THE BOX COUNTING DIMENSION OF A CLASS OF STATISTICALLY SELF-AFFINE FRACTALS

by

Steven P. Lalley*
Department of Statistics
Purdue University
West Lafayette, IN 47907-1399

and

Dimitrios Gatzouras
Department of Statistics
Purdue University
West Lafayette, IN 47907-1399

Technical Report # 92-16

Department of Statistics
Purdue University

May, 1992

* This research was partially supported by NSF Grant DMS-9005118 and partially by David Ross Grant No. 6901629

△

THE BOX COUNTING DIMENSION OF A CLASS OF STATISTICALLY SELF-AFFINE FRACTALS

by

Steven P. Lalley
Department of Statistics
Purdue University
West Lafayette, IN 47907-1399

and

Dimitrios Gatzouras
Department of Statistics
Purdue University
West Lafayette, IN 47907-1399

Abstract

The box-counting dimension of a class of random self-affine fractal sets is given. The methods used involve certain branching processes with random environments. It turns out that the dimension is a constant, depending on a probability distribution, naturally associated with the construction.

1. Introduction

Let m and n be positive integers, with $2 \le m \le n$ and set $K_0 = [0,1]^2$. Divide the unit square into the mn rectangles $[in^{-1},(i+1)n^{-1}] \times [jm^{-1},(j+1)m^{-1}]$, $i = 0,1,\ldots,n-1$, $j = 0,1,\ldots,m-1$ and choose a random subcollection of these rectangles according to some given probability distribution F. Denote the aggregate of the rectangles so chosen by K_1 and let M_1 be the number of rectangles in K_1. Note that the possibility $M_1 = 0$ (so $K_1 = \emptyset$) is not excluded. Next divide each rectangle in K_1 (if any) into mn rectangles of height m^{-2} and width n^{-2} and choose a subcollection of these according to the same distribution F as before, independently for each rectangle in K_1. Denote the aggregate of these rectangles by K_2 and let M_2 be the number of rectangles in K_2. In case $M_1 = 0$ set $K_2 = \emptyset$ and $M_1 = 0$. Continuing in this way we obtain a decreasing sequence of compact subsets of K_0

$$K_0 \supset K_1 \supset K_2 \supset \ldots$$

and a sequence of nonnegative integers $\{M_i\}$, where K_i is either empty or a (finite) union of rectangles each of which has height m^{-i} and width n^{-i} and M_i is the number of rectangles in K_i. Set $M_0 = 1$ and observe that $\{M_i\}$ is a Galton-Watson branching process.

Define

$$K = \bigcap_{i=1}^{\infty} K_i.$$

The set K is what we call a *statistically self-affine fractal* (as opposed to the strictly self-affine fractals considered in [Mc], [Be], [Fa] and [LG]) and is the object of interest in this note.

It is well known that if $E[M_1] \le 1$, then either $P(M_i \to 0) = 1$ or $P(M_i = 1, \forall\, i \in \mathbb{N}) = 1$ and hence either $K = \emptyset$ with probability one, or K is a singleton with probability one. So when $E[M_1] \le 1$, the limit set K is trivial. We therefore assume throughout this note that $E[M_1] > 1$. In this case, with probability one, $\{M_i\}$ either dies out ($M_i \to 0$) or tends to infinity and

$$P(\lim_{i\to\infty} M_i = \infty) = 1 - P(\lim_{i\to\infty} M_i = 0) > 0.$$

Thus we get a nonempty ("nontrivial") limit set K, with positive probability.

Theorem

There exists a constant d, depending on the distribution F according to which K_1 was chosen, such that, given that $K \neq \emptyset$, the box-counting dimension of K is equal to d, with probability one.

The proof of the theorem is given in Section 3, where we also give a formula for the value of d.

It is of interest to also determine the Hausdorff dimension of K and to give conditions under which these two notions of dimension agree. (See Lalley and Gatzouras, [LG], for a discussion in the deterministic case.) This will be done in a subsequent paper.

Finally, we would like to mention that random constructions have been considered before by various authors. All of them however, as far as we know, lead to "statistically" self-similar sets. (See [CCD], [DG], [DM], $[F]_1$, $[F]_2$, [Ke], [MGW] and [MW]. See also [Ma] for general definitions, examples and motivation.) We emphasize the fact that the cases considered here are drastically different than the ones already studied, the difference being that the collection of subsets of the unit square that we consider consists of *rectangles* (rather than squares), i.e., images of the unit square under *affine* transformations (rather than similarity transformations). Hence the name "self-affine" for the limit set K.

For the proof of the Theorem we will need a result about branching processes with random environments, which is given in Section 2. In Section 3 then, we use this result to prove the Theorem.

2. Branching Processes with Random Environments

Let $\overline{\zeta} = (\zeta_1, \zeta_2, \ldots)$ be a sequence of i.i.d. random variables with

$$P(\zeta_1 = j) = q_j, \qquad j = 1, \ldots, m,$$

where $\{q_j : j = 1, \ldots, m\}$ is a probability distribution on $\{1, \ldots, m\}$. For each $j \in \mathbb{N}$, let $\{p_j(i) : i = 0, 1, \ldots\}$ be a probability distribution on the nonnegative integers, satisfying

$$\sum_{i=0}^{\infty} i p_j(i) < \infty.$$

Let

$$\varphi_j(x) = \sum_{i=0}^{\infty} x^i p_j(i), \qquad 0 \leq x \leq 1,$$

be the p.g.f. of the distribution $\{p_j(i) : i = 0, 1, \ldots\}$, for each $j \in \mathbb{N}$.

Now define a branching process $\{Z_k\}$ as follows. Conditional on $\overline{\zeta}$, let the transition from Z_{k-1} to Z_k take place as follows: all the Z_{k-1} members of the $(k-1)$th generation reproduce independently of one another and according to the same offspring distribution specified by the p.g.f. φ_{ζ_k}. Then $\{Z_k\}$ is what is commonly referred to as a branching process with random environments.

Smith and Wilkinson ([SW]) were the first to study such processes. Later on Athreya and Karlin ($[\mathrm{AK}]_1$, $[\mathrm{AK}]_2$) extended and refined their results. The object of interest to us is the limiting behavior of the sequence $\{P(Z_k > 0) : k = 1, 2, \ldots\}$. In particular, in this section we determine the value of $\lim_{k\to\infty}\{P(Z_k > 0)\}^{1/k}$ (Proposition 2.1 and Corollary 2.2 below).

After this note was written we discovered that Proposition 2.1 (and its corollary) was the object of a paper by Dekking ([De]). However, he only proved Proposition 2.1 under the additional assumption that $E[Z_1^2] < \infty$. Although Dekking's theorem is sufficient for our purposes, we have included Proposition 2.1 here for two reasons: 1.) because Proposition 2.1 is more general than Dekking's theorem and 2.) because some of the arguments in the proof come up again in Section 3.

Let

$$q(\overline{\zeta}) = P(Z_k = 0, \text{ for some } k \in \mathbb{N} \mid \overline{\zeta}) \tag{2.1}$$

be the probability of extinction, given the whole environmental sequence $\overline{\zeta}$. We will make use of the following theorem, which may be found in $[\mathrm{AK}]_1$.

2.0 Theorem

Assume that $p_j(0) + p_j(1) < 1$ for each $j = 1, \ldots, m$.

(i) *If $E[\log \varphi'_{\zeta_1}(1)] \leq 0$ then $P(q(\overline{\zeta}) = 1) = 1$.*

(ii) *If $E[\log \varphi'_{\zeta_1}(1)] > 0$ then $P(q(\overline{\zeta}) = 1) = 0$.*

Observe at this point that

$$E\left[x^{Z_k}\right] = E[\varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(x)], \qquad 0 \leq x \leq 1,$$

so that

$$P(Z_k = 0) = E[\varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(0)]$$

for all $k \in \mathbf{N}$.

Now define $\psi: \mathbf{R} \to \mathbf{R}$ by

$$\psi(\theta) = \log E\left[(\varphi'_{\zeta_1}(1))^\theta\right], \qquad \theta \in \mathbf{R}.$$

and notice that

$$\psi'(0) = E\left[\log \varphi'_{\zeta_1}(1)\right]$$

In the proof of Proposition 2.1 and in Section 3 we will use a certain family of probability measures $\{P_\theta : \theta \in \mathbf{R}\}$, which we introduce here. Define

$$\mathcal{F}_k = \sigma(\zeta_1, \ldots, \zeta_k), \qquad k \in \mathbf{N}$$

and

$$\mathcal{F}_\infty = \sigma(\overline{\zeta}) = \sigma(\zeta_1, \zeta_2, \ldots).$$

Observe that for $k \in \mathbf{N}, A \in \mathcal{F}_k$

$$e^{-(k+j)\psi(\theta)} E\left[\prod_{i=1}^{k+j} (\varphi'_{\zeta_i}(1))^\theta \, 1_A\right] = e^{-k\psi(\theta)} E\left[\prod_{i=1}^{k} (\varphi'_{\zeta_i}(1))^\theta \, 1_A\right], \tag{2.2}$$

for all $j \in \mathbf{N}$. Given $\theta \in \mathbf{R}$, define for $A \in \mathcal{F}_k$

$$P_\theta(A) = e^{-k\psi(\theta)} E\left[\prod_{i=1}^{k} (\varphi'_{\zeta_i}(1))^\theta \, 1_A\right]$$

and observe that by (2.2) P_θ is a well defined probability measure on $\cup_{k=1}^\infty \mathcal{F}_k$. Since $\cup_{k=1}^\infty \mathcal{F}_k$ is an algebra, P_θ can be extended, in a unique way, on $\mathcal{F}_\infty = \sigma(\cup_{k=1}^\infty \mathcal{F}_k)$. Finally, for an arbitrary event A (in the underlying probability space) define

$$P_\theta(A) = E_\theta[P(A|\mathcal{F}_\infty)],$$

where E_θ denotes the expectation operator corresponding to the measure P_θ, i.e.,

$$E_\theta[X] = \int X dP_\theta,$$

for any random variable X.

We now give the main result of this section.

2.1 Proposition

Assume that $p_j(0) + p_j(1) < 1$, *for each* $j = 1, \ldots, m$.

(a) *If* $\psi'(0) > 0$, *then*

$$\lim_{k\to\infty} P(Z_k > 0) = 1 - E[q(\overline{\zeta})] > 0.$$

(b) (i) *If* $\psi'(0) \le 0$ *and* $\psi'(1) < 0$, *then*

$$\lim_{k\to\infty} \left\{ \frac{1}{k} \log P(Z_k > 0) \right\} = \psi(1).$$

(ii) *If* $\psi'(0) \le 0$ *but* $\psi'(1) \ge 0$, *then*

$$\lim_{k\to\infty} \left\{ \frac{1}{k} \log P(Z_k > 0) \right\} = \min_{\theta\in\mathbf{R}} \psi(\theta).$$

PROOF:

(a) This is an immediate consequence of the bounded convergence theorem, (2.1) and Theorem 2.0.

(b) $\lim_{k\to\infty} P(Z_k > 0) = 1 - E[q(\overline{\zeta})]$ is still true in this case but $1 - E[q(\overline{\zeta})] = 0$. Thus we have to proceed differently when $\psi'(0) \le 0$.

First observe that

$$\varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(x) = E(x^{Z_k} | \mathcal{F}_\infty), \qquad 0 \le x \le 1.$$

Hence $\varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}$ is a convex function, for each $k \in \mathbf{N}$. Thus

$$1 - \varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(x) \le \prod_{i=1}^{k} \varphi'_{\zeta_i}(1).(1 - x), \qquad 0 \le x \le 1.$$

and so

$$1 - \varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(0) \le \prod_{i=1}^{k} \varphi'_{\zeta_i}(1), \tag{2.3}$$

for all $k \in \mathbf{N}$.

(i) Assume $\psi'(1) < 0$. By (2.3)

$$P(Z_k > 0) = E[1 - \varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(0)] \le (E[\varphi'_{\zeta_1}(1)])^k,$$

for all $k \in \mathbb{N}$, establishing

$$\limsup_{k\to\infty}\left\{\frac{1}{k}\log P(Z_k>0)\right\}\le \log E[\varphi'_{\zeta_1}(1)]=\psi(1).$$

Now fix $\epsilon>0$ and let $\delta>0$ be such that

$$1-\varphi_j(x)\ge(1-\epsilon)\cdot(\varphi'_j(1))\cdot(1-x),$$

for all $1-\delta\le x\le 1$ and all $j=1,\dots,m$. (Such a δ, depending on ϵ, always exists by definition of the functions φ_j.) Then

$$\begin{aligned}
&E[1-\varphi_{\zeta_k}\circ\dots\circ\varphi_{\zeta_1}(0)]\\
&\ge(1-\epsilon)E[\varphi'_{\zeta_k}(1)(1-\varphi_{\zeta_{k-1}}\circ\dots\circ\varphi_{\zeta_1}(0))1_{(1-\delta,1)}(\varphi_{\zeta_{k-1}}\circ\dots\circ\varphi_{\zeta_1}(0))]\\
&\ge(1-\epsilon)E\left[\varphi'_{\zeta_k}(1)(1-\varphi_{\zeta_{k-1}}\circ\dots\circ\varphi_{\zeta_1}(0))1_{(0,\delta)}\left(\prod_{i=1}^{k-1}\varphi'_{\zeta_i}(1)\right)\right],
\end{aligned}$$

which upon iteration yields

$$E[1-\varphi_{\zeta_k}\circ\dots\circ\varphi_{\zeta_1}(0)]\ge(1-\epsilon)^{k-j}E\left[\prod_{i=j+1}^{k}\varphi'_{\zeta_i}(1)(1-\varphi_{\zeta_j}\circ\dots\circ\varphi_{\zeta_1}(0))\ 1_{F_j}\right]$$

for any $k\in\mathbb{N}$ and any $j\in\{1,\dots,k-1\}$, where

$$F_j=\bigcup_{i=j}^{k-1}\{\varphi'_{\zeta_1}(1)\dots\varphi'_{\zeta_i}(1)<\delta\}.$$

It now follows that

$$\begin{aligned}
&E[1-\varphi_{\zeta_k}\circ\dots\circ\varphi_{\zeta_1}(0)]\ge \qquad (2.4)\\
&\ge(1-\epsilon)^{k-j}E\left[\frac{\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)}{\prod_{i=1}^{j}\varphi'_{\zeta_i}(1)}(1-\varphi_{\zeta_j}\circ\dots\circ\varphi_{\zeta_1}(0))\ 1_{F_j}\right]\\
&=(1-\epsilon)^{k-j}e^{k\psi(1)}E_1\left[\prod_{i=1}^{j}(\varphi'_{\zeta_i}(1))^{-1}(1-\varphi_{\zeta_j}\circ\dots\circ\varphi_{\zeta_1}(0))\ 1_{F_j}\right]\\
&=(1-\epsilon)^{k-j}e^{k\psi(1)}E_1\left[\prod_{i=1}^{j}(\varphi'_{\zeta_i}(1))^{-1}(1-\varphi_{\zeta_j}\circ\dots\circ\varphi_{\zeta_1}(0))\ 1_{G_j}\right]
\end{aligned}$$

for all $k\in\mathbb{N}$ and $j\in\{1,\dots,k-1\}$, where now

$$G_j=\bigcup_{i=j}^{\infty}\{\varphi'_{\zeta_1}(1)\dots\varphi'_{\zeta_i}(1)<\delta\}.$$

Now since $\psi'(1) < 0$ and since, by the strong law of large numbers,

$$\frac{1}{k}\sum_{i=1}^{k}\log\varphi'_{\zeta_i}(1) \to E_1[\log\varphi'_{\zeta_1}(1)] = \psi'(1), \qquad P_1 - a.s.$$

we must have that

$$\prod_{j=1}^{k}\varphi'_{\zeta_i}(1) \to 0, \qquad P_1 - a.s. \tag{2.5}$$

Consequently there exists a $j \in \mathbf{N}$ such that $P_1(G_j) > 0$. Fix such a j and set

$$c = E_1\left[\prod_{i=1}^{j}(\varphi'_{\zeta_i}(1))^{-1}(1-\varphi_{\zeta_j}\circ\ldots\circ\varphi_{\zeta_1}(0))\,1_{G_j}\right].$$

Then $c > 0$ and by (2.4) we have that for all $k > j$

$$E[1-\varphi_{\zeta_k}\circ\ldots\circ\varphi_{\zeta_1}(0)] \geq c(1-\epsilon)^{k-j}e^{k\psi(1)}$$

Since ϵ was arbitrary it now follows that

$$\liminf_{k\to\infty}\left\{\frac{1}{k}\log P(Z_k > 0)\right\} = \liminf_{k\to\infty}\left\{\frac{1}{k}\log E[1-\varphi_{\zeta_1}\circ\ldots\circ\varphi_{\zeta_k}(0)]\right\}$$
$$= \liminf_{k\to\infty}\left\{\frac{1}{k}\log E[1-\varphi_{\zeta_k}\circ\ldots\circ\varphi_{\zeta_1}(0)]\right\} \geq \psi(1).$$

(ii) Assume $\psi'(1) \geq 0$. By the convexity of the function ψ, there exists a $t \in [0,1]$ such that

$$\psi(t) = \min_{\theta\in\mathbf{R}}\psi(\theta).$$

Then, by (2.3)

$$P(Z_k > 0) = E[1-\varphi_{\zeta_1}\circ\ldots\circ\varphi_{\zeta_k}(0)]$$
$$= E\left[(1-\varphi_{\zeta_1}\circ\ldots\circ\varphi_{\zeta_k}(0))\left\{1_{(0,1]}\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right) + 1_{(1,\infty)}\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right)\right\}\right]$$
$$\leq E\left[\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\,1_{(0,1]}\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right)\right] + P\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1) > 1\right)$$
$$\leq E\left[\prod_{i=1}^{k}(\varphi'_{\zeta_i}(1))^t\,1_{(0,1]}\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right)\right] +$$
$$+E\left[\prod_{i=1}^{k}(\varphi'_{\zeta_i}(1))^t\,1_{(1,\infty)}\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right)\right]$$
$$= E\left[\prod_{i=1}^{k}(\varphi'_{\zeta_i}(1))^t\right] = e^{k\psi(t)},$$

for each $k \in \mathbf{N}$. Consequently,

$$\limsup_{k\to\infty} \left\{ \frac{1}{k} \log P(Z_k > 0) \right\} \le \psi(t)$$

We now have to consider two separate cases.

Case 1: $\varphi_1'(1) = \varphi_2'(1) = \ldots = \varphi_m'(1)$. Notice that we must necessarily have that $\varphi_j'(1) = 1$, for all $j = 1, \ldots, m$, because $\psi'(0) = \log \varphi_1'(1) \le 0$ and $\psi'(1) = \log \varphi_1'(1) \ge 0$. Now fix $\varepsilon > 0$ and choose $\delta > 0$ so that for every $1 - \delta \le x \le 1$ we have that

$$1 - \varphi_j(x) \ge (1 - \varepsilon)(1 - x)$$

for all $j \in \{1, \ldots, m\}$. Such a δ always exists because for each j, φ_j is strictly convex (recall $p_j(0) + p_j(1) < 1$) and $\varphi_j'(1) = 1$.

Now since $E[\log \varphi_{\zeta_1}'(1)] = \log \varphi_1'(1) = 0$, we have by Theorem 2.0 and (2.1) that

$$\lim_{k\to\infty} \varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_k}(0) = \lim_{k\to\infty} P(Z_k = 0 | \zeta_1, \zeta_2, \ldots) = q(\overline{\zeta}) = 1,$$

with probability one. Hence there exists some $j \in \mathbf{N}$ such that

$$P(\varphi_{\zeta_j} \circ \ldots \circ \varphi_{\zeta_1}(0) > 1 - \delta) = P(\varphi_{\zeta_1} \circ \ldots \circ \varphi_{\zeta_j}(0) > 1 - \delta) > 0.$$

Fix such a j and observe that since $\varphi_j(x) \ge x$, for all $0 \le x \le 1$ and all j,

$$\begin{aligned} P(Z_{k+j} > 0) &= E[1 - \varphi_{\zeta_{k+j}} \circ \ldots \circ \varphi_{\zeta_1}(0)] \\ &\ge E[(1 - \varphi_{\zeta_{k+j}} \circ \ldots \circ \varphi_{\zeta_1}(0))\, 1_{(1-\delta,1)}(\varphi_{\zeta_j} \circ \ldots \circ \varphi_{\zeta_1}(0))] \\ &\ge (1-\varepsilon)^k E[(1 - \varphi_{\zeta_j} \circ \ldots \circ \varphi_{\zeta_1}(0))\, 1_{(1-\delta,1)}(\varphi_{\zeta_j} \circ \ldots \circ \varphi_{\zeta_1}(0))] \\ &\ge (1-\varepsilon)^k P(\varphi_{\zeta_j} \circ \ldots \circ \varphi_{\zeta_1}(0) > 1 - \delta)(1 - \max_{i_1,\ldots,i_j} \varphi_{i_1} \circ \ldots \circ \varphi_{i_j}(0)), \end{aligned}$$

for all $k \in \mathbf{N}$. Consequently,

$$\liminf_{k\to\infty} \left\{ \frac{1}{k} \log P(Z_k > 0) \right\} \ge \log(1 - \varepsilon)$$

and since ε was arbitrary we must have that

$$\liminf_{k\to\infty} \left\{ \frac{1}{k} \log P(Z_k > 0) \right\} \ge 0 = \psi(t).$$

Case 2: There exist integers i and j, with $i \neq j$, in $\{1, \dots, m\}$ such that $\varphi_i'(1) \neq \varphi_j'(1)$. In this case the function ψ is strictly convex. Hence for any $\theta > t$, we have that $\psi'(\theta) > 0$ and hence $E_\theta[\log \varphi'_{\zeta_1}(1)] > 0$. It then follows by Theorem 2.0 (with the measure P_θ in the place of P) that $P_\theta(\overline{q}(\zeta) = 1) = 0$ and so

$$P_\theta(Z_k > 0, \text{ for all } k \in \mathbb{N}) > 0, \tag{2.6}$$

for all $\theta > t$. In other words, under the measure $P_\theta, \theta > t$, the branching process $\{Z_k\}$ is supercritical (in the terminology Athreya and Karlin [AK]$_1$) and has therefore a positive probability of living on forever.

Now choose $\varepsilon > 0$. Fix a $\theta > t$ such that $\psi'(\theta) < \varepsilon$ and $\theta < t + \varepsilon$ (recall that $\psi'(t) = 0$ and ψ' is continuous). By the strong law of large numbers, as $k \to \infty$,

$$\left[\prod_{i=1}^{k} \varphi'_{\zeta_i}(1)\right]^{1/k} \to e^{\psi'(\theta)}, \qquad P_\theta - \text{ a.s.}$$

Consequently,

$$\begin{aligned}
P(Z_k > 0) &= E_\theta\left[\left(\prod_{i=1}^{k} \varphi'_{\zeta_i}(1)\right)^{-\theta} e^{k\psi(\theta)}\ 1\{Z_k > 0\}\right] \\
&\geq e^{k\psi(\theta)} E_\theta\left[\left(\prod_{i=1}^{k} \varphi'_{\zeta_i}(1)\right)^{-\theta} 1\{Z_j > 0, \ \ \forall\, j \in \mathbb{N}\}\right] \\
&\geq e^{k\psi(\theta)} E_\theta\Bigg[\left(\prod_{i=1}^{k} \varphi'_{\zeta_i}(1)\right)^{-\theta} 1\{Z_j > 0, \ \ \forall\, j \in \mathbb{N}\} \\
&\qquad \times 1\left\{\prod_{i=1}^{k} \varphi'_{\zeta_i}(1) \leq e^{k(\psi'(\theta)+\varepsilon)}\right\}\Bigg] \\
&\geq \exp\{k\psi(\theta) - k\theta(\psi'(\theta) + \varepsilon)\}\left[\frac{1}{2} P_\theta(Z_j > 0, \ \ \forall\, j \in \mathbb{N})\right] \\
&\geq \frac{1}{2} e^{k\psi(t)} e^{-2k\varepsilon\theta} P_\theta(Z_j > 0, \ \ \forall\, j \in \mathbb{N})
\end{aligned}$$

for all large enough k. It now follows by (2.6) that

$$\liminf_{k\to\infty} \left\{\frac{1}{k} \log P(Z_k > 0)\right\} \geq \psi(t) - 2\varepsilon\theta \geq \psi(t) - 2\varepsilon(t + \varepsilon).$$

Since ε was arbitrary the proof is now complete. □

2.2 Corollary

$$\lim_{k\to\infty}\left\{\frac{1}{k}\log P(Z_k>0)\right\}=\min_{\theta\in[0,1]}\psi(\theta)$$

PROOF: First assume that $p_j(0)+p_j(1)<1$, for all $j\in\{1,\ldots,m\}$. Then the corollary is an immediate consequence of Proposition 2.1.

For the general case define $N=\{j\in\{1,\ldots,m\}: p_j(0)+p_j(1)=1\}$ and $q=\Sigma_{j\in N}q_j$. Then

$$P(Z_k>0)=P(Z_k>0\mid \zeta_i\notin N,\ \text{for all } 1\le i\le k)(1-q)^k.$$

Now let $(\tilde\zeta_1,\tilde\zeta_2,\ldots)$ be a sequence of i.i.d. random variables with $P(\tilde\zeta_1=j)=q_j(1-q)^{-1}$, for $j\in\{1,\ldots,m\}\backslash N$ and $P(\tilde\zeta_1=j)=0$, for $j\in N$. Let $\{\tilde Z_k\}$ be a branching process with random environments corresponding to the environmental sequence $(\tilde\zeta_1,\tilde\zeta_2,\ldots)$. Then

$$P(\tilde Z_k>0)=P(Z_k>0|\zeta_i\notin N,\ \text{for all } 1\le i\le k).$$

By Proposition 2.1

$$\lim_{k\to\infty}\{P(\tilde Z_k>0)\}^{1/k}=\min_{\theta\in[0,1]}E[(\varphi'_{\tilde\zeta_1}(1))^\theta]=\frac{1}{1-q}\min_{\theta\in[0,1]}E[(\varphi'_{\zeta_1}(1))^\theta]$$

and the result now follows. □

3. The Box-Counting Dimension of K

In this section we determine the box-counting dimension $\delta(K)$ of K. Recall that if X is a bounded finite dimensional metric space and $N(\varepsilon)$ denotes the cardinality of a minimal covering of X by ϵ-balls, then the box-counting dimension $\delta(X)$ of X is

$$\delta(X)=\limsup_{\varepsilon\to 0}\frac{\log N(\epsilon)}{-\log\epsilon}$$

Instead of coverings of K by balls, we will consider coverings consisting of certain rectangles that we call *approximate squares* and which are tailored to the structure of K. These rectangles are defined as follows: given $k\in\mathbb{N}$, let $l_k=[k\log_n m]$, where the brackets denote integer part. The approximate squares are then the rectangles

$$R_k(p,q)=[pn^{-l_k},(p+1)n^{-l_k}]\times[qm^{-k},(q+1)m^{-k}]$$

where $p \in \{0, 1, \ldots, n^{l_k} - 1\}, q \in \{0, 1, \ldots, m^k - 1\}$ and $k \in \mathbb{N}$. These rectangles have height m^{-k} and width n^{-l_k} and since

$$n^{-1} < \frac{m^{-k}}{n^{-l_k}} \leq 1,$$

the rectangles $R_k(p, q)$ have sides whose ratio is bounded away from 0 and ∞; hence the term approximate squares. (These rectangles were first used in [Mc].)

Evidently we have the following:

3.1 Lemma

Let $\tilde{N}_k$ denote the cardinality of a minimal covering of K by approximate squares $R_k(p, q)$. Then

$$\delta(K) = \limsup_{k \to \infty} \frac{\log \tilde{N}_k}{k \log m}. \qquad \square$$

We now introduce some notation. Let $\mathcal{J} = \{1, \ldots, m\}$ and $\mathcal{J}_k = \{(s_1, \ldots, s_k): s_i \in \mathcal{J}\}$. Given a finite sequence $s = (s_1, \ldots, s_i) \in \mathcal{J}_i$ define N_s to be the number of rectangles in K_i that are contained in the horizontal strip

$$[0, 1] \times \left[\sum_{j=1}^{i} m^{-j}(s_j - 1) \, , \, \sum_{j=1}^{i} m^{-j}(s_j - 1) + m^{-i} \right].$$

In particular, for $j \in \mathcal{J}, N_j$ is the nubmer of rectangles in K_1 that are contained in $[0, 1] \times [(j - 1)m^{-1}, jm^{-1}]$.

For $k \in \mathbb{N}$, define

$$\mu_k = E\left[\sum_{s \in \mathcal{J}_k} 1_{(0,\infty)}(N_s) \right] = \sum_{s \in \mathcal{J}_k} P(N_s > 0). \tag{3.1}$$

Now let $(\zeta_1, \zeta_2, \ldots)$ be a sequence of i.i.d. random variables with $P(\zeta_1 = j) = m^{-1}$, for $j = 1, \ldots, m$, and such that $(\zeta_1, \zeta_2, \ldots)$ is independent of the whole process by which the fractal K was constructed. For $k \in \mathbb{N}$ define

$$Z_k = N_s \text{ on the event } (\zeta_1, \ldots, \zeta_k) = s,$$

where $s \in \mathcal{J}_k$. Set $Z_0 = 1$. Then $\{Z_k\}$ is a branching process with random environments, of the type considered in Section 2. By (3.1) we then have that for each $k \in \mathbb{N}$

$$\mu_k = m^k P(Z_k > 0)$$

and upon application of Corollary 2.2,

$$\lim_{k\to\infty} \mu_k^{1/k} = \min_{0\le\theta\le1} \left\{ \sum_{j=1}^m (E[N_j])^\theta \right\}. \tag{3.2}$$

Now set

$$d = \log_m \left(\min_{0\le\theta\le1} \left\{ \sum_{j=1}^m (E[N_j])^\theta \right\} \right) + \log_n \left(\frac{E[M_1]}{\min_{0\le\theta\le1} \{\sum_{j=1}^m (E[N_j])^\theta\}} \right).$$

We will show that, given that $K \neq \emptyset$, the box-counting dimension $\delta(K)$ of K equals d, with probability 1.

We begin by introducing some more notation. Given $k \in \mathbb{N}$ let $I_1^{(k)}, \ldots, I_{M_{l_k}}^{(k)}$ be the rectangles in K_{l_k}. Let $X_j^{(k)}$ be the number of approximate squares $R_k(p,q)$ which are contained in $I_j^{(k)}$ and do contain at least one rectangle of K_k. Then, by construction, $X_1^{(k)}, \ldots, X_{M_{l_k}}^{(k)}$ are independent and identically distributed, with common mean equal to μ_{k-l_k}. Furthermore M_{l_k} is independent of $(X_1^{(k)}, \ldots, X_{M_{l_k}}^{(k)})$. Set

$$S_k = \sum_{j=1}^{M_{l_k}} X_j^{(k)}.$$

(Set $S_k = 0$ if $M_{l_k} = 0$.) Then S_k is the number of approximate squares $R_k(p,q)$ that contain at least one of the rectangles in K_k.

3.2 Proposition

The box-counting dimension $\delta(K)$ of K satisfies

$$P(\delta(K) \le d | K \neq \emptyset) = 1.$$

PROOF: By Lemma 3.1, it suffices to show that

$$P\left(\limsup_{k\to\infty} \frac{\log S_k}{k \log m} \le d | K \neq \emptyset \right) = 1, \tag{3.3}$$

since

$$\delta(K) \le \limsup_{k\to\infty} \frac{\log S_k}{k \log m}.$$

If $\epsilon > 0$ is given, we have by Markov's inequality, that for all $k \in \mathbb{N}$

$$P\left(\frac{S_k}{\mu_{k-l_k} M_{l_k}} > e^{\varepsilon k} | K \neq \emptyset \right) \le e^{-\varepsilon k} \frac{P(M_{l_k} > 0)}{P(K \neq \emptyset)}$$

It then follows by the Borel Cantelli lemma that

$$P\left(\limsup_{k\to\infty}\left\{\frac{1}{k}\log\frac{S_k}{\mu_{k-l_k}M_{l_k}}\right\}\geq 0|K\neq\emptyset\right)=1.$$

Now using the fact that

$$\lim_{k\to\infty}\frac{M_{l_k}}{(E[M_1])^{l_k}}=M_\infty,\text{ a.s.}$$

(since $\{M_j\}$ is a Galton-Watson process and $\lim_{k\to\infty} l_k=\infty$) and the fact that

$$P(M_\infty>0|K\neq\emptyset)=P(M_\infty>0|M_j\to\infty)=1,$$

we get (3.3) from (3.2) and the definition of d. □

To show that $P(\delta(K)\geq d|K\neq\emptyset)$ we will first show that

$$\liminf_{k\to\infty}\frac{\log S_k}{k\log m}\geq d,\text{ a.s.}$$

For this we will need the following:

3.2 Lemma

There exist constants $c\in(0,\infty)$ and $\beta\in(0,1)$ such that, for all $k\in\mathbb{N}$ we have

$$E[M_k^{-1}\,1_{(0,\infty)}(M_k)]\leq c\beta^k.$$

PROOF: Let $f(x)=\Sigma_{j=0}^{\infty}s^jP(M_1=j), 0\leq s\leq 1$, be the p.g.f. of M_1; set $f_1=f$ and $f_k=f\circ f_{k-1}, k=2,3,\ldots$. Then f_k is the p.g.f. of M_k, i.e.,

$$f_k(s)=\sum_{j=0}^{\infty}s^jP(M_k=j),\ 0\leq s\leq 1,\ k\in\mathbb{N}.$$

Now

$$\begin{aligned}(3.4)\qquad E[M_k^{-1}\,1_{(0,\infty)}M_k)]&\leq 2E[(M_k+1)^{-1}\,1_{(0,\infty)}(M_k)]\\ &=2\sum_{j=1}^{\infty}\frac{1}{j+1}P(M_k=j)=2\sum_{j=1}^{\infty}P(M_k=j)\int_0^1 s^j ds\\ &=2\int_0^1(f_k(s)-f_k(0))ds=2\int_0^1(f_k(s)-q)ds+2(q-f_k(0))\\ &\leq 2\int_q^1(f_k(s)-q)ds+2(q-f_k(0)),\end{aligned}$$

for all $k \in \mathbf{N}$, where $q = P(\lim_{i\to\infty} M_i = 0)$ is the probability of extinction of the branching process $\{M_i\}$.

Now choose $\varepsilon > 0$ so that $f'(1) - \varepsilon > 1$. This is always possible since we have assumed that $f'(1) = E[M_1] > 1$. Let t be the unique point in (0, 1) for which

$$f(t) = (f'(1) - \varepsilon)(t - 1) + 1;$$

such a t always exists and is unique since $f\colon [0,1] \to [0,1]$ is strictly convex and $f(1) = 1$. Then $f(s) \le (f'(1) - \varepsilon)(s - 1) + 1$, for all $s \in [t, 1]$, by convexity, and hence

$$\begin{aligned}
&\int_q^1 (f_k(s) - q)ds \\
&= \int_q^t (f_k(s) - q)ds + \int_t^1 (f_k(s) - q)ds \\
&= \int_q^t (f_k(s) - q)ds + \int_t^1 (f_{k-1}(f(s)) - q)ds \\
&\le \int_q^t (f_k(s) - q)ds + \int_t^1 [f_{k-1}((f'(1) - \varepsilon)(s - 1) + 1) - q)ds \\
&= \int_q^t (f_k(s) - q)ds + \frac{1}{f'(1) - \varepsilon} \int_{f(t)}^1 (f_{k-1}(u) - q)du \\
&\le \int_q^t (f_k(s) - q)ds + \frac{1}{f'(1) - \varepsilon} \left[\int_q^t (f_{k-1}(s) - q)ds + \int_t^1 (f_{k-1}(s) - q)ds \right]
\end{aligned}$$

which upon iteration yields

$$\int_q^1 (f_k(s) - q)ds \le \sum_{j=0}^{k} \left(\int_q^t (f_{k-j}(s) - q)ds \right) \left(\frac{1}{f'(1) - \varepsilon} \right)^j + \frac{\int_t^1 (s - q)ds}{(f'(1) - \varepsilon)^k}. \tag{3.5}$$

Now by standard Galton-Watson process theory (e.g. see [AN], pp. 38–40)

$$Q(s) = \lim_{k\to\infty} \frac{f_k(s) - q}{[f'(q)]^k}$$

exists and $Q(s) \in (0, \infty)$, for all $s \in [0, 1)$. Hence by the bounded convergence theorem (since $[f_k(s) - q][f'(q)]^{-k} \le [f_k(t) - q][f'(q)]^{-k}$, for all $q \le s \le t$ and $[f_k(t) - q][f'(q)]^{-k} \to Q(t) < \infty$)

$$\int_q^t (f_i(s) - q)ds \sim [f'(q)]^i, \text{ as } i \to \infty.$$

We then have by (3.5) that for all sufficiently large k,

$$\int_q^1 (f_k(s) - q)ds$$
$$\leq \sum_{j=0}^{[k/2]} \int_q^t (f_{k-j}(s) - q)ds + \sum_{j=[k/2]+1}^{k} (f'(1) - \varepsilon)^{-j} + (f'(1) - \epsilon)^{-k}$$
$$\leq C_1(f'(q))^{k/2} + C_2(f'(1) - \varepsilon)^{-k/2}.$$

Since $q - f_k(0) \sim (f'(q))^k$, the result now follows from (3.4) by setting $\beta = \max\{(f'(q))^{1/2}, (f'(1) - \varepsilon)^{-1/2}\}$. □

3.4 Proposition

$$P\left(\liminf_{k\to\infty} \frac{\log S_k}{k \log m} \geq d | K \neq \emptyset\right) = 1.$$

PROOF: We will need to consider three separate cases.

Case 1: $\Sigma_{j=1}^{m} \log E[N_j] > 0$. Observe that in this case (3.2) reduces to $\lim_j \mu_j^{1/j} = m$.

Let $\epsilon > 0$ be given. By Markov's inequality

(3.6)
$$P(S_k < e^{-\varepsilon k} M_{l_k} \mu_{k-l_k} | K \neq \emptyset)$$
$$= P(M_{l_k}\mu_{k-l_k} - S_k > (1 - e^{-\varepsilon k})\mu_{k-l_k} M_{l_k} | K \neq \emptyset)$$
$$\leq \frac{1}{P(K \neq \emptyset)} E\left[\frac{(S_k - \mu_{k-l_k} M_{l_k})^2}{(1 - e^{-\varepsilon k})^2 M_{l_k}^2 \mu_{k-l_k}^2} 1_{(0,\infty)}(M_{l_k})\right]$$
$$= \frac{1}{P(K \neq \emptyset)} E\left[\frac{E[(S_k - \mu_{k-l_k} M_{l_k})^2 | M_{l_k}]}{(1 - e^{-\varepsilon k})^2 M_{l_k}^2 \mu_{k-l_k}^2} 1_{(0,\infty)}(M_{l_k})\right]$$
$$= \frac{1}{P(K \neq \emptyset)} E\left[\frac{E(S_k^2 | M_{l_k}) - \mu_{k-l_k}^2 M_{l_k}^2}{(1 - e^{-\varepsilon k})^2 M_{l_k}^2 \mu_{k-l_k}^2} 1_{(0,\infty)}(M_{l_k})\right]$$
$$= \frac{1}{P(K \neq \emptyset)} E\left[\frac{E[(\sum_{j=1}^{M_{l_k}} X_j^{(k)})^2 | M_{l_k}] - \mu_{k-l_k}^2 M_{l_k}^2}{(1 - e^{-\varepsilon k})^2 M_{l_k}^2 \mu_{k-l_k}^2} 1_{(0,\infty)}(M_{l_k})\right]$$
$$= \frac{1}{P(K \neq \emptyset)} E\left[\frac{M_{l_k}\sigma_{k-l_k}^2 + M_{l_k}(M_{l_k} - 1)\mu_{k-l_k}^2 - \mu_{k-l_k}^2 M_{l_k}^2}{(1 - e^{-\varepsilon k})^2 M_{l_k}^2 \mu_{k-l_k}^2} 1_{(0,\infty)}(M_{l_k})\right]$$
$$= \frac{1}{P(K \neq \emptyset)} E[M_{l_k}^{-1} 1_{(0,\infty)}(M_{l_k})] \left(\frac{\sigma_{k-l_k}^2}{\mu_{k-l_k}^2} - 1\right) (1 - e^{-\varepsilon k})^{-2},$$

where

$$\sigma_{k-l_k}^2 = E[(X_1^{(k)})^2] = E\left[\left(\sum_{s \in \mathcal{J}_{k-l_k}} 1_{(0,\infty)}(N_s)\right)^2\right].$$

Now since $\sigma^2_{k-l_k} \le m^{2(k-l_k)}$ and since $\mu_j^{1/j} \to m$ in this case,

$$\left(\frac{\sigma^2_{k-l_k}}{\mu^2_{k-l_k}}\right)^{\frac{1}{k-l_k}} \to 1, \text{ as } k \to \infty.$$

By Lemma 3.4 and the estimate (3.6) we then have that

$$\sum_{k=1}^{\infty} P(S_k < e^{-\varepsilon k}\mu_{k-l_k} M_{l_k} | K \neq \emptyset) < \infty.$$

The Borel-Cantelli lemma now implies that

$$P\left(\liminf_{k\to\infty}\{\frac{1}{k}\log\frac{S_k}{\mu_{k-l_k}M_{l_k}}\} \ge 0 | K \neq \emptyset\right) = 1.$$

But by standard branching process theory,

$$\frac{M_{l_k}}{E[M_1]^{l_k}} \to M_\infty, \text{ a.s.}$$

and $P(M_\infty > 0|K \neq \emptyset) = P(M_\infty > 0|M_i \to \infty) = 1$. Hence, given that $K \neq \emptyset$, we have that, with probability 1,

$$\begin{aligned}
&\liminf_{k\to\infty}\frac{\log S_k}{k\log m} \\
&= \liminf_{k\to\infty}\left\{\frac{1}{k}\log\frac{S_k}{\mu_{k-l_k}M_{l_k}}\right\} + \liminf_{k\to\infty}\left\{\frac{1}{k}\frac{\log\mu_{k-l_k}+\log M_{l_k}}{\log m}\right\} \\
&\ge \lim_{k\to\infty}\left\{\frac{1}{k}\frac{\log\mu_{k-l_k}}{\log m}\right\} + \lim_{k\to\infty}\left\{\frac{1}{k}\frac{\log M_{l_k}}{\log m}\right\} = d.
\end{aligned}$$

Case 2: $\Sigma_{j=1}^m \log E[N_j] \ge 0$ and $\Sigma_{j=1}^m E[N_j]\log E[N_j] \le 0$. Observe that in this case $\lim_{j\to\infty}\mu_j^{1/j} = E[M_1]$.

Let again

$$\sigma_k^2 = E\left[\left(\sum_{s\in\mathcal{J}_k} 1_{(0,\infty)}(N_s)\right)^2\right],$$

for $k \in \mathbf{N}$. Then

$$\sigma_k^2 \le E[M_k^2] \le \text{ (constant) } \times (E[M_1])^{2k},$$

for all $k \in \mathbf{N}$, since $\{M_i\}$ is a Galton-Watson process. Consequently, we again have that

$$\left(\frac{\sigma^2_{k-l_k}}{\mu^2_{k-l_k}}\right)^{\frac{1}{k-l_k}} \to 1, \text{ as } k \to \infty$$

and the same argument as in Case 1 can be used.

Case 3: $\Sigma_{j=1}^m \log E[N_j] \leq 0$ but $\Sigma_{j=1}^m E[N_j] \log E[N_j] > 0$.

We will show that given any $\varepsilon > 0$,

$$P\left(\liminf_{k\to\infty} \frac{\log S_k}{k \log m} \geq d - \varepsilon | K \neq \emptyset\right) = 1,$$

which, of course, is sufficient to prove the result.

So fix an arbitrary $\varepsilon > 0$. We will truncate the $X_j^{(k)}$ in a suitable way and work with the truncated random variables.

Recall the braching process with random environments $\{Z_k\}$ introduced in the beginning of this section. It is a process of the type considered in Section 2, with

$$\varphi_j(x) = E[x^{N_j}], \qquad 0 \leq x \leq 1$$

and such that $q_j = P(\zeta_1 = j) = m^{-1}$, for $j \in \mathcal{J}(= \{1, \ldots, m\})$. Now observe that since we assumed $\Sigma_{j=1}^m E[N_j] \log E[N_j] > 0$ and $\Sigma_{j=1}^m \log E[N_j] \leq 0$, we cannot have $\varphi_1'(1) = \ldots = \varphi_m'(1)$ and consequently the function

$$\psi(\theta) = \log E[(\varphi_{\zeta_1}'(1))^\theta], \qquad \theta \in \mathsf{R},$$

is strictly convex in this case. Let, as in Section 2, t be the unique number in $[0, 1]$, with the property $\psi(t) = \min_{\theta \in [0,1]} \psi(\theta)$. Then $\psi'(\theta) > 0, \ \forall\, \theta > t$ and consequently

$$P_\theta(Z_j \to \infty) > 0,$$

for all $\theta > t$, where the probability measures P_θ are the ones introduced in Section 2. Now fix a $\theta > t$. θ will be chosen appropriately at the end of the proof and will depend on ε. Let

$$A = \left\{P_\theta(Z_j \to \infty | \mathcal{F}_\infty) \geq \frac{1}{2} P_\theta(Z_j \to \infty)\right\}$$

and observe that $P_\theta(A) > 0$, since $E_\theta[P_\theta(Z_j \to \infty | \mathcal{F}_\infty)] = P_\theta(Z_j \to \infty) > 0$. Now let

(3.7) $$A_k = \left\{P_\theta(Z_j \to \infty | \mathcal{F}_k) \geq \frac{1}{4} P_\theta(Z_j \to \infty)\right\},$$

for $k \in \mathsf{N}$ and observe that since $P_\theta(Z_j \to \infty | \mathcal{F}_k) \to P_\theta(Z_j \to \infty | \mathcal{F}_\infty)$, a.s., as $k \to \infty$, we must have that for some $k_0 \in \mathsf{N}$,

(3.8) $$P_\theta(A_k) \geq \frac{1}{2} P_\theta(A), \ \ \forall\, k \geq k_0.$$

Set

$$\alpha = \frac{1}{4} P_\theta(Z_j \to \infty), \tag{3.9}$$

$$\gamma = \frac{1}{2} P_\theta(A) \tag{3.10}$$

and for $k \in \mathbb{N}$

$$B_k = \left\{ s \in \mathcal{J}_k : P_\theta(Z_j \to \infty | (\zeta_1, \ldots, \zeta_k) = s) \geq \frac{1}{4} P_\theta(Z_j \to \infty) \right\} \tag{3.11}$$

and observe that

$$P_\theta((\zeta_1, \ldots, \zeta_k) \in B_k) = P_\theta(A_k). \tag{3.12}$$

Finally, define

$$Y_k = \sum_{s \in B_k} 1_{(0,\infty)}(N_s),$$

for $k \in \mathbb{N}$. The random variable Y_k is a truncation of $\Sigma_{s \in \mathcal{J}_k} 1_{(0,\infty)}(N_s)$ and this is the way in which we will truncate the $X_j^{(k)}$'s. Before doing so observe that

$$\begin{aligned} \alpha &= \frac{1}{4} P_\theta(Z_j \to \infty) \leq P_\theta(Z_j \to \infty | (\zeta_1, \ldots, \zeta_k) = s) \\ &\leq P_\theta(Z_k > 0 | (\zeta_1, \ldots, \zeta_k) = s) = P_\theta(N_s > 0 | (\zeta_1, \ldots, \zeta_k) = s) \\ &= P_\theta(N_s > 0) = E_\theta[P(N_s > 0 | \mathcal{F}_\infty)] = P(N_s > 0), \end{aligned}$$

for all $s \in B_k$. Therefore

$$\begin{aligned} \tilde{\sigma}_k^2 := E[Y_k^2] &= E\left[\left(\sum_{s \in B_k} 1_{(0,\infty)}(N_s) \right)^2 \right] \\ &\leq \alpha^{-2} E\left[\left(\sum_{s \in B_k} P(N_s > 0) \right)^2 \right] = \alpha^{-2} (E[Y_k])^2 \end{aligned}$$

Letting $\tilde{\mu}_k = E[Y_k]$, we then have that for each $k \in \mathbb{N}$

$$\frac{\tilde{\sigma}_k^2}{\tilde{\mu}_k^2} \leq \alpha^{-2}.$$

We are now going to truncate the $X_j^{(k)}$. Recall that if $I_1^{(k)}, \ldots, I_{M_{l_k}}^{(k)}$ are the distinct rectangles in K_{l_k}, then $X_j^{(k)}$ is the number of approximate squares $R_k(p,q)$ which are contained in $I_j^{(k)}$ and do contain at least one of the rectangles in K_k. So if $s = (s_1, \ldots, s_{l_k})$ is the unique sequence in $\mathcal{J}_{l_k}$ for which the corresponding horizontal strip (row) contains $I_j^{(k)}$, i.e.,

$$I_j^{(k)} \subseteq [0,1] \times \left[\sum_{j=1}^{l_k} m^{-j}(s_j - 1)\ ,\ \sum_{j=1}^{l_k} m^{-j}(s_j - 1) + m^{-l_k} \right]$$

then

$$X_j^{(k)} = \sum_{t \in \mathcal{J}_{k-l_k}} Z_{j,t},$$

where for $t = (t_1, \ldots, t_{k-l_k}) \in \mathcal{J}_{k-l_k}, Z_{j,t} = 1$, if the part of the row corresponding to $(s_1, \ldots, s_{l_k}, t_1, \ldots, t_{k-l_k})$, that is in $I_j^{(k)}$, contains a rectangle of K_k and $Z_{j,t} = 0$ otherwise; more precisely, $Z_{j,t} = 1$ if

$$I_j^{(k)} \cap \Bigg([0,1] \times \Bigg[\sum_{j=1}^{l_k} m^{-1}(s_j - 1) + \sum_{j=1}^{k-l_k} m^{-(j+l_k)}(t_j - 1), \\ \sum_{j=1}^{l_k} m^{-j}(s_j - 1) + \sum_{j=1}^{k-l_k} m^{-(j+l_k)}(t_j - 1) + m^{-k} \Bigg] \Bigg)$$

contains at least one of the rectangles in K_k. We now truncate $X_j^{(k)}$ by defining

$$Y_j^{(k)} = \sum_{t \in B_{k-l_k}} Z_{j,t}.$$

Clearly $Y_1^{(k)}, \ldots, Y_{M_{l_k}}^{(k)}$ are independent and identically distributed, all having the distribution of Y_{k-l_k} and furthermore $(Y_1^{(k)}, \ldots, Y_{M_{l_k}}^{(k)})$ is independent of M_{l_k}.

Define $\tilde{S}_k = \Sigma_{j=1}^{M_{l_k}} Y_j^{(k)}$. Then, since $Y_j^{(k)} \leq X_j^{(k)}$, for all j, we have that

$$\tilde{S}_k \leq S_k, \qquad k = 1, 2, \ldots\ .$$

Hence to show that

$$P\left(\liminf_{k \to \infty} \frac{\log S_k}{k \log m} \geq d - \varepsilon | K \neq \emptyset \right) = 1,$$

it suffices to show that

$$P\left(\liminf_{k \to \infty} \frac{\log \tilde{S}_k}{k \log m} \geq d - \epsilon | K \neq \emptyset \right) = 1.$$

Using the same arguments as in Case 1, (recall that $\tilde{\sigma}_k^2 \le \alpha^{-2}\tilde{\mu}_k, k = 1, 2, \ldots$) we obtain

$$P\left(\liminf_{k\to\infty}\left\{\frac{1}{k}\log\frac{\tilde{S}_k}{\tilde{\mu}_{k-l_k}M_{l_k}}\right\} \ge 0 | K \ne \emptyset\right) = 1$$

and consequently, given that $K \ne \emptyset$, we have that with probability 1,

$$\liminf_{k\to\infty}\frac{\log\tilde{S}_k}{k\log m} \ge \liminf_{k\to\infty}\left\{\frac{\log\tilde{\mu}_{k-l_k}}{k\log m} + \frac{\log M_{l_k}}{k\log m}\right\}$$
$$= d - \limsup_{k\to\infty}\frac{\log\mu_{k-l_k} - \log\tilde{\mu}_{k-l_k}}{k\log m}$$

We will now show that if we choose θ appropriately (recall $\theta > t$ and the sequence $\{\tilde{\mu}_j\}$ depends on θ) then

$$\limsup_{k\to\infty}\frac{\log\mu_{k-l_k} - \log\tilde{\mu}_{k-l_k}}{k\log m} < \varepsilon \tag{3.13}$$

and this will complete the proof.

Observe that by the strong law of large numbers

$$\left[\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right]^{1/k} \to e^{\psi'(\theta)}, \qquad P_\theta - \text{ a.s.}$$

Hence, if we set $C_k = \{\prod_{i=1}^k \varphi'_{\zeta_i}(1) \le e^{k(\psi'(\theta)+\delta)}\}$, where $\delta > 0$ is arbitrary, then

$$\lim_{k\to\infty} P_\theta(C_k) = 1. \tag{3.14}$$

But then

$$\tilde{\mu}_k = E[Y_k] = \sum_{s\in B_k} P(Z_k > 0 | (\zeta_1, \ldots, \zeta_k) = s)$$
$$= m^k P(Z_k > 0; (\zeta_1, \ldots, \zeta_k) \in B_k)$$
$$= m^k e^{k\psi(\theta)} E_\theta\left[\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right)^{-\theta} 1_{(0,\infty)}(Z_k)\, 1_{B_k}(\zeta_1, \ldots, \zeta_k)\right]$$
$$\ge m^k e^{k\psi(t)} E_\theta\left[\left(\prod_{i=1}^{k}\varphi'_{\zeta_i}(1)\right)^{-\theta} 1\{Z_j > 0, \quad \forall\, j \in \mathbb{N}\}\, 1_{B_k}(\zeta_1, \ldots, \zeta_k)\, 1_{C_k}\right]$$
$$\ge m^k e^{k\psi(t)} e^{-\theta k(\psi'(\theta)+\delta)} P_\theta(Z_j > 0, \quad \forall\, j \in \mathbb{N}; (\zeta_1, \ldots, \zeta_k) \in B_k; C_k)$$
$$\ge m^k e^{k\psi(t)} e^{-\theta k(\psi'(\theta)+\delta)} \times$$
$$\times\, [P_\theta(Z_j > 0, \quad \forall\, j \in \mathbb{N}; (\zeta_1, \ldots, \zeta_k) \in B_k) - (1 - P_\theta(C_k))]$$
$$= m^k e^{k\psi(t)} e^{-\theta k(\psi'(\theta)+\delta)} \times$$
$$\times \left[\sum_{s\in B_k} P_\theta(Z_j \to \infty | (\zeta_1, \ldots, \zeta_k) = s) P_\theta((\zeta_1, \ldots, \zeta_k) = s) - (1 - P_\theta(C_k))\right]$$

(by (3.7), (3.9), (3.11), (3.12))

$$\geq m^k e^{k\psi(t)} e^{-\theta k(\psi'(\theta)+\delta)}[\alpha P_\theta(A_k) - (1 - P_\theta(C_k))]$$

(by (3.8), (3.10))

$$\geq m^k e^{k\psi(t)} e^{-\theta k(\psi'(\theta)+\delta)}[\alpha\gamma - (1 - P_\theta(C_k))]$$

(by 3.14)

$$\geq (me^{\psi(t)})^k e^{-\theta k(\psi'(\theta)+\delta)} \left(\frac{\alpha\gamma}{2}\right)$$

for all sufficiently large k. Since $me^{\psi(t)} = \lim_{j\to\infty} \mu_j^{1/j}$, (3.13) now follows by choosing δ small enough and by choosing θ close enough to t so that $\psi'(\theta)$ is sufficiently small (recall $\psi'(t) = 0$). □

As a by product of the proofs of Propositions 3.2 and 3.4 we also get that

$$P\left(\lim_{k\to\infty}\left\{\frac{1}{k}\log\frac{S_k}{M_{l_k}\mu_{k-l_k}}\right\} = 0|K \neq \emptyset\right) = 1$$

We will only need a weaker fact namely Corollary 3.5, for which we also give a short proof in terms of Proposition 3.4. (So the reader will not have to go through the proofs of Propositions 3.2 and 3.4 again.)

3.5 Corollary

$$P\left(\liminf_{k\to\infty}\left\{\frac{1}{k}\log\frac{S_k}{\mu_{k-l_k}M_{l_k}}\right\} \geq 0|K \neq \emptyset\right) = 1.$$

PROOF: By Proposition 3.4

$$P\left(\liminf_{k\to\infty}\frac{\log S_k}{k\log m} \geq d|K \neq \emptyset\right) = 1.$$

But since $M_{l_k}(E[M_1])^{-l_k} \to M_\infty$, as $k \to \infty$ and $P(M_\infty > 0|K \neq \emptyset) = 1$, we have that

$$P\left(\lim_{k\to\infty}\frac{\log \mu_{k-l_k}M_{l_k}}{k\log m} = d|K \neq \emptyset\right) = 1. \qquad \square$$

3.6 Theorem

The box-counting dimension $\delta(K)$ of K satisfies

$$P(\delta(K) = d|K \neq \emptyset) = 1,$$

where

$$d = \log_m \left(\min_{0 \le \theta \le 1} \left\{ \sum_{j=1}^{m} (E[N_j])^\theta \right\} \right) + \log_n \frac{E[M_1]}{\min_{0\le\theta\le1} \{\Sigma_{j=1}^m (E[N_j])^\theta\}}.$$

PROOF: By Proposition 3.2 it suffices to show that

$$P(\delta(K) \ge d | K \ne \emptyset) = 1. \tag{3.15}$$

Let $R_1^{(k)}, \ldots, R_{S_k}^{(k)}$ be the distinct approximate squares $R_k(p,q)$ that contain at least one of the rectangles of K_k. Let I be one of the $n^{-k} \times m^{-k}$ rectangles of K_k that are contained in $R_j^{(k)}$. Then I will "reproduce" rectangles of size $n^{-(k+1)} \times m^{-(k+1)}$ within it and each of them is going to reproduce rectangles of size $n^{-(k+2)} \times m^{-(k+2)}$ and so on. If M_1^I is the number of $n^{-(k+1)} \times m^{-(k+1)}$ rectangles that I reproduces within it and M_2^I is the number of $n^{-(k+2)} \times m^{-(k+2)}$ rectangles that each of the $n^{-(k+1)} \times m^{-(k+1)}$ rectangles within I reproduces, etc., then $\{M_i^I\}$ is a Galton-Watson process with the same distribution as the original process $\{M_i\}$. In particular $P(M_i^{(I)} \to \infty) = P(M_i \to \infty) =: p$.

Now let $W_j^{(k)} = 1$ if *at least* one of the $n^{-k} \times m^{-k}$ rectangles of K_k, which is within $R_j^{(k)}$, say rectangle I, satisfies $M_i^I \to \infty$, as $i \to \infty$ and let $W_j^{(k)} = 0$ otherwise. Then $p_{j,k} := P(W_j^{(k)} = 1) \ge p$. Define $\tilde{S}_k = 0$ if $S_k = 0$ and

$$\tilde{S}_k = \sum_{j=1}^{S_k} W_j^{(k)},$$

if $S_k > 0$ and observe that the $W_j^{(k)}$ are independent (not necessarily identically distributed) Bernoulli random variables and that $(W_1^{(k)}, \ldots, W_{S_k}^{(k)})$ is independent of S_k.

Now observe that if $\tilde{N}_k$ is the cardinality of a minimal covering of K by approximate squares $R_k(p,q)$ then

$$4\tilde{N}_k \ge \tilde{S}_k. \tag{3.16}$$

The reason for this is the following: an approximate square among $R_1^{(k)}, \ldots, R_{S_k}^{(k)}$, say $R_j^{(k)}$, will be in a minimal covering of K if at least one of the $n^{-k} \times m^{-k}$ rectangles of K_k that it contains, say I, satisfies $M_i^I \to \infty$, as $i \to \infty$, except for the case in which I "reproduces" rectangles that converge to a point $x \in K$ that lies on the boundary of $R_j^{(k)}$; in this case $R_j^{(k)}$ may not be needed in a minimal covering because x may be covered by a neighboring $R_i^{(k)}$. Since an $x \in K$ can only be covered by at most *four* neighboring approximate squares (when x lies in the intersection of 4 different $R_k(p,q)$) we get (3.16).

By (3.16) it now suffices to show that

$$P\left(\liminf_{k\to\infty} \frac{\log \tilde{S}_k}{k \log m} \geq d | K \neq \emptyset\right) = 1, \tag{3.17}$$

in order to show (3.15) and complete the proof. By Proposition 3.4 then, (3.17) will be shown once we have established

$$P\left(\lim_{k\to\infty} \left\{\frac{1}{k} \log \frac{\tilde{S}_k}{S_k}\right\} = 0 | K \neq \emptyset\right) = 1.$$

But since $\tilde{S}_k \leq S_k$, for all $k \in \mathbb{N}$, it is enough to show that for any $\epsilon > 0$

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{\tilde{S}_k < e^{-\varepsilon k} S_k\} | K \neq \emptyset\right) = 0.$$

By Corollary 3.5 however, this is equivalent to

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{\tilde{S}_k < e^{-\varepsilon k} S_k; S_k \geq e^{-\varepsilon k} \mu_{k-l_k} M_{l_k}\} | K \neq \emptyset\right) = 0$$

and this follows from the Borel-Cantelli lemma and Lemma 3.3, since

$$P(\tilde{S}_k < e^{-\varepsilon k} S_k; S_k \geq e^{-\varepsilon k} \mu_{k-l_k} M_{l_k} | K \neq \emptyset) \leq$$

$$\begin{aligned}
&\leq \frac{1}{P(K \neq \emptyset)} P\left(\sum_{j=1}^{S_k} p_{j,k} - \tilde{S}_k > \sum_{j=1}^{S_k} p_{j,k} - e^{-\varepsilon k} S_k; S_k \geq e^{-\varepsilon k} \mu_{k-l_k} M_{l_k}; S_k > 0\right) \\
&\leq \frac{1}{P(K \neq \emptyset)} P\left(\sum_{j=1}^{S_k} p_{j,k} - \tilde{S}_k > (p - e^{-\varepsilon k}) S_k; S_k \geq e^{-\varepsilon k} \mu_{k-l_k} M_{l_k}; S_k > 0\right) \\
&\leq \frac{(p - e^{-\varepsilon k})^{-2}}{P(K \neq \emptyset)} E\left[\frac{E[(\tilde{S}_k - \sum_{j=1}^{S_k} p_{j,k})^2 | \mathcal{G}_k]}{S_k^2} 1\left\{S_k \geq \frac{\mu_{k-l_k} M_{l_k}}{e^{\varepsilon k}}\right\} 1\{S_k > 0\}\right] \\
&\leq \frac{(p - e^{-\varepsilon k})^{-2}}{P(K \neq \emptyset)} E\left[S_k^{-1} 1\{S_k \geq e^{-\varepsilon k} \mu_{k-l_k} M_{l_k}\} 1\{S_k > 0\}\right] \\
&\leq \frac{(p - e^{-\varepsilon k})^{-2}}{P(K \neq \emptyset)} \frac{e^{\varepsilon k}}{\mu_{k-l_k}} E[M_{l_k}^{-1} 1\{S_k > e^{-\varepsilon k} \mu_{k-l_k} M_{l_k}\} 1\{S_k > 0\}] \\
&\leq \frac{(p - e^{-\varepsilon k})^{-2}}{P(K \neq \emptyset)} \frac{e^{\varepsilon k}}{\mu_{k-l_k}} E[M_{l_k}^{-1} 1_{(0,\infty)}(M_{l_k})]
\end{aligned}$$

for all sufficiently large k, where

$$\mathcal{G}_k = \sigma(M_1, \ldots, M_k; S_1, \ldots, S_k). \qquad \square$$

4. The Dimension of the Projections of K

We remark that with the results of Section 2 about the branching process $\{Z_k\}$, introduced in the beginning of Section 3, one can easily obtain the dimension of $\mathrm{proj}_2(K)$, the projection of K onto the y-axis. In particular one can use the arguments in the proof of the Theorem on page 69 in $[\mathrm{F}]_2$ to show that

$$\begin{aligned} \dim_H(\mathrm{proj}_2 K) = \delta(\mathrm{proj}_2 K) &= \lim_{k\to\infty} \left\{ \frac{1}{k} \log_m \mu_k \right\} \\ &= \log_m \left(\min_{0\le\theta\le 1} \left\{ \sum_{j=1}^{m} (E[N_j])^\theta \right\} \right), \end{aligned}$$

with probability one, given of course that $K \neq \emptyset$; here $\dim_H(\mathrm{proj}_1 K)$ and $\delta(\mathrm{proj}_2 K)$ denote the Hausdorff and box-counting dimensions of $\mathrm{proj}_2 K$, respectively.

Finally, observe that similar considerations apply to $\mathrm{proj}_1 K$, the projection of K onto the x-axis.

REFERENCES

$[\mathrm{AK}]_1$ K.B. Athreya, and S. Karlin: On branching processes in random enviroments: I. Extinction probability, *Ann. Math. Stat.*, **42** (1971), 1499–1520.

$[\mathrm{AK}]_2$ K.B. Athreya, and S. Karlin: On branching processes in random environments: II. Limit theorems, *Ann. Math. Stat.*, **42** (1971), 1843–1858.

[AN] K.B. Athreya, and P.E. Ney: Branching processes, *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstelung*, **196**, Springer-Verlag, New York, Heidelberg Berglin (1972).

[Be] T. Bedford: On Weierstrass-like functions and random recurrent sets, *Math. Proc. Camb. Phil. Soc.*, **106** (1989), 325–342.

[CCD] J.T. Chayes, L. Chayes, and R. Durrett: Connecticivity properties of Mandelbrot's percolation process, *Prob. Theory and Rel. Fields*, **77** (1988), 307–324.

[De] F.M. Dekking: On the survival probability of a branching process in a finite state i.i.d. random environment, *Stoch. Proc. and Appl.*, **27** (1988), 151–157.

[DG] F.M. Dekking, and G.R. Grimmett: Superbranching processes and projections of random Cantor sets, *Prob. Theory and Rel. Fields*, **78** (1988), 335–355.

[DM] F.M. Dekking, and R.W. Meester: On the structure of Mandelbrot's percolation process and other random Cantor sets, *J. of Stat. Physics*, **58** (1990), 1109–1126.

[Fa] K.J. Falconer: The Hausdorff dimension of self-affine fractals, *Math. Proc. Camb. Phil. Soc.*, **103** (1988), 339–350.

$[F]_1$ K.J. Falconer: Random Fractals, *Math. Proc. Camb. Phil. Soc.*, **100** (1986), 559–582.

$[F]_2$ K.J. Falconer: Projections of random Cantor sets, *J. of Theor. Prob.*, **2** (1989), 65–70.

[Ke] H. Kesten: Supercritical branching processes with countably many types and the size or random Cantor sets, *Probability, Statistics and Mathematics, Papers in honor of Samuel Karlin*, Academic Press (1989).

[LG] S.P. Lalley, and D. Gatzouras: Hausdorff and box dimensions of certain self-affine fractals, *Indiana Univ. Math. J.*, to appear.

[Ma] B. Mandelbrot: The Fractal Geometry of Nature, Freeman, San Francisco (1983).

[Mc] C. McMullen: The Hausdorff dimension of general Sierpinski Carpets, *Nagoya Math J.*, **96** (1984), 1–9.

[MW] R.D. Mauldin, and S.C. Williams: Random recursive constructions: Asymptotic geometric and topological properties, *Trans. of the AMS*, **295** (1986), 325–346.

[MGW] R.D. Mauldin, S. Graff, and S.C. Williams: Exact Hausdorff dimension in random recursive constructions, **Memoir #381**, *AMS* (1987); see also *Proceedings of the National Academy of Sciences*, **84** (1987), 3959–3961.

[SW] W.L. Smith, and W.E. Williams: On branching processes in random environments, *Ann. Math. Stat.*, **40** (1969), 814–827.