ABSORPTION PROBABILITIES OF RANDOM PATHS FROM DICHOTOMOUS POPULATIONS

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ABSTRACT

The problem of computing the probability of random paths, as outcomes of sampling from dichotomous population either with replacement or without replacement, being absorbed by any given set of points is considered. A recursive formula for computation of a function defined on the set is derived. The absorption probability of random path at a point in the set is shown to be the product of easily computable function $\psi(\cdot)$ and the probability at this point. This result holds when the random path is binomially distributed or hypergeometrically distributed. Above investigations are done for the cases when absorption of paths is defined as first hitting (or second hitting and etc.) to some points in the set.

Key Words and Phrases: absorption probability; random paths; sequential test; hypergeometric distribution; binomial distribution; first hitting; dichotomous population.

AMS 1991 subject classification: Primary 62E30; Secondary 62L15, 60G17, 62L10.

1 Introduction

In many practical problems, a random variable is under investigation. Suppose the distribution of this interested random variable belongs to certain class $\{F_\theta,\ \theta \in \Theta\}$, but the true θ is unknown to us. We are interested in testing hypothesis $H_0:\ \theta \in \Theta_0$ *v.s.* $H_1:\ \theta \in \Theta_1$. To make a statistical decision, a number of observations from this random number are sampled to provide information about the underlying true θ. More observations are sampled, more information is obtained. But in real life, more observations means higher costs and longer time needed. A challenge to statisticians is to find ways to get more information

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from less observations. Sequentially gathering observations and making decision whenever information provided by the gathered observations is enough, this practice provides efficient means to achieve the goal mentioned above. Now suppose the observations are $X_1, X_2, \cdots$ and $S_n = S_n(X_1, \cdots, X_n)$ is a sufficient statistics for θ. We can always imagine that the sampling is ever going thus the infinite sequence $S_1, S_2, \cdots$, can be observed. In sequential procedure, we have an opportunity to look at $S_1, S_2, \cdots$ in sequence one by one, stop and make decision whenever the early stages of this sequence shows strong evidence either in favor of Θ_0 or in favor of Θ_1. The observations in later stages of this sequence are ignored at all. In fixed sampling size procedure (assuming sampling size is n_0), we making our decision only depends on S_{n_0}, so the observations in stages, up to $n_0 - 1$ and after n_0, of the sequence are ignored at all. Looking at these two kinds of decision procedures from this unique point of view, pinpoints their similarities and differences.

Next we give some definitions to summarize above idea. Let $\tilde{S} = (S_1, S_2, \cdots)$. Through out this paper, we will call $\tilde{S}$ the random path. Let $\mathcal{X}$ be the set of all sample paths of $\tilde{S}$. Let $P_\theta(\cdot)$ be a probability measure on $\mathcal{X}$, derived from F_θ. Then $\{P_\theta(\cdot), \theta \in \Theta\}$, derived from $\{F_\theta, \theta \in \Theta\}$, is a class of probability measures on $\mathcal{X}$. In order to test $H_0 : \ \theta \in \Theta_0$ *v.s.* $H_1 : \ \theta \in \Theta_1$, we divide $\mathcal{X}$ into a partition $\{H_\alpha, \alpha \in \mathcal{A}\}$ ($\mathcal{A}$ is some proper index set) according to $\{P_\theta(\cdot), \theta \in \Theta\}$, the probability measures on $\mathcal{X}$. We say $\{H_\alpha, \alpha \in \mathcal{A}\}$ is a partition of $\mathcal{X}$ if $\bigcup_{\alpha \in \mathcal{A}} H_\alpha = \mathcal{X}$ and $H_\alpha \cup H_{\alpha'} = \emptyset$ for any $\alpha \neq \alpha'$. $\{H_\alpha, \alpha \in \mathcal{A}\}$ should be such that for each H_α, if $P_\theta(\tilde{S} \in H_\alpha)$ is large (small) for $\theta \in \Theta_0$, then it is small (large) for $\theta \in \Theta_1$. When doing the test, we observe which H_α that $\tilde{S}$ falls in, accordingly we make decision in favor of Θ_0 if $P_\theta(\tilde{S} \in H_\alpha)$ is large for $\theta \in \Theta_0$, in favor of Θ_1 otherwise. Obviously there are many different partitions meet above requirements. If the partition is made only according to the possible values of S_{n_0}, then we have a procedure of fixed sampling size n_0. If the partition is made according to possible values of S_n for some different n's, then we have a sequential procedure. We further illustrate this idea in later discussion of random path from dichotomous population.

Even though sequential procedures are more efficient than the fixed sampling size procedures, there are two difficulties which disencourage practitioners to prefer the former to the latter. One difficulty is that in most cases, it is difficult (or too complicated), sometimes impossible, to compute OC (operation characteristic) and expected sampling size for a given sequential procedure. The other difficulty is that in most cases, there doesn't exist a sequential procedure which is superior than other procedures (sequential or not) uniformly on Θ.

In this paper, we only discuss sampling from dichotomous populations. We will develop a method of computing the probabilities of random paths, as outcomes of sampling from dichotomous population either *with* or *without* replacement, being absorbed by a specified set of points. This method, tractable and easily computable, not only overcomes the first difficulty mentioned above but also helps to select proper procedures by providing easily computed power function and expected sampling sizes for any tentative sequential procedure.

A dichotomous population is defined as a population $\mathcal{P}$ consists of two classes of items such that Np items of which are of one class (which designated as 1) and $N(1-p)$ of which are of another class (which designated as 0). Here N is the population size thus a positive integer, p is the proportion of 1's in the population thus a positive fraction number between 0 and 1.

Sampling with replacement from a dichotomous population, we denote X_i as the outcome of ith observation, *i.e.* X_i takes the value 1 or 0. Then $X_1, \cdots, X_n, \cdots$ are independent Bernoulli random variables with same success rate p.

Let $S_n = \sum_{i=1}^{n} X_i$. Sampling with replacement, $\tilde{S} = (S_1, S_2, \cdots, S_n, \cdots)$ is a random path with increment $X_n = 0$ or 1, which is independent of passage position $S_{n-1} = s_{n-1}$ for $n = 1, 2, \cdots$.

Let

$$\mathcal{X} = \left\{ \tilde{s} = (s_1, \cdots, s_n, \cdots) : s_n = \sum_{i=1}^{n} x_i, \ x_i = 0 \text{ or } 1, \quad n = 1, 2, \cdots \right\}.$$

If sampling is done infinite times with replacement, then $\tilde{S}$ is the *random path*, and $\mathcal{X}$ is the set of all possible *sample paths*. In other words, $\mathcal{X}$ is the sample space of $\tilde{S}$.

Let $\{H(1), H(2), \cdots, H(k)\}$ be a partition of $\mathcal{X}$. Since elements in each $H(i)$ are sample paths, for the reason of intuitive thus easier conception, we call $H(i)$ a bunch of paths. So partitioning of $\mathcal{X}$ is to divide all paths in $\mathcal{X}$ into a number of bunches in a way such that each of the paths in $\mathcal{X}$ belong to some bunch, and no single path belong to two bunches.

One of the simple partitions of $\mathcal{X}$ is $\{H_{n_0}(0), \cdots, H_{n_0}(n_0)\}$ where $H_{n_0}(i) = \{\tilde{s} \in \mathcal{X} : s_{n_0} = i\}$ and n_0 is any fixed positive integer. It is easy to check that $\{H_{n_0}(i), i = 0, \cdots, n_0\}$ is a partition of $\mathcal{X}$ and $P(\tilde{S} \in H_{n_0}(i)) = P(S_{n_0} = i)$ for $i = 0, \cdots, n_0$.

When sampling is carried out with replacement, population $\mathcal{P}$ is exhausted at the Nth step of sampling, and $X_1, X_2, \cdots, X_N$ are not *i.i.d* . $\tilde{S}_N = (S_1, S_2, \cdots, S_N)$ is a random path with increment $X_n = 0$ or 1, which is dependent of passage position $S_{n-1} = s_{n-1}$ for

$n = 1, 2, \cdots, N$. Let $\mathcal{X}_N$ be the set of all possible sample paths of $\tilde{S}_N$, then

$$\mathcal{X}_N = \left\{ \tilde{s}_N = (s_1, \cdots, s_n, \cdots, s_N) : s_n = \sum_{i=1}^{n} x_i, \ x_i = 0 \ \text{ or } 1, \quad n = 1, \cdots, N \right\}.$$

$\mathcal{X}_N$ is the sample space of $\tilde{S}_N$.

Our goal is to derive a method of computing absorption probabilities of random paths which arise by sampling *with* or *without* replacement from dichotomous population, with respect to any specified set of points. Computation of absorption probability is of broad interests. Especially in sequential tests, computation of absorption probability is critical for computation of O.C. and expected sampling sizes for the tests. Even though this method is motivated in sequential tests, here we treat it as general as possible in hope it can find applications widely. First the discussion is given when the absorption of a random path is defined as the usual sense, *i.e.* the random path first hitting a point in the specified set of points. Then we will generalize this method for the cases when absorption of random path is defined as second hitting (third hitting, etc.) points in a specified set of points.

Random path $\tilde{S}$ can be graphed on two-dimension coordinates (n, s_n). The first coordinate n indicates the sampling time for S_n, the second coordinate s_n indicates the possible value for S_n. Let $\mathcal{B}$ be a set of some interested points on this coordinates that $\tilde{S}$ might hit. We call $\mathcal{B}$ the set of barrier points. Hence $\mathcal{B} = \{\tilde{b} = (b_1, b_2) : b_1$ is positive integer, b_2 is integer $\}$. For example, $\mathcal{B} = \{(5,1), (5,3), (8,2), (8,4), (8,6), (11,5), (11,8)\}$.

Let $B(n)$ be the subset of $\mathcal{B}$, points in which random path $\tilde{S}$ might hit at time n (during the nth sampling), that is $B(n) = \{\tilde{b}_n(i) = (n, b_n(i)) : \ i = 1, \cdots, I_n\}$, where $b_n(i)$ for $i = 1, \cdots, I_n$ are integers; I_n is the number of points in $B(n)$, $I_n = 0$ if $B(n)$ is empty. We call $B(n)$ the set of barrier points at time n. Hence $\mathcal{B} = \bigcup_n B(n)$, and $B(1), B(2), \cdots$ are disjoint.

In last example, $B(5) = \{(5,1),(5,3)\}$, $B(8) = \{(8,2),(8,4),(8,6)\}$, $B(11) = \{(11,5),(11,8)\}$, and $B(n) = \emptyset$ for $n \neq 5, 8, 11$. $\mathcal{B}$ is the set of all barrier points which random path $\tilde{S}$ might hit during the whole process of sampling.

Let $\mathcal{X}^{\mathcal{B}}$ be the subset of $\mathcal{X}$, which includes all sample paths which pass through some points in $\mathcal{B}$. $\mathcal{X}^{\mathcal{B}}$ can be partitioned into bunches of paths $\{H(\tilde{b}_n(i)), \ \tilde{b}_n(i) \in \mathcal{B}\}$ where

$$H(\tilde{b}_n(i)) = \{\tilde{s} \in \mathcal{X} : s_n = b_n(i); \quad s_l \notin B_l, \quad l = 1, 2, \cdots, n-1\}$$
$$i = 1, \cdots, I_n; \quad n = 1, 2, \cdots \qquad (1)$$

where $B_l = \{b_l(i),\ i = 1, \cdots, I_l\}$. It is not difficult to check that $\left\{H(\tilde{b}_n(i)),\quad i = 1, \cdots, I_n;\ n = 1, 2, \cdots.\right\}$ is a partition of $\mathcal{X}^{\mathcal{B}}$ and

$$P(\tilde{S} \in H(\tilde{b}_n(i))) = P(S_n = b_n(i),\ S_l \notin B_l,\ \ l = 1, \cdots, n-1). \tag{2}$$

The right side of above equation indicates $P(\tilde{S} \in H(\tilde{b}_n(i)))$ is the probability that random path $\tilde{S}$ hits $\tilde{b}_n(i)$ before hitting other barrier points in $\mathcal{B}$, or the absorption probability of $\tilde{S}$ by $\tilde{b}_n(i)$. For this moment, absorption of $\tilde{S}$ by $\tilde{b}_n(i)$ is defined as $\tilde{S}$ hits $\tilde{b}_n(i)$ for the first hitting in $\mathcal{B}$. Later on, we will discuss the cases when the absorption of $\tilde{S}$ by $\tilde{b}_n(i)$ is defined as $\tilde{b}_n(i)$ is the second point in $\mathcal{B}$ being hitted by $\tilde{S}$, *etc.*.

With same idea, ${\mathcal{X}_N}^{\mathcal{B}}$ can be partitioned into bunches of paths $\{H_N(\tilde{b}_n(i)),\ \tilde{b}_n(i) \in \mathcal{B}\}$ where

$$H_N(\tilde{b}_n(i)) = \{\tilde{s}_N = (s_1, \cdots, s_N) :\ s_n = b_n(i);\ \ s_l \notin B_l,\ \ l = 1, 2, \cdots, n-1\}$$
$$i = 1, \cdots, I_n; \quad n = 1, \cdots, N. \tag{3}$$

It is easy to see that $\left\{H_N(\tilde{b}_n(i)) \quad i = 1, \cdots, I_n; n = 1, \cdots, N.\right\}$ is a partition of ${\mathcal{X}_N}^{\mathcal{B}}$ and

$$P(\tilde{S}_N \in H_N(\tilde{b}_n(i))) = P(S_n = b_n(i),\ S_l \notin B_l,\ \ l = 1, \cdots, n-1). \tag{4}$$

2 Main Results

Given a barrier set $\mathcal{B}$, sampling with replacement, the absorption probability at $\tilde{b}_n(i) \in \mathcal{B}$, for any $p \in [0, 1]$, is

$$\begin{aligned} P_p(\tilde{S} \in H(\tilde{b}_n(i))) &= \psi(\tilde{b}_n(i)) P_p(S_n = b_n(i)) \\ &= \psi(\tilde{b}_n(i)) \binom{n}{b_n(i)} p^{b_n(i)} (1-p)^{n-b_n(i)}, \end{aligned} \tag{5}$$

where $\psi(\cdot)$ is a function defined on $\mathcal{B}$, and can be computed recursively by following formula

$$\psi(\tilde{b}_n(k)) = 1 - \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i) \in B(l)} \psi(\tilde{b}_l(i)) \frac{\binom{b_n(k)}{b_l(i)} \binom{n-b_n(k)}{l-b_l(i)}}{\binom{n}{l}}, \tag{6}$$

where a convention is assumed : $\binom{m}{t} = 0$ if $m < t$ or $t < 0$.

For the same $\mathcal{B}$, sampling without replacement, the absorption probability of $\tilde{S}_N$ by $\tilde{b}_n(i) \in \mathcal{B}$, for any N and $p = \frac{1}{N}, \cdots, \frac{N}{N}$, is

$$\begin{aligned} P_{p,N}(\tilde{S}_N \in H_N(\tilde{b}_n(i))) &= \psi(\tilde{b}_n(i)) P_{p,N}(S_n = b_n(i)) \\ &= \psi(\tilde{b}_n(i)) \frac{\binom{pN}{b_n(i)}\binom{(1-p)N}{n-b_n(i)}}{\binom{N}{n}}. \end{aligned} \tag{7}$$

3 Illustrative Example

Assuming $\mathcal{P}$ is a dichotomous population with size N and p (the proportion of 1's in $\mathcal{P}$). Let the set of barrier points $\mathcal{B} = \{(5,1),(5,3),(8,2),(8,4),(8,6),(11,5),(11,8)\}$. Let $B(5) = \{(5,1),(5,3)\}$, $B(8) = \{(8,2),(8,4),(8,6)\}$, $B(11) = \{(11,5),(11,8)\}$. And let $B(n) = \emptyset$ for $n \neq 5,8,11$. We have $\mathcal{B} = \bigcup_n B(n)$.

Barrier function $\psi(\cdot,\cdot)$ can be computed by following formula (6).

$\psi(5,1) = 1$, $\psi(5,3) = 1$ because $B(l) = \emptyset$ for $l = 1,\cdots,4$.

$$\psi(8,2) = 1 - \left\{1 \cdot \frac{\binom{2}{1}\binom{6}{4}}{\binom{8}{5}} + 1 \cdot \frac{\binom{2}{3}\binom{6}{2}}{\binom{8}{5}}\right\} = \frac{13}{28} = 0.46429,$$

$$\psi(8,4) = 1 - \left\{1 \cdot \frac{\binom{4}{1}\binom{4}{4}}{\binom{8}{5}} + 1 \cdot \frac{\binom{4}{3}\binom{4}{2}}{\binom{8}{5}}\right\} = \frac{1}{2} = 0.5,$$

$$\psi(8,6) = 1 - \left\{1 \cdot \frac{\binom{6}{1}\binom{2}{4}}{\binom{8}{5}} + 1 \cdot \frac{\binom{6}{3}\binom{2}{2}}{\binom{8}{5}}\right\} = \frac{9}{14} = 0.64286,$$

$$\psi(11,5) = 1 - \left\{1 \cdot \frac{\binom{5}{1}\binom{6}{4}}{\binom{11}{5}} + 1 \cdot \frac{\binom{5}{3}\binom{6}{2}}{\binom{11}{5}} + \frac{13}{28} \cdot \frac{\binom{5}{2}\binom{6}{6}}{\binom{11}{8}} + \frac{1}{2} \cdot \frac{\binom{5}{4}\binom{6}{4}}{\binom{11}{8}} + \frac{9}{14} \cdot \frac{\binom{5}{6}\binom{6}{2}}{\binom{11}{8}}\right\} = 0.25758,$$

$$\psi(11,8) = 1 - \left\{1 \cdot \frac{\binom{8}{1}\binom{3}{4}}{\binom{11}{5}} + 1 \cdot \frac{\binom{8}{3}\binom{3}{2}}{\binom{11}{5}} + \frac{13}{28} \cdot \frac{\binom{8}{2}\binom{3}{6}}{\binom{11}{8}} + \frac{1}{2} \cdot \frac{\binom{8}{4}\binom{3}{4}}{\binom{11}{8}} + \frac{9}{14} \cdot \frac{\binom{8}{6}\binom{3}{2}}{\binom{11}{8}}\right\} = 0.30909.$$

Sampling with replacement, the underlying distribution is binomial with parameter p.

Assuming $p = 0.65$, then by (5)

$$P_p(\tilde{S} \in H(\tilde{b}_{11}(5))) = \psi(11,5) \cdot P_p(S_{11} = 5) = 0.25758 \cdot \binom{11}{5} 0.65^5 0.35^6 = 0.02538.$$

Likewise the absorption probabilities with respect to points in $\mathcal{B}$ are computed as following

Absorption Probability:	0.04877	0.3364	0.0101	0.09375	0.1663	0.02538	0.06968
Barrier Points in $\mathcal{B}$:	(5,1)	(5,3)	(8,2)	(8,4)	(8,6)	(11,5)	(11,8)

The probability that $\tilde{S}$ hits at least one point in $\mathcal{B}$ is

$$P_{0.65}(\tilde{S} \in \mathcal{X}^{\mathcal{B}}) = \sum_{\tilde{b}_n(i) \in \mathcal{B}} P_{0.65}\left(\tilde{S} \in H(\tilde{b}_n(i))\right) = 0.750.$$

Sampling without replacement, the underlying distribution is hypergeometric with parameters N and p. Assuming $N = 20$ and $p = 0.65$, so $pN = 13$ then by (7)

$$P_{p,N}(\tilde{S}_N \in H_N(\tilde{b}_{11}(5))) = \psi(11,5) P_{p,N}(S_{11} = 5) = 0.25758 \cdot \frac{\binom{13}{5}\binom{7}{6}}{\binom{20}{11}} = 0.01382.$$

Likewise the absorption probabilities with respect to points in $\mathcal{B}$ are computed as following

Absorption Probability:	0.02935	0.38738	0.00201	0.09933	0.1839	0.01382	0.08289
Barrier Points in $\mathcal{B}$:	(5,1)	(5,3)	(8,2)	(8,4)	(8,6)	(11,5)	(11,8)

The probability that $\tilde{S}_N$ hits at least one point in $\mathcal{B}$ is

$$P_{0.65,20}(\tilde{S}_N \in \mathcal{X}_N{}^B) = \sum_{\tilde{b}_n(i) \in \mathcal{B}} P_{0.65,20}\left(\tilde{S}_N \in H_N(\tilde{b}_n(i))\right) = 0.799.$$

4 Enclosed Boundary

We say barrier set $\mathcal{B}$ is enclosed for random path $\tilde{S}$ if $P_p(\tilde{S} \in \mathcal{X}^{\mathcal{B}}) = 1$ for any $p \in [0,1]$. Actually if $\mathcal{B}$ is enclosed for $\tilde{S}$, then $\mathcal{X}^{\mathcal{B}} = \mathcal{X}$. In sequential tests, sampling stops whenever random path goes acrossing an enclosed boundary. In that case, the set of barrier points $\mathcal{B}$

consists of an enclosed boundary, so we call $\mathcal{B}$ the enclosed boundary. Let

$$\mathcal{B} = \left\{(i,j) : (i,j) \in \{(n,a_n)\}_{n=1}^{m-1} \cup \{(n,b_n)\}_{n=1}^{m-1} \cup \{(m,k)\}_{k=a_m}^{b_m}\right\}$$

where a_n, b_n are integers such that $a_n \le b_n$ for $n = 1, \cdots, m$.

We call $\{(n,a_n)\}_{n=1}^{m-1}$ as the lower boundary, $\{(n,b_n)\}_{n=1}^{m-1}$ as the upper boundary, and $\{(m,k)\}_{k=a_n}^{b_n}$ as the truncation boundary.

The barrier function $\psi(\cdot,\cdot)$ becomes boundary functions $\psi_a(\cdot)$, $\psi_b(\cdot)$, $\psi_m(\cdot)$, where $\psi_a(n) = \psi(n, a_n)$, $\psi_b(n) = \psi(n, b_n)$ for $n = 1, \cdots, m-1$; $\psi_m(k) = \psi(m,k)$ for $k = a_m, \cdots, b_m$.

By formula (6), boundary functions can be computed recursively by

$$\psi_a(n) = 1 - \sum_{l=1}^{n-1} \left\{ \psi_a(l) \frac{\binom{a_n}{a_l}\binom{n-a_n}{l-a_l}}{\binom{n}{l}} + \psi_b(l) \frac{\binom{a_n}{b_l}\binom{n-a_n}{l-b_l}}{\binom{n}{l}} \right\}, \tag{8}$$

$$\psi_b(n) = 1 - \sum_{l=1}^{n-1} \left\{ \psi_a(l) \frac{\binom{b_n}{a_l}\binom{n-b_n}{l-a_l}}{\binom{n}{l}} + \psi_b(l) \frac{\binom{b_n}{b_l}\binom{n-b_n}{l-b_l}}{\binom{n}{l}} \right\}, \tag{9}$$

for $n = 1, \cdots, m-1$;

$$\psi_m(k) = 1 - \sum_{l=1}^{m-1} \left\{ \psi_a(l) \frac{\binom{k}{a_l}\binom{m-k}{l-a_l}}{\binom{m}{l}} + \psi_b(l) \frac{\binom{k}{b_l}\binom{m-k}{l-b_l}}{\binom{m}{l}} \right\}, \tag{10}$$

for $k = a_m, \cdots, b_m$.

To test hypothesis $H_0 : p < p^*$ *v.s.* $H_1 : p \ge p^*$, the decision rule should be defined as rejecting H_0 if random path first hits upper boundary or upper portion of the truncation boundary on which the cut off point is (m, k_c). So the stopping time is $T \wedge m$ where $T = inf\{n : S_n \ge b_n \text{ or } \le a_n\}$.

Sampling with replacement, the power function is, for any $p \in [0,1]$

$$\begin{aligned} \beta(p) &= \sum_{n=1}^{m-1} \psi_b(n) \cdot P_p(S_n = b_n) + \sum_{k=k_c}^{b_m} \psi_m(k) \cdot P_p(S_m = k) \\ &= \sum_{n=1}^{m-1} \psi_b(n) \cdot \binom{n}{b_n} p^{b_n}(1-p)^{n-b_n} + \sum_{k=k_c}^{b_m} \psi_m(k) \cdot \binom{m}{k} p^k (1-p)^{m-k} \end{aligned} \tag{11}$$

The expected sampling size $S(p)$ as a function of p on $[0,1]$ is

$$S(p) = E_p T \wedge m$$

$$= \sum_{n=1}^{m-1} n \left\{ \psi_b(n) \cdot \binom{n}{b_n} p^{b_n}(1-p)^{n-b_n} + \psi_a(n) \cdot \binom{n}{a_n} p^{a_n}(1-p)^{n-a_n} \right\}$$
$$+m \sum_{k=k_c}^{b_m} \psi_m(k) \cdot \binom{m}{k} p^k (1-p)^{m-k} \quad (12)$$

Sampling without replacement, the power function is, for any $p = \frac{0}{N}, \frac{1}{N}, \cdots, \frac{N}{N}$

$$\begin{aligned}
\beta(p) &= \sum_{n=1}^{m-1} \psi_b(n) \cdot P_{p,N}(S_n = b_n) + \sum_{k=k_c}^{b_m} \psi_m(k) \cdot P_{p,N}(S_m = k) \\
&= \sum_{n=1}^{m-1} \psi_b(n) \cdot \frac{\binom{pN}{b_n}\binom{(1-p)N}{n-b_n}}{\binom{N}{n}} + \sum_{k=k_c}^{b_m} \psi_m(k) \cdot \frac{\binom{pN}{k}\binom{(1-p)N}{m-k}}{\binom{N}{m}} \quad (13)
\end{aligned}$$

The expected sampling size $S(p)$ as a function of p is

$$\begin{aligned}
S(p) &= E_{p,N} T \wedge m \\
&= \sum_{n=1}^{m-1} n \left\{ \psi_b(n) \cdot \frac{\binom{pN}{b_n}\binom{(1-p)N}{n-b_n}}{\binom{N}{n}} + \psi_a(n) \cdot \frac{\binom{pN}{a_n}\binom{(1-p)N}{n-a_n}}{\binom{N}{n}} \right\} \\
&\quad +m \sum_{k=k_c}^{b_m} \psi_m(k) \cdot \frac{\binom{pN}{k}\binom{(1-p)N}{m-k}}{\binom{N}{m}}. \quad (14)
\end{aligned}$$

Similarly one can construct sequential test rule for testing hypothesis $H_0 : p = p^*$ *v.s.* $H_1 : p \neq p^*$. Similar formula for $\beta(p)$ and $S(p)$ can be set up for the cases of sampling *with* and *without* replacement from a dichotomous population.

5 Proof of Main Results

First let us consider the case of sampling with replacement, in which, S_n has binomial distribution $B(n, p)$ for $n = 1, 2, \cdots$. Now we define, for all possible $\tilde{b}_n(k)) \in \mathcal{B}$,

$$\psi(\tilde{b}_n(k)) = \frac{P_p(\tilde{S} \in H(\tilde{b}_n(k)))}{P_p(S_n = b_n(k))}. \quad (15)$$

By this definition, obviously equation (5) holds. Then we only need to show $\psi(\cdot, \cdot)$ can be determined by recursive formula (6). Hence $\psi(\cdot, \cdot)$, as a function defined on $\mathcal{B}$, doesn't depend on p.

Since $P_p(\tilde{S} \in H(\tilde{b}_n(k))) = P_p\left(\tilde{S} \in H(\tilde{b}_n(k)), S_n = b_n(k)\right)$, so

$$\begin{aligned}
\psi(\tilde{b}_n(k)) &= P_p\left(\tilde{S} \in H(\tilde{b}_n(k)) | S_n = b_n(k)\right) \\
&= P_p\left(S_n = b_n(k), S_l \notin B_l, \ \ l = 1, \cdots, n-1 | S_n = b_n(k)\right) \\
&= P_p\left(S_l \notin B_l, \ \ l = 1, \cdots, n-1 | S_n = b_n(k)\right) \\
&= P_p\left(\tilde{X}_n \in D_n | S_n = b_n(k)\right), \qquad (16)
\end{aligned}$$

where $D_n \in I_n = \sigma\left(\{(x_1, \cdots, x_n) : x_i = 0 \text{ or } 1 \ \ i = 1, \cdots, n\}\right)$ such that

$$\left(\tilde{X}_n \in D_n\right) = \left(S_l \notin B_l, \ \ l = 1, \cdots, n-1\right). \qquad (17)$$

$\psi(\tilde{b}_n(k))$ doesn't depend on p because S_n is a sufficient statistics of p. So we write $\psi(\tilde{b}_n(k)) = P\left(\tilde{X}_n \in D_n | S_n = b_n(k)\right)$, the intuitive meaning of which can be explained as following. The sample paths in $\mathcal{X}$ passing through $\tilde{b}_n(k)$ can take $\binom{n}{b_n(k)}$ different passages to reach $\tilde{b}_n(k)$. Each passage is taken equally likely by *random path* $\tilde{S}$. Of these passages, some met other barrier points in $\mathcal{B}$ before reaching $\tilde{b}_n(k)$, while the rest didn't. Actually $\psi(\tilde{b}_n(k))$ is the ratio of number of passages which didn't meet other barrier points before reaching $\tilde{b}_n(k)$ and the number of all passages reaching $\tilde{b}_n(k)$. So obviously, $\psi(\tilde{b}_n(k))$ doesn't depend on p.

Next we consider the case of sampling without replacement, in which S_n has hypergeometric distribution $H(pN, N, n)$ for $n = 1, \cdots, N$. Now we want to show, for any p, N,

$$P_{p,N}\left(\tilde{S}_N \in H_N(\tilde{b}_n(k)\right) = \psi(\tilde{b}_n(k)) \cdot P_{p,N}(S_n = b_n(k)) \qquad (18)$$

holds for any $\tilde{b}_n(k)$ in $\mathcal{B}$ such that $n \leq N$.

Claim 5.1 *If* $P_{p,N}(S_n = b_n(k)) > 0$ *then*

$$P_{p,N}\left(\tilde{S}_N \in H_N(\tilde{b}_n(k)) | S_n = b_n(k)\right) = \psi(\tilde{b}_n(k)). \qquad (19)$$

Proof of Claim 5.1:

If $P_{p,N}(S_n = b_n(k)) > 0$, then by definition of $H_N(\tilde{b}_n(k))$ in (3),

$$\begin{aligned}
&P_{p,N}\left(\tilde{S}_N \in H_N(\tilde{b}_n(k)) | S_n = b_n(k)\right) \\
=\ &P_{p,N}\left(S_n = b_n(k), S_l \notin B_l, \ \ l = 1, \cdots, n-1 | S_n = b_n(k)\right) \\
=\ &P_{p,N}\left(S_l \notin B_l, \ \ l = 1, \cdots, n-1 | S_n = b_n(k)\right)
\end{aligned}$$

$P_{p,N}\left(S_l \notin B_l,\ \ l=1,\cdots,n-1|S_n=b_n(k)\right)$ is well defined. For D_n in (17), we have

$$P_{p,N}(S_l \notin B_l,\ \ l=1,\cdots,n-1|S_n=b_n(k)) = P_{p,N}(\tilde{X}_n \in D_n|S_n=b_n(k)).$$

If we denote $P_p(\cdot)$ as the probability measure of binomial distribution for sampling with replacement, and denote $P_{p,N}(\cdot)$ as the probability measure of hypergeometric distribution for sampling without replacement, then for any $\tilde{x}_n \in I_n$,

$$\begin{aligned} P_{p,N}(\tilde{X}_n=\tilde{x}_n|S_n=b_n(k)) &= P_p(\tilde{X}_n=\tilde{x}_n|S_n=b_n(k)) \\ &= \begin{cases} \frac{1}{\binom{n}{b_n(k)}} & s_n=b_n(k); \\ 0 & s_n \neq b_n(k). \end{cases} \end{aligned}$$

where $s_n=\sum_{i=1}^n x_i$. So consequently we have

$$P_{p,N}(\tilde{X}_n \in D_n|S_n=b_n(k)) = P_p(\tilde{X}_n \in D_n|S_n=b_n(k)) = \psi(\tilde{b}_n(k)).$$

Thus by (16) we have

$$P_{p,N}(\tilde{S}_N \in H_N(\tilde{b}_n(k))|S_n=b_n(k)) = \psi(\tilde{b}_n(k)).$$

∎

If $P_{p,N}(S_n=b_n(k))=0$, then $P_{p,N}\left(\tilde{S}_N \in H_N(\tilde{b}_n(k)\right)=0$ because

$$\begin{aligned} P_{p,N}\left(\tilde{S}_N \in H_N(\tilde{b}_n(k))\right) &= P_{p,N}\left(S_n=b_n(k), S_l \notin B_l,\ \ l=1,\cdots,n-1\right) \\ &\leq P_{p,N}\left(S_n=b_n(k)\right). \end{aligned}$$

By Claim 5.1, we can see that for any p, N and $\tilde{b}_n(k) \in \mathcal{B}$, whether $P_{p,N}(S_n=b_n(k)) > 0$ or $=0$, it always holds

$$P_{p,N}(\tilde{S}_N \in H_N(\tilde{b}_n(k)) = \psi(\tilde{b}_n(k))P_{p,N}(S_n=b_n(k)).$$

Now we are ready to show that the $\psi(\tilde{b}_n(k))$ defined by (15) also satisfies (6). Fix n and point $\tilde{b}_n(k)$ in $\mathcal{B}$, let $N=n$ and $p=\frac{b_n(k)}{n}$, then

$$P_{p,N}(S_l=b_l(i)) = \frac{\binom{pN}{b_l(i)}\binom{(1-p)N}{l-b_l(i)}}{\binom{N}{l}} = \frac{\binom{b_n(k)}{b_l(i)}\binom{n-b_n(k)}{l-b_l(i)}}{\binom{n}{l}}.$$

Under above assumption, random path $\tilde{S}_N$ hits barrier point $(n, b_n(k))$ at time $N = n$ with probability one. Thus

$$\sum_{l=1}^{N} P_{p,N}(\tilde{S}_N \in H_N(l)) = 1 \tag{20}$$

where $H_N(l) = \bigcup_{i=1}^{I_l} H_N(\tilde{b}_l(i))$. By (18) we have, for $l = 1, \cdots, N$

$$\begin{aligned} P_{p,N}(\tilde{S}_N \in H_N(l)) &= \sum_{\tilde{b}_l(i)\in B(l)} P_{p,N}(\tilde{S}_N \in H_N(\tilde{b}_l(i))) \\ &= \sum_{\tilde{b}_l(i)\in B(l)} \psi(\tilde{b}_l(i)) P_{p,N}(S_l = b_l(i)) \\ &= \sum_{\tilde{b}_l(i)\in B(l)} \psi(\tilde{b}_l(i)) \frac{\binom{b_n(k)}{b_l(i)}\binom{n-b_n(k)}{l-b_l(i)}}{\binom{n}{l}}. \end{aligned} \tag{21}$$

In above equation, let $l = N$ then

$$P_{p,N}(\tilde{S}_N \in H_N(N)) = \psi(\tilde{b}_n(k)), \tag{22}$$

which is because of

$$\binom{b_n(k)}{b_n(i)}\binom{n-b_n(k)}{n-b_n(i)} = \begin{cases} 1 & i = k; \\ 0 & i \neq k. \end{cases}$$

By (20), (21) and (22), we have, for any $\tilde{b}_n(k)$ in $\mathcal{B}$,

$$\psi(\tilde{b}_n(k)) = 1 - \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i)\in B(l)} \psi(\tilde{b}_l(i)) \frac{\binom{b_n(k)}{b_l(i)}\binom{n-b_n(k)}{l-b_l(i)}}{\binom{n}{l}} \qquad n = 1, 2, \cdots. \tag{23}$$

$\psi(\tilde{b}_n(k))$ can be computed recursively for all barrier points $\tilde{b}_n(k)$ in $\mathcal{B}$.

6 Generalization of Main Results

In previous sections, we have discussed the method of computing the probabilities of random path $\tilde{S}$, as outcome of sampling *with* or *without* replacement from a dichotomous population, being absorbed by points in $\mathcal{B}$ which is any specified set of points on (n, s_n) plane. There the event that $\tilde{S}$ is absorbed by a point $\tilde{b}_n(i)$ in $\mathcal{B}$ was defined as that $\tilde{S}$ hits $\tilde{b}_n(i)$ before hitting any other points in $\mathcal{B}$, or in brief words, as that $\tilde{b}_n(i)$ is the point of first hitting in $\mathcal{B}$.

In this section, we will discuss the cases, as the generalization of main results, in which absorption of $\tilde{S}$ by a point $\tilde{b}_n(i)$ in $\mathcal{B}$ is defined as that $\tilde{S}$ hits $\tilde{b}_n(i)$ after hitting other $k-1$ points in $\mathcal{B}$, or in other words, $\tilde{b}_n(i)$ is the point of kth hitting in $\mathcal{B}$. For easier demonstration, we will deal the second hitting in detail, then give the formula for general case.

6.1 Absorption defined as second hitting

If we denote $H^{(2)}(\tilde{b}_n(i))$ the event that $\tilde{S}$ hits $\tilde{b}_n(i)$ as the second hitting in $\mathcal{B}$, then

$$H^{(2)}(\tilde{b}_n(i)) = \bigcup_{\substack{j<n \\ B(j)\neq\emptyset}} \{\tilde{s} \in \mathcal{X} :\ s_n = b_n;\ s_j \in B_j;\ s_l \notin B_l,\ \ l < n,\ \ l \neq j\ \}. \tag{24}$$

Let $\mathcal{X}^{\mathcal{B}}(2)$ be the subset of $\mathcal{X}$, which includes all sample paths which pass through at least two points in $\mathcal{B}$. $\left\{H^{(2)}(\tilde{b}_n(i)),\quad i = 1, \cdots, I_n;\ n = 1, 2, \cdots.\right\}$ is a partition of $\mathcal{X}^{\mathcal{B}}(2)$.

We are interested in computation of

$$P(\tilde{S} \in H^{(2)}(\tilde{b}_n(i))) = P\left(\bigcup_{\substack{j<n \\ B(j)\neq\emptyset}} \left(S_n = \tilde{b}_n(i);\ S_j \in B_j;\ S_l \notin B_l,\ \ l < n,\ \ l \neq j\ \right)\right),$$

the probability that $\tilde{b}_n(i)$ is the exact second point in $\mathcal{B}$ being hitted by random path $\tilde{S}$. Also we are interested in computing $P(\tilde{S} \in \mathcal{X}^{\mathcal{B}}(2)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P(\tilde{S} \in H^{(2)}(\tilde{b}_n(i)))$, the probability that $\tilde{S}$ hits at least two points in $\mathcal{B}$.

Before giving computation formula for $P(\tilde{S} \in H^{(2)}(\tilde{b}_n(i)))$, we need definitions of relative barrier function $\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$ and barrier-II function $\psi^{(2)}(\tilde{b}_n(k))$ which are defined on $\mathcal{B}$.

Definition 6.1 *For any pair of points $(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$ such that $n' < n$ and $\tilde{b}_{n'}(k')$, $\tilde{b}_n(k) \in \mathcal{B}$, relative barrier function $\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$ is defined recursively on $\mathcal{B}$ by*

$$\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k)) = 1 - \sum_{l=n'+1}^{n-1} \sum_{\tilde{b}_l(i)\in B(l)} \varphi(\tilde{b}_{n'}(k'); \tilde{b}_l(i)) \frac{\binom{b_n(k)-b_{n'}(k')}{b_l(i)-b_{n'}(k')}\binom{n-n'-b_n(k)+b_{n'}(k')}{l-n'-b_l(i)+b_{n'}(k')}}{\binom{n-n'}{l-b_{n'}(k')}}. \tag{25}$$

■

Fix each $\tilde{b}_{n'}(k')$ in $\mathcal{B}$, $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ can be computed recursively in n for all $n > n'$, all $\tilde{b}_n(k))$ in $\mathcal{B}$. In this way, for all pairs of $(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ such that $n' < n$ and $\tilde{b}_{n'}(k')$, $\tilde{b}_n(k) \in \mathcal{B}$, $\varphi(\tilde{b}_n(k);\tilde{b}_{n'}(k'))$ is computable. Actually the definition of $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ is a generalization of that of $\psi(\tilde{b}_n(k))$. It's easy to check $\varphi((0,0);\tilde{b}_n(k)) = \psi(\tilde{b}_n(k))$. $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ is the conditional probability that $\tilde{S}$ doesn't hit any points in $\mathcal{B}$ after time n' and before time n given $\tilde{S}$ hits $\tilde{b}_{n'}(k')$ at time n' and $\tilde{b}_n(k))$ at time n. Or in other words, $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ is the percentage of sample paths passing through $\tilde{b}_{n'}(k')$ and $\tilde{b}_n(k)$ that don't pass any points in $\mathcal{B}$ after time n' and before time n.

Once the values of $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ are known for all pairs of $(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ such that $n' < n$ and $\tilde{b}_{n'}(k')$, $\tilde{b}_n(k) \in \mathcal{B}$, We may define barrier-II function on $\mathcal{B}$ by following definition.

Definition 6.2 *For any point $\tilde{b}_n(k))$ in $\mathcal{B}$, barrier-II function $\psi^{(2)}(\tilde{b}_n(k))$ is defined recursively by*

$$\psi^{(2)}(\tilde{b}_n(k)) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i)\in B(l)} \psi(\tilde{b}_l(i))\varphi(\tilde{b}_l(i);\tilde{b}_n(k)) \frac{\binom{b_n(k)}{b_l(i)}\binom{n-(b_n(k))}{l-b_l(i)}}{\binom{n}{l}}. \tag{26}$$

∎

$\psi^{(2)}(\tilde{b}_n(k))$ is the percentage of sample paths passing through $\tilde{b}_n(k)$ that had passed exact one point in $\mathcal{B}$ before time n. We see that $\psi^{(2)}(\tilde{b}_n(k))$ is easily computable for all points in $\mathcal{B}$ with above recursive formula.

Theorem 6.1 *Given any barrier set $\mathcal{B}$, sampling with replacement, the absorption probability at $\tilde{b}_n(i) \in \mathcal{B}$, for any $p \in [0,1]$, is*

$$\begin{aligned} P_p(\tilde{S} \in H^{(2)}(\tilde{b}_n(i))) &= \psi^{(2)}(\tilde{b}_n(i))P_p(S_n = b_n(i)) \\ &= \psi^{(2)}(\tilde{b}_n(i))\binom{n}{b_n(i)} p^{b_n(i)}(1-p)^{n-b_n(i)}, \end{aligned} \tag{27}$$

where $\psi^{(2)}(\cdot)$ is a function on $\mathcal{B}$ given by (26). ∎

For the case of sampling without replacement, the event that $\tilde{b}_n(i)$ is the second point in $\mathcal{B}$ being hitted by random path $\tilde{S}_N$ is denoted as $H_N^{(2)}(\tilde{b}_n(i))$, then

$$H_N^{(2)}(\tilde{b}_n(i)) = \bigcup_{\substack{j<n \\ B(j)\neq\emptyset}} \{\tilde{s}_N \in \mathcal{X}_N : \; s_n = b_n; \; s_j \in B_j; \; s_l \notin B_l, \;\; l < n, \;\; l \neq j \;\} \tag{28}$$

$P_{p,N}(\tilde{S}_N \in H_N^{(2)}(\tilde{b}_n(i)))$, the probability that $\tilde{S}_N$ is absorbed by $\tilde{b}_n(i)$ as the second hitting in $\mathcal{B}$, can be computed by following theorem.

Theorem 6.2 *Given any barrier set $\mathcal{B}$, sampling without replacement, the absorption probability of $\tilde{S}_N$ by $\tilde{b}_n(i) \in \mathcal{B}$, for any N and $p = \frac{1}{N}, \cdots, \frac{N}{N}$, is*

$$\begin{aligned} P_{p,N}(\tilde{S}_N \in H_N^{(2)}(\tilde{b}_n(i))) &= \psi^{(2)}(\tilde{b}_n(i)) P_{p,N}(S_n = b_n(i)) \\ &= \psi^{(2)}(\tilde{b}_n(i)) \frac{\binom{pN}{b_n(i)}\binom{(1-p)N}{n-b_n(i)}}{\binom{N}{n}}. \end{aligned} \tag{29}$$

where $\psi^{(2)}(\cdot)$ is a function on $\mathcal{B}$ given by (26). ∎

Let $\mathcal{X}_N^{\mathcal{B}}(2)$ be the subset of $\mathcal{X}_N$, which includes all sample paths which pass through at least two points in $\mathcal{B}$. $\left\{H_N^{(2)}(\tilde{b}_n(i)), \quad i = 1, \cdots, I_n;\ n = 1, 2, \cdots.\right\}$ is a partition of $\mathcal{X}_N{}^{\mathcal{B}}(2)$. The probability that $\tilde{S}_N$ hits at least two points in $\mathcal{B}$ is $P(\tilde{S}_N \in \mathcal{X}_N{}^{\mathcal{B}}(2)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P(\tilde{S}_N \in H_N^{(2)}(\tilde{b}_n(i)))$.

6.2 Illustrative Example Continued

Let $\mathcal{B}$ be the barrier set given in Section 3, *i.e.* $\mathcal{B} = \{(5,1), (5,3), (8,2), (8,4), (8,6), (11,5), (11,8)\}$. For all pairs of $(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$ such that $n' < n$ and $\tilde{b}_{n'}(k'), \tilde{b}_n(k) \in \mathcal{B}$, relative barrier function $\varphi(\tilde{b}_{n'}(k')); \tilde{b}_n(k)$ can be computed by following (25).

$$\varphi((5,1);(8,2)) = 1 - \sum_{l=6}^{7} \sum_{\tilde{b}_l(i)\in B(l)} \varphi((5,1);\tilde{b}_l(i)) \frac{\binom{3}{b_l(i)-1}\binom{2}{l-b_l(i)-4}}{\binom{3}{l-1}} = 1$$

because $B(6) = B(7) = \emptyset$. Similarly $\varphi((5,1);(8,4)) = \varphi((5,1);(8,6)) = \varphi((5,3);(8,2)) = \varphi((5,3);(8,4)) = \varphi((5,3);(8,6)) = 1$.

$$\begin{aligned} \varphi((5,1);(11,5)) &= 1 - \sum_{l=6}^{10} \sum_{\tilde{b}_l(i)\in B(l)} \varphi((5,1);\tilde{b}_l(i)) \frac{\binom{4}{b_l(i)-1}\binom{2}{l-b_l(i)-4}}{\binom{6}{l-5}} \\ &= 1 - \left\{ 1 \cdot \frac{\binom{4}{1}\binom{2}{2}}{\binom{6}{3}} + 1 \cdot \frac{\binom{4}{3}\binom{2}{0}}{\binom{6}{3}} + 1 \cdot \frac{\binom{4}{5}\binom{2}{-2}}{\binom{6}{3}} \right\} \\ &= \frac{3}{5}. \end{aligned}$$

Likewise for other pairs of $(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ such that $n' < n$ and $\tilde{b}_{n'}(k'), \tilde{b}_n(k) \in \mathcal{B}$, $\varphi(\tilde{b}_n(k);\tilde{b}_{n'}(k'))$ are computed and listed as below.

$\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ (only for $n' < n$)		$\tilde{b}_n(k) = (n, b_n(k))$ (8,2)	(8,4)	(8,6)	(11,5)	(11,8)
	(5,1)	1	1	1	$\frac{3}{5}$	1
	(5,3)	1	1	1	$\frac{2}{5}$	$\frac{1}{2}$
$\tilde{b}_{n'}(k') =$	(8,2)				1	1
$(n', b_{n'}(k'))$	(8,4)				1	1
	(8,6)				1	1

Barrier-II function $\psi^{(2)}(\cdot,\cdot)$ can be computed by following formula (26).
$\psi^{(2)}(5,1) = 0$, $\psi^{(2)}(5,3) = 0$ because $B(l) = \emptyset$ for $l = 1,\cdots,4$.

$$\psi^{(2)}(8,2) = 1\cdot 1\cdot \frac{\binom{2}{1}\binom{6}{4}}{\binom{8}{5}} + 1\cdot 1\cdot \frac{\binom{2}{3}\binom{6}{2}}{\binom{8}{5}} = \frac{15}{28} = 0.53571,$$

$$\psi^{(2)}(8,4) = 1\cdot 1\cdot \frac{\binom{4}{1}\binom{4}{4}}{\binom{8}{5}} + 1\cdot 1\cdot \frac{\binom{4}{3}\binom{4}{2}}{\binom{8}{5}} = \frac{1}{2} = 0.5,$$

$$\psi^{(2)}(8,6) = 1\cdot 1\cdot \frac{\binom{6}{1}\binom{2}{4}}{\binom{8}{5}} + 1\cdot 1\cdot \frac{\binom{6}{3}\binom{2}{2}}{\binom{8}{5}} = \frac{5}{14} = 0.35714,$$

$$\psi^{(2)}(11,5) = 1\cdot\frac{3}{5}\cdot\frac{\binom{5}{1}\binom{6}{4}}{\binom{11}{5}} + 1\cdot\frac{2}{5}\cdot\frac{\binom{5}{3}\binom{6}{2}}{\binom{11}{5}} + \frac{13}{28}\cdot 1\cdot\frac{\binom{5}{2}\binom{6}{6}}{\binom{11}{8}} + \frac{1}{2}\cdot 1\cdot\frac{\binom{5}{4}\binom{6}{4}}{\binom{11}{8}} + \frac{9}{14}\cdot 1\cdot\frac{\binom{5}{6}\binom{6}{2}}{\binom{11}{8}} = 0.48268,$$

$$\psi^{(2)}(11,8) = 1\cdot 1\cdot\frac{\binom{8}{1}\binom{3}{4}}{\binom{11}{5}} + 1\cdot\frac{1}{2}\cdot\frac{\binom{8}{3}\binom{3}{2}}{\binom{11}{5}} + \frac{13}{28}\cdot 1\cdot\frac{\binom{8}{2}\binom{3}{6}}{\binom{11}{8}} + \frac{1}{2}\cdot 1\cdot\frac{\binom{8}{4}\binom{3}{4}}{\binom{11}{8}} + \frac{9}{14}\cdot 1\cdot\frac{\binom{8}{6}\binom{3}{2}}{\binom{11}{8}} = 0.50909.$$

Sampling with replacement, the underlying distribution is binomial with parameter p. Assuming $p = 0.65$, then by (27)

$$P_p(\tilde{S} \in H^{(2)}(\tilde{b}_{11}(5))) = \psi^{(2)}(11,5)\cdot P_p(S_{11} = 5) = 0.48268\cdot\binom{11}{5}0.65^5 0.35^6 = 0.04756.$$

Likewise the absorption probabilities with respect to points in $\mathcal{B}$ are computed as following

Absorption Probability:	0.0	0.0	0.01165	0.09375	0.09239,	0.04756	0.11476
Barrier Points in $\mathcal{B}$:	(5,1)	(5,3)	(8,2)	(8,4)	(8,6)	(11,5)	(11,8)

.

The probability that $\tilde{S}$ hits at least two points in $\mathcal{B}$ is

$$P_{0.65}(\tilde{S} \in \mathcal{X}^{\mathcal{B}}(2)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P_{0.65}\left(\tilde{S} \in H^{(2)}(\tilde{b}_n(i))\right) = 0.360.$$

Sampling without replacement, the underlying distribution is hypergeometric with parameters N and p. Assuming $N = 20$ and $p = 0.65$, so $pN = 13$ then by (29)

$$P_{p,N}(\tilde{S}_N \in H_N^{(2)}(\tilde{b}_{11}(5))) = \psi(11,5)P_{p,N}(S_{11} = 5) = 0.48268 \cdot \frac{\binom{13}{5}\binom{7}{6}}{\binom{20}{11}} = 0.02589.$$

Likewise the absorption probabilities with respect to points in $\mathcal{B}$ are computed as following

Absorption Probability:	0.0	0.0	0.00232	0.09933	0.10217	0.02589	0.13653
Barrier Points in $\mathcal{B}$:	(5,1)	(5,3)	(8,2)	(8,4)	(8,6)	(11,5)	(11,8)

The probability that $\tilde{S}_N$ hits at least two points in $\mathcal{B}$ is

$$P_{0.65,20}(\tilde{S}_N \in {\mathcal{X}_N}^{B}(2)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P_{0.65,20}\left(\tilde{S}_N \in H_N^{(2)}(\tilde{b}_n(i))\right) = 0.366.$$

6.3 Absorption defined as mth hitting

Now we consider the general problem of computing the absorption probability of $\tilde{S}$ by points in $\mathcal{B}$ when absorption of $\tilde{S}$ by $\tilde{b}_n(i)$ in $\mathcal{B}$ is defined as $\tilde{S}$ hits $\tilde{b}_n(i)$ for the mth hitting in $\mathcal{B}$. If we define $\psi^{(1)}(\tilde{b}_n(i)) = \psi(\tilde{b}_n(i))$ whose values are available by (6), with relative function $\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$ whose values are available by (25), barrier-m function $\psi^{(m)}(\cdot)$ on $\mathcal{B}$ is defined as following.

Definition 6.3 *For $m = 2, 3, \cdots$, for any point $\tilde{b}_n(k)$ in $\mathcal{B}$, a function $\psi^{(m)}(\tilde{b}_n(k))$ is defined recursively by*

$$\psi^{(m)}(\tilde{b}_n(k)) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i)\in B(l)} \psi^{(m-1)}(\tilde{b}_l(i))\varphi(\tilde{b}_l(i); \tilde{b}_n(k)) \frac{\binom{b_n(k)}{b_l(i)}\binom{n-b_n(k)}{l-b_l(i)}}{\binom{n}{l}}. \qquad (30)$$

■

Above definition indicates that $\left\{\psi^{(m)}(\tilde{b}_n(i))\right\}_{\tilde{b}_n(i)\in\mathcal{B}}$ can be computed inductively for $m = 2, 3, \cdots$.

We denote $H^{(m)}(\tilde{b}_n(k))$ the event that $\tilde{S}$ hits $\tilde{b}_n(k)$ as the m*th* hitting in $\mathcal{B}$, then

$$H^{(m)}(\tilde{b}_n(k)) = \bigcup_{\substack{l_1<\cdots<l_{m-1}<n \\ B(l_t)\neq\emptyset,\ t=1,\cdots,m-1}} \left\{ \tilde{s} \in \mathcal{X} : \begin{array}{c} s_n = b_n(k);\ s_{l_t} \in B_l,\ t=1,\cdots,m-1; \\ s_l \notin B_l,\ \ l<n,\ \ l \neq l_1,\cdots,l_{m-1}. \end{array} \right\}. \quad (31)$$

Let $\mathcal{X}^{\mathcal{B}}(m)$ be the subset of $\mathcal{X}$, which consists of all sample paths $\tilde{s}$ which pass through at least m points in $\mathcal{B}$. $\left\{H^{(m)}(\tilde{b}_n(i)),\quad i=1,\cdots,I_n;\ n=1,2,\cdots.\right\}$ is a partition of $\mathcal{X}^{\mathcal{B}}(m)$.

We denote $H_N^{(m)}(\tilde{b}_n(i))$ the event that $\tilde{S}_N$ hits $\tilde{b}_n(i)$ as the m*th* hitting in $\mathcal{B}$, then

$$H_N^{(m)}(\tilde{b}_n(i)) = \bigcup_{\substack{l_1<\cdots<l_{m-1}<n \\ B(l_t)\neq\emptyset,\ t=1,\cdots,m-1}} \left\{ \tilde{s}_N \in \mathcal{X}_N : \begin{array}{c} s_n = b_n(k);\ s_{l_t} \in B_l,\ t=1,\cdots,m-1; \\ s_l \notin B_l,\ \ l<n,\ \ l \neq l_1,\cdots,l_{m-1}. \end{array} \right\}. \quad (32)$$

Let $\mathcal{X}_N{}^{\mathcal{B}}(m)$ be the subset of $\mathcal{X}_N$, which consists of all sample paths which pass through at least m points in $\mathcal{B}$. $\left\{H_N^{(m)}(\tilde{b}_n(i)),\quad i=1,\cdots,I_n;\ n=1,2,\cdots.\right\}$ is a partition of $\mathcal{X}_N{}^{\mathcal{B}}(m)$.

Theorem 6.3 *Let absorption of $\tilde{S}$ by points in a given set $\mathcal{B}$ be defined as* m*th hitting in $\mathcal{B}$. Sampling with replacement, the absorption probability at $\tilde{b}_n(i) \in \mathcal{B}$, for any $p \in [0,1]$, is*

$$\begin{aligned} P_p(\tilde{S} \in H^{(m)}(\tilde{b}_n(i))) &= \psi^{(m)}(\tilde{b}_n(i))P_p(S_n = b_n(i)) \\ &= \psi^{(m)}(\tilde{b}_n(i))\binom{n}{b_n(i)}p^{b_n(i)}(1-p)^{n-b_n(i)}, \end{aligned} \quad (33)$$

where $\psi^{(m)}(\cdot)$ is a function on $\mathcal{B}$ given by (30). The probability that $\tilde{S}$ hits at least m points in $\mathcal{B}$ is

$$P_p(\tilde{S} \in \mathcal{X}^{\mathcal{B}}(m)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P_p\left(\tilde{S} \in H^{(m)}(\tilde{b}_n(i))\right). \quad (34)$$

Sampling without replacement, the absorption probability of $\tilde{S}_N$ by $\tilde{b}_n(i) \in \mathcal{B}$, for any N and $p = \frac{1}{N},\cdots,\frac{N}{N}$, is

$$\begin{aligned} P_{p,N}(\tilde{S}_N \in H_N^{(m)}(\tilde{b}_n(i))) &= \psi^{(m)}(\tilde{b}_n(i))P_{p,N}(S_n = b_n(i)) \\ &= \psi^{(m)}(\tilde{b}_n(i))\frac{\binom{pN}{b_n(i)}\binom{(1-p)N}{n-b_n(i)}}{\binom{N}{n}}. \end{aligned} \quad (35)$$

The probability that $\tilde{S}_N$ hits at least m points in $\mathcal{B}$ is

$$P_{p,N}(\tilde{S}_N \in \mathcal{X}_N^{\mathcal{B}}(m)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P_{p,N}\left(\tilde{S}_N \in H_N^{(m)}(\tilde{b}_n(i))\right). \tag{36}$$

■

Let's continue the example given in Section 3 and Section 6.3 to illustrate above theorem. Let m=3, by (30) we have

$$\psi^{(3)}(\tilde{b}_n(k)) = \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i)\in B(l)} \psi^{(2)}(\tilde{b}_l(i))\varphi(\tilde{b}_l(i);\tilde{b}_n(k)) \frac{\binom{b_n(k)}{b_l(i)}\binom{n-(b_n(k))}{l-b_l(i)}}{\binom{n}{l}}. \tag{37}$$

Since relative barrier function $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$ and barrier-II function $\psi^{(2)}(\tilde{b}_l(i))$ for this example have been already computed in Section 6.1, the barrier-III function $\psi^{(3)}(\tilde{b}_n(k))$ can be easily computed as following.

$\psi^{(3)}(5,1) = 0$, $\psi^{(3)}(5,3) = 0$ because $B(l) = \emptyset$ for $l = 1,\cdots,4$. $\psi^{(3)}(8,2) = \psi^{(3)}(8,4) = \psi^{(3)}(8,6) = 0$ because $B(l) = \emptyset$ for $l = 1,2,3,4,6,7$ and $\psi^{(2)}(5,1) = \psi^{(2)}(5,3) = 0$.

$$\psi^{(3)}(11,5) = 0\cdot\frac{3}{5}\cdot\frac{\binom{5}{1}\binom{6}{4}}{\binom{11}{5}} + 0\cdot\frac{2}{5}\cdot\frac{\binom{5}{3}\binom{6}{2}}{\binom{11}{5}} + \frac{15}{28}\cdot 1\cdot\frac{\binom{5}{2}\binom{6}{6}}{\binom{11}{8}} + \frac{1}{2}\cdot 1\cdot\frac{\binom{5}{4}\binom{6}{4}}{\binom{11}{8}} + \frac{5}{14}\cdot 1\cdot\frac{\binom{5}{6}\binom{6}{2}}{\binom{11}{8}} = 0.25974,$$

$$\psi^{(3)}(11,8) = 0\cdot 1\cdot\frac{\binom{8}{1}\binom{3}{4}}{\binom{11}{5}} + 0\cdot\frac{1}{2}\cdot\frac{\binom{8}{3}\binom{3}{2}}{\binom{11}{5}} + \frac{13}{28}\cdot 1\cdot\frac{\binom{8}{2}\binom{3}{6}}{\binom{11}{8}} + \frac{1}{2}\cdot 1\cdot\frac{\binom{8}{4}\binom{3}{4}}{\binom{11}{8}} + \frac{9}{14}\cdot 1\cdot\frac{\binom{8}{6}\binom{3}{2}}{\binom{11}{8}} = 0.18182.$$

Sampling with replacement, the underlying distribution is binomial with parameter p. Assuming $p = 0.65$, then by (33) the absorption probabilities with respect to points in $\mathcal{B}$ are computed as following

Absorption Probability:	0.0	0.0	0.0	0.0	0.0	0.02560	0.04099
Barrier Points in $\mathcal{B}$:	(5,1)	(5,3)	(8,2)	(8,4)	(8,6)	(11,5)	(11,8)

The probability that $\tilde{S}$ hits at least three points in $\mathcal{B}$ is

$$P_{0.65}(\tilde{S} \in \mathcal{X}^{\mathcal{B}}(3)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P_{0.65}\left(\tilde{S} \in H^{(3)}(\tilde{b}_n(i))\right) = 0.067.$$

Sampling without replacement, the underlying distribution is hypergeometric with parameters N and p. Assuming $N = 20$ and $p = 0.65$, so $pN = 13$ then by (35) the absorption probabilities with respect to points in $\mathcal{B}$ are computed as following

Absorption Probability:	0.0	0.0	0.0	0.0	0.0	0.01393	0.04876
Barrier Points in $\mathcal{B}$:	(5,1)	(5,3)	(8,2)	(8,4)	(8,6)	(11,5)	(11,8)

The probability that $\tilde{S}_N$ hits at least three points in $\mathcal{B}$ is

$$P_{0.65,20}(\tilde{S}_N \in \mathcal{X}_N{}^{\mathcal{B}}(3)) = \sum_{\tilde{b}_n(i)\in\mathcal{B}} P_{0.65,20}\left(\tilde{S}_N \in H_N^{(3)}(\tilde{b}_n(i))\right) = 0.063.$$

6.4 Proof of Theorem 6.3

Similar to the definition of $\psi(\tilde{b}_n(k))$ given in (15) and (16), here we define

$$\begin{aligned}\psi^{(m)}(\tilde{b}_n(k)) &= P_p\left(\tilde{S} \in H^{(m)}(\tilde{b}_n(k)) | S_n = b_n(k)\right) \\ &= P\left(\tilde{X}_n \in D_n | S_n = b_n(k)\right) \end{aligned} \tag{38}$$

where $H^{(m)}(b_n(k))$ were given in (31) and D_n is such that

$$(\tilde{X}_n \in D_n) = \bigcup_{\substack{l_1<\cdots<l_{m-1}<n \\ B(l_t)\neq\emptyset,\ t=1,\cdots,m-1}} \left\{ \begin{array}{c} S_n = b_n(k);\ S_{l_t} \in B_{l_t},\ t = 1,\cdots,m-1; \\ S_r \notin B_r,\ \ l<n,\ \ l \neq l_1,\cdots,l_{m-1}. \end{array} \right\}.$$

By this definition, $\psi^{(m)}(\tilde{b}_n(k))$ satisfies equations (33). With justifications similar to those leads to (7), $\psi^{(m)}(\tilde{b}_n(k))$ also satisfies (35). Now we only need to show $\psi^{(m)}(\tilde{b}_n(k))$ defined by (38) satisfies equation (30), thus can be computed recursively with that equation. Before doing this, we need to give an intuitive presentation of relative barrier function $\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$ which is defined in (25) and plays an important role in equation (30).

We give another definition of $\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k))$. For $n' < n$, $\tilde{b}_{n'}(k')$, $\tilde{b}_n(k) \in \mathcal{B}$, let

$$\varphi(\tilde{b}_{n'}(k'); \tilde{b}_n(k)) = P_p\left(S_l \notin B_l,\ l = n'+1,\cdots,n-1 | S_{n'} = \tilde{b}_{n'}(k'), S_n = \tilde{b}_n(k)\right). \tag{39}$$

Then we show this definition agrees with that in (25).

Conditioned on $S_{n'} = b_{n'}(k')$, random walk $\tilde{S}^* = (S_{n'+1} - S_{n'}, S_{n'+2} - S_{n'}, \cdots)$ has same stochastic behavior as that of $\tilde{S}$. Let $l^* = l - n'$, $n^* = n - n'$, $S^*_{l^*} = S_l - S_{n'}$,

$b^*_{l^*}(i) = b_l(i) - b_{n'}(k')$, then equation (39) became

$$\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k)) = P_p\left(S^*_{l^*} \notin B^*(l^*),\ l^*=1,\cdots,n^*-1 | S^*_{n^*} = b^*_{n^*}(k), S_{n'} = \tilde{b}_{n'}(k')\right). \qquad (40)$$

Let $\tilde{S}^* = (S^*_1, S^*_2, \cdots)$, $\mathcal{B}^* = \left\{\tilde{b}^*_{l^*}(i) = b_l(i) - b_{n'}(k') :\ b_l(i) \in \mathcal{B}, l^* = l - n', l = n'+1, n'+2\cdots\right\}$, then (25) can be derived from (40) and (6) just by replacing $\psi(\tilde{b}_n(k))$ by $\varphi(\tilde{b}_{n'}(k');\tilde{b}_n(k))$, n by $n^* = n - n'$, $b_l(i)$ by $b^*_{l^*}(i) = b_l(i) - b_{n'}(k')$ *etc.* in equation (6). Justification of this derivation is analogous to the proof of (6) given in Section 5.

Now we proceed to show that $\psi^{(m)}(\tilde{b}_n(k))$ defined by (38) satisfies (30). Recall that $H^{(m)}(\tilde{b}_n(k)$ is a bunch of paths in $\mathcal{X}$ that just pass the $\tilde{b}_n(k)$ as the mth point in $\mathcal{B}$.

$$\begin{aligned}
& P_p\left(\tilde{S} \in H^{(m)}(\tilde{b}_n(k))\right) = P_p\left(\tilde{S} \in H^{(m)}(\tilde{b}_n(k)), S_n = b_n(k)\right) \\
= & \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i) \in B(l)} P_p\left(\tilde{S} \in H^{(m-1)}(\tilde{b}_l(i)); S_l = b_l(i); S_r \notin B_r, r = l+1,\cdots,n-1; S_n = b_n(k)\right) \\
= & \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i) \in B(l)} P_p\left(\tilde{S} \in H^{(m-1)}(\tilde{b}_l(i)) | S_l = b_l(i); S_r \notin B_r, r = l+1,\cdots,n-1; S_n = b_n(k)\right) \cdot \\
& \cdot P_p\left(S_l = b_l(i);\ S_r \notin B_r,\ r = l+1,\cdots,n-1;\ S_n = b_n(k)\right). \qquad (41)
\end{aligned}$$

But in (41),

$$\begin{aligned}
& P_p\left(\tilde{S} \in H^{(m-1)}(\tilde{b}_l(i)) | S_l = b_l(i); S_r \notin B_r, r = l+1,\cdots,n-1; S_n = b_n(k)\right) \\
= & P\left(\tilde{S} \in H^{(m-1)}(\tilde{b}_l(i)) | S_l = b_l(i)\right) \\
= & \psi^{(m-1)}(\tilde{b}_l(i)) \qquad (42)
\end{aligned}$$

and

$$\begin{aligned}
& P_p\left(S_l = b_l(i);\ S_r \notin B_r,\ r = l+1,\cdots,n-1;\ S_n = b_n(k)\right) \\
= & P_p(S_n = b_n(k)) P_p(S_l = b_l(i) |\ S_n = b_n(k)) \cdot \\
& \cdot P_p\left(S_r \notin B_r,\ r = l+1,\cdots,n-1 |\ S_l = b_l(i),\ S_n = b_n(k)\right). \qquad (43)
\end{aligned}$$

With the fact $P_p(S_l = b_l(i) | S_n = b_n(k)) = \frac{\binom{b_n(k)}{b_l(i)}\binom{n-b_n(k)}{l-b_l(i)}}{\binom{n}{l}}$ and because of (42), (43) and (39), equation (41) became

$$P_p\left(\tilde{S} \in H^{(m)}(\tilde{b}_n(k))\right)$$

$$= \sum_{l=1}^{n-1} \sum_{\tilde{b}_l(i) \in B(l)} \psi^{(m-1)}(\tilde{b}_n(k))\varphi(\tilde{b}_l(i); \tilde{b}_n(k)) \frac{\binom{b_n(k)}{b_l(i)}\binom{n-b_n(k)}{r-b_l(i)}}{\binom{n}{l}} P_p(S_n = b_n(k)). \tag{44}$$

Divided both sides by $P_p(S_n = b_n(k))$, equation (44) became (30). ∎

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