z_s^n \mid$ is uniformly small, for n big enough. In the vector case, $(\Delta Z_s)^2$ stands for $\Delta Z_s \Delta Z_s^T$ (T denotes the transpose). □

Theorem 44 *If (*) holds, $(X_0^n, Z^n) \Rightarrow (X_0, Z)$ in the Skorohod topology, and (f_n, f_n') converges to (f, f') uniformly, then X^n converges weakly to X.*

The proof is preceded by six lemmas.

As $\{Z^n\}$ is good we know that Z is a semimartingale, and by results of the previous lemma we have that $[Z^n, Z^n]^d \Rightarrow [Z, Z]^d$.

The proof of Theorem 44 consists of transforming each equation into a continuous type equation and use stability results for the continuous case (i.e. Theorem 38).

Define:

$$\begin{aligned} \Gamma_n(t) &= \sum_{0<s\leq t} |\Delta Z_s|, \\ \gamma_n(t) &= \sum_{0<s\leq t} |\Delta Z_t^n| + t, \\ \eta_1^n(t) &= sup\{s : \gamma_n^{-1}(s) < \gamma_n^{-1}(t)\}, \\ \eta_2^n(t) &= inf\{u : \gamma_n^{-1}(u) > \gamma_n^{-1}(t)\}. \end{aligned} \tag{95}$$

$$V_t^n = \begin{cases} Z^n_{\gamma_n^{-1}(t)} & \text{if } \eta_1^n(t) = \eta_2^n(t); \\ Z^n_{\gamma_n^{-1}(t)} \dfrac{t - \eta_1^n(t)}{\eta_2^n(t) - \eta_1^n(t)} + Z^n_{\gamma_n^{-1}(t)-} \dfrac{\eta_2^n(t) - t}{\eta_2^n(t) - \eta_1^n(t)} & \text{if } \eta_2^n(t) \neq \eta_1^n(t) \end{cases} \tag{96}$$

V^n is a semimartingale because it is the sum of the semimartingale $Z^n_{\gamma_n^{-1}(t)}$ with the process of bounded variation U^n,

$$U_t^n = -\Delta Z^n_{\gamma_n^{-1}(t)} \frac{\eta_2^n(t) - t}{\eta_2^n(t) - \eta_1^n(t)} 1_{[\eta_1^n(t), \eta_2^n(t)[}(t). \tag{97}$$

We also have the following result:

Lemma 45 $[V^n, V^n]_t = [Z^n, Z^n]^c_{\gamma_n^{-1}(t)}$

Proof.

It is enough to realize that there are only a countable number of intervals $[\eta_1^n(t), \eta_2^n(t)]$ that contain more than one point. In any of those intervals the quadratic variation of V^n is zero because V^n is a continuous bounded variation function in each such interval. The rest are intervals where Z^n is continuous and therefore the result follows. □

It is not difficult to obtain that if Y_t^n is defined as:

$$Y_t^n = \begin{cases} X^n_{\gamma_n^{-1}(t)} & \text{if } \eta_1^n(t) = \eta_2^n(t); \\ X^n_{\gamma_n^{-1}(t)-} + \displaystyle\int_{\eta_1^n(t)}^t f(Y^n(u)) \frac{\Delta Z_{\gamma_n^{-1}(t)}}{\eta_2^n(t) - \eta_1^n(t)} du - \kappa^n \left(\frac{\eta_2^n(t) - t}{\eta_2^n(t) - \eta_1^n(t)} \right) & \begin{array}{l} \text{if } \eta_1^n(t) \leq t \leq \eta_2^n(t) \\ \text{and } \eta_2^n(t) \neq \eta_1^n(t), \end{array} \end{cases} \tag{98}$$

then Y^n is the unique solution to the following Stratonovich type stochastic differential equation with reflection driven by the continuous semimartingale V^n (as in Lemma 25):

$$Y_t^n = X_0^n + \int_0^t f_n(Y_s^n)dV_s^n + \frac{1}{2}\int_0^t f_n' f_n(Y_s^n)d[Z^n, Z^n]^c_{\gamma_n^{-1}(s)} - \Psi_t^n. \qquad (99)$$

Ψ^n is defined as in Lemma 25.

Lemma 46 *The sequence V^n is good.*

Proof.
If $Z^n = M^n + A^n$ where M^n is a local martingale and A^n is a process of bounded variation, then $V^n = M^n + (A^n + U^n)$. Here, $A^n + U^n$ is a process of bounded variation, and

$$T_t(U^n) \leq C \sum_{0<s\leq\gamma_n^{-1}(t)} \mid \Delta Z_s \mid .$$

As Z^n is good, it is clear that

$$sup_n E[[M^n, M^n]_{\sigma_n^\alpha \wedge t} + T_{t\wedge\sigma_n^\alpha}(A^n + U^n))] < \infty,$$

for some sequence of stopping times σ_n^α. Therefore V^n is good. □

Lemma 47 *The sequence $\{V^n\}$ is C-tight.*

Proof.
Take $\delta \in R$, and choose a partition $\{t_i\}$ so that $t_{i+1} - t_i > \delta$ and

$$\max_i w(Z^n, [t_i, t_{i+1}[) < \epsilon.$$

Define $\delta' = \min_i |t_{i+1} - t_i|$, with these definitions is clear that

$$w(V^n, \delta') = \sup_t w(V^n, [t, t+\delta']) < 2\epsilon + \delta' \max_{0\leq s\leq 1} |\Delta Z_s|.$$

From here, one could easily obtain the tightness of V^n. □

Lemma 48

$$|\gamma_n^{-1}(t) - \gamma_n^{-1}(s)| < |t - s|.$$

Proof.
Define $t_n = \gamma_n^{-1}(t)$, $s_n = \gamma_n^{-1}(s)$ and suppose that $s < t$. Then, there exists $\alpha_n, \beta_n \in [0,1]$ such that

$$t = \alpha_n\gamma_n(t_n) + (1-\alpha_n)\gamma_n(t_n)_-,$$

$$s = \beta_n\gamma_n(s_n) + (1-\beta_n)\gamma_n(s_n)_-.$$

Using the definition of γ_n, we have:

$$\begin{aligned} t - s &= \alpha_n|\Delta Z_{t_n}^n| + (\sum_{s_n\leq u<t_n} |\Delta Z_u^n| - \beta_n|\Delta Z_{s_n}^n|) \\ &\quad +(t_n - s_n) \end{aligned} \qquad (100)$$

The positivity of all the bracketed terms above ensures that $t - s > t_n - s_n \geq 0$, which is enough to conclude the proof. □

Lemma 49

$$\gamma_n^{-1} \Rightarrow \gamma_0^{-1}, \textit{ in the uniform topology.} \tag{101}$$

Proof.
First, the following inequality (proved in the previous Lemma):

$$| \gamma_n^{-1}(t) - \gamma_n^{-1}(s) |<| t - s|,$$

proves that γ_n^{-1} is C-tight. For $u \in R$, let $s = \gamma_0^{-1}(u)$. Let $u_n = \gamma_n^{-1}(u)$, we intend to prove that $u_n \to \gamma_0^{-1}(u)$. We already proved that u_n has at least one accumulation point, we only need to prove that $\gamma_0^{-1}(s)$ is the only one. By definition of u_n, there exists $\alpha_n \in [0,1]$ such that $u = \alpha_n \gamma_n(u_n) + (1-\alpha_n)\gamma_n(u_n)_-$. Denote by the same index n a subsequence of u_n converging to some real value w such that $\alpha_n \to \alpha \in [0,1]$. Then (see Ethier and Kurtz (1986)) $\gamma_n(u_n)$ and $\gamma_n(u_n)_-$ should converge to either one of $\gamma_0(w)$, $\gamma_0(w)_-$.

Therefore, $u \in [\gamma_0(w)_-, \gamma_0(w)]$ or $\gamma_0^{-1}(u) = w$. □

Lemma 50

$$V^n \Rightarrow V, \textit{ in the uniform topology.}$$

Proof.
As proved in Lemma 47, the tightness of V^n has already been obtained , therefore the only part left to prove is the characterization of the limit. First, take any converging subsequence and denote it with the same superscript. Let $\lambda_n(t)$ converge uniformly to t strictly increasing and continuous, such that $Z^n_{\lambda_n(t)} \to Z_t$, uniformly.

As noted in Jacod and Shiryaev (1989) for each t such that $\Delta Z_t \neq 0$, there exists a sequence $t_n := \lambda_n(t)$ such that $\Delta Z^n_{t_n} \to \Delta Z_t$. If the sequence $\gamma_n^{-1}(t)$ converges to a continuity point of Z, there is nothing to prove. Therefore, we only analyze the case of covergence to V_t for $t \in [\eta_1(t), \eta_2(t)]$ and $\eta_1(t) \neq \eta_2(t)$. Let $t = \alpha\gamma_0(u) + (1-\alpha)\gamma_0(u)$, define $u_n := \alpha\gamma_n(t_n) + (1-\alpha)\gamma_n(t_n)_-$. Therefore $u_n \to t$, and as:

$$\begin{aligned} \eta_2^n(t) &= \gamma_n(t_n) \to \eta_2(t) \\ \eta_1^n(t) &= \gamma_n(t_n)_- \to \eta_1(t) \end{aligned}$$

is not difficult to prove that $V^n_{u_n} \to V_t$. From these facts is not difficult to obtain that any subsequence will have to converge to V. □

Proof of Theorem 44.

The limit theorems are obtained by using Theorem 38. Take limits in (99) following closely the ideas used in Section 3. After the limit process Y has been obtained, define $X_t = Y_{\gamma_0(t)}$ and as in Section 3 obtain that $(X^n, \Phi^n) \Rightarrow (X, \Phi)$ where $\Phi_t = \Psi_{\gamma_0(t)}$. □

Remark 51.

A. This technique would work for the existence and uniqueness for the summable case , but it is not just a particular case of Section 4 because in the proof of Lemma 17 the smoothness of ∂D was strongly used.

B. Here, as well as in the next sections we need to prove tightness first, and then identify the limit. In order to prove the tightness of $\int_0^t f_n(Y_s^n)dZ_s^n$ we use Theorem 4, which gives the tightness of $\int_0^{a_n(t)} f_n(Y_s^n)dZ_s^n$. This proves the tightness of $(Y^n_{a_n(s)}, \Psi^n_{a_n(s)})$, which proves the tightness of (Y_s^n, Ψ_s^n) by using Theorem 4.

5.4 Wong-Zakai corrections

In this subsection we study stability problems for sequences Z^n which are not good, but nearly good in a sense that will be explained later. Wong and Zakai (1965) showed that in a very natural range of approximating sequences we have that this sequence is not good therefore generating an extra term in the limit (for examples, see Kurtz and Protter (1991)). Let D be a smooth domain.

Theorem 52 *Suppose $Z^n = Y_n + W_n$, where Y_n and W_n are also semimartingales adapted to the same filtration to which Z_n is adapted (these could change with n). Let X_n be the unique solution to the following Stratonovich type stochastic differential equation with reflection:*

$$\begin{aligned} X_t^n &= X_0^n + \int_0^t f_n(X_{s-}^n)dZ_s^n + \frac{1}{2}\int_0^t f_n' f_n(X_{s-}^n)d[Z^n, Z^n]_s^c - \Phi^n(t) \\ &\quad + \sum_{0<s\le t}\{\varphi(\Delta Z_s^n f_n, X_{s-}^n) - X_{s-}^n - f_n(X_{s-}^n)\Delta Z_s^n\}, \end{aligned} \tag{102}$$

where $f_n \in C_b^2$. Define H_n and K_n by

$$H_n^{\beta\gamma}(t) = \int_0^t W_n^\beta(s-)dW_n^\gamma(s),$$

and

$$K_n^{\beta\gamma}(t) = [Y_n^\beta, W_n^\gamma]_t.$$

Suppose that $\{Y_n\}$ and $\{H_n\}$ satisfy (*) and that

$$(X_n(0), Y_n, W_n, H_n, K_n) \Rightarrow (X(0), Y, 0, H, K),$$

and that $(f_n, f_n') \to (f, f')$ uniformly. Then$(X_n(0), Y_n, W_n, H_n, K_n, \Phi^n)$ is relatively compact, and any limit point
$(X(0), Y, 0, H, K, \Phi)$ satisfies

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-})dY_s + \sum_{\alpha,\beta,\gamma}\int_0^t \partial_\alpha f_\beta(X_{s-})f_{\alpha\gamma}(X_{s-})d(H^{\gamma\beta}(s) - K^{\gamma\beta}(s)) \\ &\quad + \frac{1}{2}\int_0^t f'f(X_{s-})d[Y,Y]_s^c - \Phi(t) \\ &\quad + \sum_{0<s\le t}\{\varphi(\Delta Y_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Y_s\}, \end{aligned} \tag{103}$$

where ∂_α denotes the partial derivative with respect to the αth variable and f_β denotes the βth column of f.

The following remarks are clearly explained in Kurtz and Protter (1991).
Remark 53. 1. The boundedness assumptions on f_n and its derivatives can be weakened by using a localization argument.
2. H and K must be continuous.
3. $[Z^n, Z^n]_t^{\beta\gamma}$ and $K_n^{\beta\gamma}$ are good.
4. $[Z^n, Z^n]^{\beta\gamma} \Rightarrow I := -(H^{\beta\gamma} + H^{\gamma\beta})$.

Proof of Theorem 52. We will do the proof in the one dimensional case, the extension to the multidimensional case is straightforward.

First, we use the Skorohod representation theorem assuming that

$$(X_n(0), Y_n, W_n, H_n, K_n) \to (X(0), Y, 0, H, K) \ a.s.$$

We will prove that the assertion of the theorem holds *a.s.*

Step 1. Tightness of (Y^n, Ψ^n). As in Section 5.2 we obtain a new continuous process Y^n from each X^n which holds the following stochastic equation with reflection process Ψ^n:

$$Y_t^n = S_t^n - \Psi_t^n, \tag{104}$$

$$\begin{aligned} S_t^n &= \int_0^t f_n(Y_s^n) dZ^n_{\gamma_n^{-1}(s)} + f_n(Y_t^n)U_t^n - \int_0^t U_s^n f_n' f_n(Y_s^n) dZ^n_{\gamma_n^{-1}(s)} \\ &- \int_0^t f_n' f_n(Y_s^n) d(\frac{(U_s^n)^2}{2} - \frac{[Z^n, Z^n]_{\gamma_n^{-1}(s)}}{2}) + \int_0^t U_s^n f_n'(Y_s^n) d\Psi_s^n. \end{aligned} \tag{105}$$

The tightness of

$$\int_0^t U_s^n f_n' f_n(Y_s^n) dZ^n_{\gamma_n^{-1}(s)} - \int_0^t f_n' f_n(Y_s^n) d(\frac{(U_s^n)^2}{2}) + \int_0^t U_s^n f_n'(Y_s^n) d\Psi_s^n. \tag{106}$$

is easily obtained because each of the above terms is the sum of terms of the order $o((\Delta Z_s)^2)$. As $[Z^n, Z^n]$ is a good sequence the tightness of

$$\int_0^t f_n' f_n(Y_s^n) d[Z^n, Z^n]_{\gamma_n^{-1}(s)}$$

follows. The proof of the tightness of $f_n(Y_t^n)U_t^n$ is similar to the proof of Lemma 13.

The remain of this step is devoted to prove the tightness of $\int_0^t f_n(Y_s^n) dZ^n_{\gamma_n^{-1}(s)}$.

$$\int_0^t f_n(Y_s^n) dZ^n_{\gamma_n^{-1}(s)} = \int_0^t f_n(Y_s^n) dY_n(\gamma_n^{-1}(s)) + \int_0^t f_n(Y_s^n) dW_n(\gamma_n^{-1}(s)). \tag{107}$$

The first term above is tight because Y_n is good. The second term becomes:

$$\begin{aligned} \int_0^t f_n(Y_s^n) dW_n(\gamma_n^{-1}(s)) &= W_n(\gamma_n^{-1}(s)) f_n(Y_s^n) - \int_0^t W_n(\gamma_n^{-1}(s)) df_n(Y_s^n) \\ &\quad -[W_n(\gamma_n^{-1}(s)), f_n(Y_s^n)]. \end{aligned} \tag{108}$$

The first term goes to zero and the third term can be rewritten as:

$$\begin{aligned} [W_n(\gamma_n^{-1}(s)), f_n(Y_s^n)]_t &= \int_0^t f_n'(Y_s^n) d[W_n(\gamma_n^{-1}(s)), Y_s^n] \\ &= \int_0^t f_n' f_n d[W_n(\gamma_n^{-1}(\cdot)), Z^n_{\gamma_n^{-1}(\cdot)}]_s^c \end{aligned}$$

This last equality shows that this term is also tight. Now decompose the integral term in (108), by using an extension of Itô's formula used in equation (51):

$$\begin{aligned}\int_0^t W_n(\gamma_n^{-1}(s))df_n(Y_s^n) &= \int_0^t W_n(\gamma_n^{-1}(s))f_n'f_n(Y_s^n)dZ^n_{\gamma_n^{-1}(s)} \\ &+\frac{1}{2}\int_0^t W_n(\gamma_n^{-1}(s))(f_n'f_n'f_n(Y_s^n)+f_n''f_n(Y_s^n))d[Z^n,Z^n]^c_{\gamma_n^{-1}(s)} \\ &+\sum_{0<u\leq(\gamma_n^{-1}(\eta_2^n(t)))}(\int_{\eta_1^n(\gamma_n(u))}^{\eta_2^n(\gamma_n(u))}\frac{W_n(\gamma_n^{-1}(s))f_n'f_n(Y_s^n)\Delta Z_u^n}{\eta_2^n(\gamma_n(u))-\eta_1^n(\gamma_n(u))}ds \\ &-W_n(\gamma_n^{-1}(s))f_n'f_n(Y^n_{\eta_1^n(\gamma_n(u))})\Delta Z_u^n)) \\ &-\int_t^{\eta_2^n(t)} W_n(\gamma_n^{-1}(s))f_n'f_n(Y_s^n)\frac{\Delta Z^n_{\gamma_n^{-1}(t)}}{\eta_2^n(t)-\eta_1^n(t)}ds \\ &-\int_0^t W_n(\gamma_n^{-1}(s))f_n'(Y_s^n)d\Psi_s^n.\end{aligned}$$

All the above terms are clearly tight with the exception of the last term. In short we have proven the following inequality:

$$\sup_{u\leq t_1<t_2\leq t}|S_{t_1}^n-S_{t_2}^n|\leq\epsilon'+\sup_{u\leq t_1<t_2\leq t}|\int_{t_1}^{t_2}W_n(\gamma_n^{-1}(s))f_n'(Y_s^n)d\Psi_s^n|, \tag{109}$$

for $|t-u|$ and ϵ' small enough. As $W_n\to 0$, we can assume that $\sup_{0\leq s\leq 1}|W_n(s)|<\epsilon$, therefore inequality (109) becomes

$$\sup_{u\leq t_1<t_2\leq t}|S_{t_1}^n-S_{t_2}^n|\leq\epsilon'+C\epsilon(|\Psi^n|_t-|\Psi^n|_u). \tag{110}$$

Using inequality (20) we have:

$$\sup_{u\leq t_1<t_2\leq t}|S_{t_1}^n-S_{t_2}^n|\leq\epsilon'+C\epsilon K(\omega)\sup_{u\leq t_1<t_2\leq t}(|S_{t_1}^n-S_{t_2}^n|). \tag{111}$$

Therefore

$$\sup_{u\leq t_1<t_2\leq t}|S_{t_1}^n-S_{t_2}^n|\leq\frac{\epsilon'}{1-CK(\omega)\epsilon}$$

is small because ϵ can be made as small as desired.

This ends the proof of the tightness of S_t^n therefore proving the tightness of (Y^n,S^n,Ψ^n).

Step 2. Identification of the limit.

In order to identify the limit of S_t^n, one has to follow the decomposition that we used in Step 1. The only term that will bring the Wong-Zakai correction term is $\int_0^t f_n(Y_s^n)dW_n(\gamma_n^{-1}(s))$, whose limit is

$$-\int_0^t f'f(Y_s)d(H_s+K_s+I_s).$$

Using Remark 53.4, we obtain the desired expression for the limit of Y^n. As in Section 5.2 we obtain the theorem by transforming Y into X by $X_t=Y_{\gamma_0(t)}$. □

The same methodology used in Section 5.3 can be adapted to give an analogous result (Theorem 52) for semimartingales Z_n with summable jumps for each n.

6 Time reversal of solutions.

In this section we study the time reversal of solutions of Stratonovich type stochastic differential equations with reflection driven by Lèvy processes. In the classical framework, Cattiaux (1988) has proven the reversibility of solutions of SDE's with reflection driven by Lebesgue measure and Brownian motion. Withouth reflection, Sundar (1989) has proven the reversibility of solutions for SDE's driven by Lévy processes.

We will use theorems concerning the reversibility of stochastic integrals obtained by Jacod and Protter (1988). To be able to use them we will require to prove the injectivity of the flows of the solution process.

6.1 Injectivity of the flows.

Let X be the solution of the Stratonovich type stochastic differential equation with reflection:

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-})dZ_s + \frac{1}{2}\int_0^t f'f(X_{s-})d[Z,Z]_s^c - \Phi(t) \\ &\quad + \sum_{0<s\le t} \{\varphi(\Delta Z_s f, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\}, \end{aligned} \tag{112}$$

$$\Phi(t) = \int_0^t n(X_s)d \mid \Phi \mid (s),\ X_t \in \bar{D},$$

and $\varphi(g,x) = y(1)$, where $y(t)$ is the solution of

$$y(t) = x + \int_0^t g(y(s))ds - \kappa(t),\ y(t) \in \bar{D}\ ,$$

$$\kappa(t) = \int_0^t n(y(s))d \mid \kappa \mid (s).$$

Here, we assume that the boundary of D is smooth. $X^x(t,\omega)$ denotes the above solution for $X_0 = x$ a.s., when X is considered as a function of x, $X^x(t,\omega)$ is called the flow of the stochastic differential equation with reflection.

We will now prove that the flow has the following property:

$$\forall x \neq y\ P(\{\omega : \exists t : X^x(t,\omega) = X^y(t,\omega)\}) = 0. \tag{113}$$

When the above property holds the flow is called weakly injective (sometimes this property is called the non-confluence of paths property). Our final goal is to prove that the flow is strongly injective, which is defined as:

$$P(\{\omega : x \to X^x(t,\omega) \text{ is injective}\}) = 1. \tag{114}$$

Lemma 54 *The flow defined by the equation (112) is weakly injective while the paths are in the domain of a system of coordinates.*

Proof.

If the paths of X_t^x and X_t^y are in the domain of the system of coordinates defined by g (see Section 3.2), we define $W_t^x = g(Y_t^x)$, $W_t^y = g(Y_t^y)$ (Y_t^x and Y_t^y were defined in equation (54)).

Applying Itô's formula (as in Section 3.2) to $\|g(Y_t^x) - g(Y_t^y)\|^2$ we have

$$
\begin{aligned}
\|W_t^x - W_t^y\|^2 &= \|w_0^x - w_0^y\|^2 + 2\int_0^t (g(Y_s^x) - g(Y_s^y))(g'f(Y_s^x) - g'f(Y_s^y))dZ_{\gamma_0^{-1}(s)} \\
&+ \int_0^t (g'f(Y_s^x) - g'f(Y_s^y))d[Z,Z]^c_{\gamma_0^{-1}(s)}(g'f(Y_s^x) - g'f(Y_s^y))^t \\
&\quad + \int_0^t (g(Y_s^x) - g(Y_s^y))(g'f)'f(Y_s^x) - (g'f)'f(Y_s^y))d[Z,Z]^c_{\gamma_0^{-1}(s)} \\
&+ 2\sum_{0<u\le(\gamma_0^{-1}(\eta_2(t)))} (\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{(g(Y_s^x) - g(Y_s^y))(g'f(Y_s^x) - g'f(Y_s^y))\Delta Z_u}{\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u))} ds \\
&- (g(Y^x_{\eta_1(\gamma_0(u))}) - g(Y^y_{\eta_1(\gamma_0(u))}))(g'f(Y^x_{\eta_1(\gamma_0(u))}) - g'f(Y^y_{\eta_1(\gamma_0(u))})\Delta Z_u)) \\
&- 2\int_t^{\eta_2(t)} (g(Y_s^x) - g(Y_s^y))(g'f(Y_s^x) - g'f(Y_s^y))\frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t) - \eta_1(t)} ds \\
&- 2\int_0^t (g(Y_s^x) - g(Y_s^y))g'(Y_s^x)d\Psi_s^x + 2\int_0^t (g(Y_s^x) - g(Y_s^y))g'(Y_s^y)d\Psi_s^y
\end{aligned} \tag{115}
$$

Then,

$$
\|W_t^x - W_t^y\|^2 = \|w_0^x - w_0^y\|^2 exp(S_t), \tag{116}
$$

$$
\begin{aligned}
S_t &= 2\int_0^t \frac{(g(Y_s^x) - g(Y_s^y))(g'f(Y_s^x) - g'f(Y_s^y))}{\|g(Y_s^x) - g(Y_s^y)\|^2} dZ_{\gamma_0^{-1}(s)} \\
&+\int_0^t \frac{(g''f^2(Y_s^x) - g''f^2(Y_s^y)) + (g(Y_s^x) - g(Y_s^y))(g'f'f(Y_s^x) - g'f'f(Y_s^x))}{\|g(Y_s^x) - g(Y_s^y)\|^2} d[Z,Z]^c_{\gamma_0^{-1}(s)} \\
&+ 2\sum_{0<u\le(\gamma_0^{-1}(\eta_2(t)))} (\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))} \frac{(g(Y_s^x) - g(Y_s^y))(g'f(Y_s^x) - g'f(Y_s^y))\Delta Z_u}{(\eta_2(\gamma_0(u)) - \eta_1(\gamma_0(u)))(g(Y_s^x) - g(Y_s^y))^2} ds \\
&- \frac{(g(Y^x_{\eta_1(\gamma_0(u))}) - g(Y^y_{\eta_1(\gamma_0(u))}))(g'f(Y^x_{\eta_1(\gamma_0(u))} - g'f(Y^y_{\eta_1(\gamma_0(u))}))\Delta Z_u)}{\|g(Y_s^x) - g(Y_s^y)\|^2}) \\
&- 2\int_t^{\eta_2(t)} \frac{(g(Y_s^x) - g(Y_s^y))(g'f(Y_s^x) - g'f(Y_s^y))\Delta Z_{\gamma_0^{-1}(t)}}{(\eta_2(t) - \eta_1(t))(g(Y_s^x) - g(Y_s^y))^2} ds \\
&- 2\int_0^t (g(Y_s^x) - g(Y_s^y))\frac{g'(Y_s^x)}{\|g(Y_s^x) - g(Y_s^y)\|^2} d\Psi_s^x \\
&+ 2\int_0^t (g(Y_s^x) - g(Y_s^y))\frac{g'(Y_s^y)}{\|g(Y_s^x) - g(Y_s^y)\|^2} d\Psi_s^y.
\end{aligned} \tag{117}
$$

S_t is continuous and is bounded if

$$
\int_0^t (g(Y_s^x) - g(Y_s^y))\frac{g'(Y_s^x)}{\|g(Y_s^x) - g(Y_s^y)\|^2} d\Psi_s^x,
$$

is bounded. The rest of the proof is done in the half-space R^d_+, for the processes W_t^x and W_s^y.

$$
\int_0^t (g(Y_s^x) - g(Y_s^y))\frac{g'(Y_s^x)}{\|g(Y_s^x) - g(Y_s^y)\|^2} d\Psi_s^x \le \int_0^t \frac{d|\Psi^x|_s}{(\sum_{i=1}^{d-1}(W_s^{x,i} - W_s^{y,i})^2)^{\frac{1}{2}}}, \tag{118}
$$

But it is known that (see Sundar (1989))

$$E[\sup_{0\leq s\leq 1}\frac{1}{(\sum_{i=1}^{d-1}(W_s^{x,i}-W_s^{y,i})^2)^{\frac{1}{2}}}]\leq C\frac{1}{|x-y|^2}, \tag{119}$$

by Theorem 6 we know that

$$\begin{aligned}|\Psi^x|_T &\leq \sup_{0\leq t\leq T}|x+\int_0^t f(Y_s^x)dZ_{\gamma_0^{-1}(s)}+\frac{1}{2}\int_0^t f'f(Y_s^x)d[Z,Z]^c_{\gamma_0^{-1}(s)} \\ &+\sum_{0<u\leq(\gamma_0^{-1}(\eta_2(t)))}(\int_{\eta_1(\gamma_0(u))}^{\eta_2(\gamma_0(u))}\frac{f(Y_s^x)\Delta Z_u}{\eta_2(\gamma_0(u))-\eta_1(\gamma_0(u))}ds-f(Y^x_{\eta_1(\gamma_0(u))})\Delta Z_u)) \\ &-\int_t^{\eta_2(t)}f(Y_s^x)\frac{\Delta Z_{\gamma_0^{-1}(t)}}{\eta_2(t)-\eta_1(t)}ds|. \end{aligned}\tag{120}$$

Using (120) and (119) it is not difficult to obtain:

$$E[\int_0^t(g(Y_s^x)-g(Y_s^y))\frac{g'(Y_s^x)}{\|g(Y_s^x)-g(Y_s^y)\|^2}d\Psi_s^x]\leq\frac{Ct}{|x-y|^2}, \tag{121}$$

this proves that S_t is bounded and therefore $(W_t^x-W_t^y)$ is never zero a.s. □

Lemma 55 *The flow defined by equation (112) is strongly injective while the paths are in the domain of a system of coordinates.*

Proof.
The proof follows the same procedure as in the proof of Theorem 44 in Protter's book toguether with the idea used for the proof of the previous Lemma. □

To obtain the reversibility of X_t we apply Theorem 3.3 of Jacod and Protter (1988). Here, we have:

$$\begin{aligned}[f(X),Z]_t &= \int_0^t f'f(X_{s-})d[Z,Z]_s+ \\ &+\sum_{0<s\leq t}(\{\varphi(\Delta Z_s f,X_{s-})-X_{s-}-f(X_{s-})\Delta Z_s\}f'(X_{s-})\Delta Z_s) \\ &\sum_{0<s\leq t}(\{f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s\}\Delta Z_s). \end{aligned}\tag{122}$$

Now, define:

$$\tilde{Z}_t=\begin{cases}0 & \text{if } t=0; \\ Z_{(1-t)-}-Y_{1-} & \text{if } 0<t<1,; \\ Z_0-Z_{1-} & \text{if } t=1\end{cases}$$

analogously define $\tilde{\Phi}_t$, $[f(X),Z]_t^{\tilde{}}$ and also $\tilde{X}_t=X_{(1-t)-}$.

Theorem 56 *X is a reversible semimartingale and holds the following SDE:*

$$\begin{aligned}\tilde{X}_t &= X_{1-}+\int_0^t f(\tilde{X}_{s-})d\tilde{Z}_s-[f(X),Z]_t^{\tilde{}}+\frac{1}{2}\int_0^t f'f(\tilde{X}_{s-})d[\tilde{Z},\tilde{Z}]_s^c-\tilde{\Phi}(t) \\ &+\sum_{0<s\leq t}\{\varphi(\Delta\tilde{Z}_s f,\tilde{X}_{s-})-\tilde{X}_{s-}-f(\tilde{X}_{s-})\Delta\tilde{Z}_s\}, \end{aligned}\tag{123}$$

$$\tilde{\Phi}(t) = \int_0^t n(\tilde{X}_s) d \mid \tilde{\Phi} \mid (s), \;\; \tilde{X}_t \in \bar{D},$$

and $\varphi(g, x) = y(1)$, *where* $y(t)$ *is the solution of*

$$y(t) = x + \int_0^t g(y(s)) ds - \kappa(t), \;\; y(t) \in \bar{D} \;,$$

$$\kappa(t) = \int_0^t n(y(s)) d \mid \kappa \mid (s).$$

Proof.
The proof is a direct application of Theorem 3.3 in Jacod and Protter (1988) and the reversibility property of ODE's. □

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