HIGHER ORDER MOMENTS OF ORDER STATISTICS FROM EXPONENTIAL AND RIGHT-TRUNCATED EXPONENTIAL DISTRIBUTIONS AND APPLICATIONS TO LIFE-TESTING PROBLEMS

by

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Higher Order Moments of Order Statistics From Exponential and Right-truncated Exponential Distributions and Applications to Life-testing Problems

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Abstract

In this paper we first establish several recurrence relations satisfied by the single, double, triple and quadruple moments of order statistics from the standard exponential distribution. We show that these relations will enable one to find all moments (of order up to 4) of order statistics from all sample sizes in a simple recursive way. These results may, therefore, be used to determine the mean, variance and coefficients of skewness and kurtosis of a general linear function of exponential order statistics. We demonstrate this by considering the best linear unbiased estimator of the mean life–time based on doubly Type–II censored samples and justify a chi–square approximation for its distribution.

Next, we consider a right–truncated exponential distribution and derive similar recurrence relations for the single, double, triple and quadruple moments of order statistics. Finally, we consider two examples from life–testing situation and illustrate an important application of the results derived in this paper.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a standard exponential population with probability density function

$$f(x) = e^{-x},\ 0 \le x < \infty, \tag{1.1}$$

and cumulative distribution function

$$F(x) = 1 - e^{-x},\ 0 \le x < \infty. \tag{1.2}$$

It is important to observe here that

$$f(x) = 1 - F(x),\ 0 \le x < \infty. \tag{1.3}$$

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics obtained by arranging the above sample in increasing order of magnitude. Let us denote the single moments $E\left[X_{r:n}^{a}\right]$ by $\mu_{r:n}^{(a)}$ for $1 \le r \le n$ and $a \ge 1$, the double moments $E\left[X_{r:n}^{a}\, X_{s:n}^{b}\right]$ by $\mu_{r,s:n}^{(a,b)}$ for $1 \le r < s \le n$ and $a,b \ge 1$, the triple moments $E\left[X_{r:n}^{a}\, X_{s:n}^{b}\, X_{t:n}^{c}\right]$ by $\mu_{r,s,t:n}^{(a,b,c)}$ for $1 \le r < s < t \le n$ and $a,b,c \ge 1$, and the quadruple moments $E\left[X_{r:n}^{a}\, X_{s:n}^{b}\, X_{t:n}^{c}\, X_{u:n}^{d}\right]$ by $\mu_{r,s,t,u:n}^{(a,b,c,d)}$ for $1 \le r < s < t < u \le n$ and $a,b,c,d \ge 1$.

It is well known that the normalized spacings

$S_1 = nX_{1:n}$, $S_2 = (n-1)(X_{2:n} - X_{1:n}), \ldots, S_n = X_{n:n} - X_{n-1:n}$

are all statistically independent and have standard exponential distributions; see Sukhatme (1937). This result, as pointed out by David (1981) and Balakrishnan and Cohen (1990), will enable one to express the r^{th} order statistic in a sample of size n as a linear function of r independent standard exponential random variables and thus to prove that the exponential order statistics form an additive Markov chain. One may use this representation to derive exact and explicit expressions for moments of exponential order statistics.

Alternatively, by using the relation in (1.3) Joshi (1978, 1982) derived some simple recurrence relations satisfied by the single and product moments of order statistics from a standard exponential distribution. Joshi pointed out that these recurrence relations are very easy to use and that one could write a simple computer program to evaluate the first and second single moments and the product moments of all order statistics without introducing serious rounding errors, at least up to moderately large sample sizes. These results have been generalized by Joshi (1979, 1982) and Balakrishnan and Joshi (1984) for the case when the order statistics arise from right–truncated and doubly truncated exponential distributions. All these results for the exponential and truncated exponential distributions and also similar results for some other distributions including the power function, Pareto, logistic, half logistic, log–logistic and truncated logistic are presented in the recent monograph by Arnold and Balakrishnan (1989) on this topic of research.

In this paper, we derive several recurrence relations satisfied by the single, double, triple and quadruple moments of order statistics from the standard exponential distribution in Sections 2 – 5, respectively. In Section 6, we consider the scale–parameter exponential distribution and use the results derived in Sections 2 – 5 to determine the mean, variance and coefficients of skewness and kurtosis of the best linear unbiased estimator of the scale parameter based on doubly Type–II censored samples. While it is well known that for the case when the available sample is Type–II right censored the best linear unbiased estimator of the scale parameter has exactly a chi–square distribution, we show in Section 6 that a chi–square distribution provides a very close approximation to the distribution of the best linear unbiased estimator even when the available sample is doubly Type–II censored. In Section 7, we generalize all the results presented in Sections 2 – 5 to the case when the order statistics arise from a right–truncated exponential distribution.

Finally, in Section 8 we consider two examples given by Lawless (1982) involving life–testing data and illustrate the usefulness of the chi–square approximation for the distribution of the best linear unbiased estimator discussed in Section 6 by constructing approximate confidence intervals for the mean life–time.

2. Relations for Single Moments

The probability density function of $X_{r:n}$ is (see David, 1981, pp. 9) given by

$$f_{r:n}(w) = \frac{n!}{(r-1)!\,(n-r)!}\left[F(w)\right]^{r-1}\left[1-F(w)\right]^{n-r} f(w),\ 0 \le w < \infty,\qquad 1 \le r \le n, \tag{2.1}$$

using which the single moments of $X_{r:n}$ may be computed by

$$\mu_{r:n}^{(a)} = \int_0^\infty w^a f_{r:n}(w)\,dw, \qquad a = 1,2,\ldots,\ 1 \le r \le n. \tag{2.2}$$

Then, the single moments $\mu_{r:n}^{(a)}$ satisfy the recurrence relations presented in the following two theorems.

__Theorem 1__: For $n \ge 1$ and $a = 1,2,\ldots,$

$$\mu_{1:n}^{(a)} = a\,\mu_{1:n}^{(a-1)}/n \tag{2.3}$$

with $\mu_{1:n}^{(0)} \equiv 1$.

__Proof__: From Eqs. (2.1) and (2.2), for $n \ge 1$ and $a \ge 1$ let us consider

$$\mu_{1:n}^{(a-1)} = n\int_0^\infty w^{a-1}\left[1-F(w)\right]^{n-1} f(w)\,dw$$

$$= \frac{n}{a}\int_0^\infty \left[1-F(w)\right]^{n} d(w^a) \tag{2.4}$$

upon using the relation in (1.3). The recurrence relation in Eq. (2.3) follows upon integrating the right hand side of (2.4) by parts and simplifying the resulting expression.

<u>**Theorem 2**</u>: For $n \geq 2$, $2 \leq r \leq n$ and $a = 1,2,\ldots$,

$$\mu_{r:n}^{(a)} = \mu_{r-1:n}^{(a)} + a\,\mu_{r:n}^{(a-1)}/(n - r + 1) \tag{2.5}$$

with $\mu_{r:n}^{(0)} \equiv 1$.

<u>Proof</u>: From Eqs. (2.1) and (2.2), for $n \geq 2$, $2 \leq r \leq n$ and $a \geq 1$ let us consider

$$\mu_{r:n}^{(a-1)} = \frac{n!}{(r-1)!(n-r)!} \int_0^\infty w^{a-1} \left[F(w)\right]^{r-1} \left[1 - F(w)\right]^{n-r} f(w)\,dw$$

$$= \frac{n!}{(r-1)!(n-r)!a}\, I, \tag{2.6}$$

where I is the integral

$$I = \int_0^\infty \left[F(w)\right]^{r-1} \left[1 - F(w)\right]^{n-r+1} d(w^a)$$

upon using the relation in (1.3). Integration by parts now yields

$$I = (n - r + 1) \int_0^\infty w^a \left[F(w)\right]^{r-1} \left[1 - F(w)\right]^{n-r} f(w)\,dw$$

$$- (r - 1) \int_0^\infty w^a \left[F(w)\right]^{r-2} \left[1-F(w)\right]^{n-r+1} f(w)\,dw$$

$$= \frac{(r-1)!(n-r+1)!}{n!} \left\{\mu_{r:n}^{(a)} - \mu_{r-1:n}^{(a)}\right\}.$$

Upon substituting this expression of I in (2.6) and simplifying the resulting equation, we derive the recurrence relation in Eq. (2.5). It should be mentioned here that Theorems 1 and 2 have been proved by Joshi (1978) and have been presented here for the sake of completeness and a better understanding of the results derived in subsequent sections. The recurrence relations presented in Theorems 1 and 2 will enable one to compute all the single moments of order statistics for all sample sizes in a simple recursive way.

<u>3. Relations for Double Moments</u>

The joint density function of $X_{r:n}$ and $X_{s:n}$ is (see David, 1981, pp. 10) given by

$$f_{r,s:n}(w,x) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-1} \left[1-F(x)\right]^{n-s} f(w)f(x),$$

$$0 \leq w < x < \infty,\ 1 \leq r < s \leq n, \tag{3.1}$$

using which we may compute the double moments as

$$\mu_{r,s:n}^{(a,b)} = \iint_{0\le w<x<\infty} w^a x^b f_{r,s:n}(w,x)\,dw\,dx,\ 1 \le r < s \le n,\ a,b \ge 1. \qquad (3.2)$$

Then, the double moments $\mu_{r,s:n}^{(a,b)}$ satisfy the recurrence relations presented in the following two theorems.

<u>Theorem 3:</u> For $1 \le r \le n-1$ and $a,b = 1,2,\ldots,$

$$\mu_{r,r+1:n}^{(a,b)} = \mu_{r:n}^{(a+b)} + b\,\mu_{r,r+1:n}^{(a,b-1)}/(n-r); \qquad (3.3)$$

and for $1 \le r < s \le n$, $s - r \ge 2$ and $a,b = 1,2,\ldots,$

$$\mu_{r,s:n}^{(a,b)} = \mu_{r,s-1:n}^{(a,b)} + b\,\mu_{r,s:n}^{(a,b-1)}/(n-s+1), \qquad (3.4)$$

where $\mu_{r,s:n}^{(a,0)} \equiv \mu_{r:n}^{(a)}$.

<u>Proof:</u> From Eqs. (3.1) and (3.2), for $1 \le r < s \le n$ and $a,b \ge 1$ let us consider

$$\mu_{r,s:n}^{(a,b-1)} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!b} \int_0^\infty w^a \left[F(w)\right]^{r-1} f(w)\,I(w)\,dw, \qquad (3.5)$$

where

$$I(w) = \int_w^\infty \left[F(x) - F(w)\right]^{s-r-1} \left[1 - F(x)\right]^{n-s+1} d(x^b)$$

upon using the relation in (1.3). Integrating by parts, we obtain for $s = r + 1$ that

$$I(w) = (n-r) \int_w^\infty x^b \left[1 - F(x)\right]^{n-r-1} f(x)dx - x^b\left[1 - F(x)\right]^{n-r},$$

and for $s - r \ge 2$ that

$$I(w) = (n-s+1) \int_w^\infty x^b\left[F(x) - F(w)\right]^{s-r-1} \left[1 - F(x)\right]^{n-s} f(x)\,dx.$$

$$- (s-r-1) \int_w^\infty x^b\left[F(x) - F(w)\right]^{s-r-2} \left[1 - F(x)\right]^{n-s+1} f(x)\,dx.$$

Upon substituting the above expressions of I(w) in Eq. (3.5) and simplifying the resulting equations, we derive the recurrence relations in Eqs. (3.3) and (3.4).

Theorem 4: For $n \geq 2$ and $a,b = 1,2,\ldots,$

$$\mu_{1,2:n}^{(a,b)} = \mu_{2:n}^{(a+b)} + a\,\mu_{1,2:n}^{(a-1,b)} - n\,\mu_{1:n-1}^{(a+b)}; \tag{3.6}$$

for $n \geq 3$, $2 \leq r \leq n-1$ and $a,b = 1,2,\ldots,$

$$\mu_{r,r+1:n}^{(a,b)} = \mu_{r+1:n}^{(a+b)} + \frac{1}{r}\left[a\,\mu_{r,r+1:n}^{(a-1,b)} - n\left\{\mu_{r:n-1}^{(a+b)} - \mu_{r-1,r:n-1}^{(a,b)}\right\}\right]; \tag{3.7}$$

for $n \geq 3$, $3 \leq s \leq n$ and $a,b = 1,2,\ldots,$

$$\mu_{1,s:n}^{(a,b)} = \mu_{2,s:n}^{(a,b)} + a\,\mu_{1,s:n}^{(a-1,b)} - n\mu_{1,s-1:n-1}^{(a,b)}; \tag{3.8}$$

and for $n \geq 4$, $2 \leq r < s \leq n$, $s - r \geq 2$ and $a,b = 1,2,\ldots,$

$$\mu_{r,s:n}^{(a,b)} = \mu_{r+1,s:n}^{(a,b)} + \frac{1}{r}\left[a\,\mu_{r,s:n}^{(a-1,b)} - n\left\{\mu_{r,s-1:n-1}^{(a,b)} - \mu_{r-1,s-1:n-1}^{(a,b)}\right\}\right], \tag{3.9}$$

where $\mu_{r,s:n}^{(0,b)} \equiv \mu_{s:n}^{(b)}$.

Proof: From Eqs. (3.1) and (3.2), for $1 \leq r < s \leq n$ and $a,b \geq 1$ let us consider

$$\mu_{r,s:n}^{(a-1,b)} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!a} \int_0^\infty x^b \left[1 - F(x)\right]^{n-s} f(x)\, J(x)dx, \tag{3.10}$$

where

$$\begin{aligned} J(x) &= \int_0^x \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-1} f(w)\, d(w^a) \\ &= \int_0^x \left[F(w)\right]^{r-1}\left[F(x) - F(w)\right]^{s-r-1} d(w^a) - \int_0^x \left[F(w)\right]^{r}\left[F(x) - F(w)\right]^{s-r-1} d(w^a) \end{aligned}$$

upon using the relation in (1.3). Integrating by parts, we obtain for $r = 1$ and $s = 2$ that

$$J(x) = x^a - x^a F(x) + \int_0^x w^a f(w)\, dw,$$

for $r \geq 2$ and $s = r + 1$ that

$$J(x) = x^a \left[F(x)\right]^{r-1} - (r-1) \int_0^x w^a \left[F(w)\right]^{r-2} f(w)\, dw$$

$$- x^a\left[F(x)\right]^r + r\int_0^x w^a\left[F(w)\right]^{r-1} f(w)\, dw,$$

for r = 1 and s ≥ 3 that

$$J(x) = (s-2)\int_0^x w^a \left[F(x) - F(w)\right]^{s-3} f(w)dw + \int_0^x w^a\left[F(x) - F(w)\right]^{s-2} f(w)\, dw$$

$$- (s-2)\int_0^x w^a\, F(w) \left[F(x) - F(w)\right]^{s-3} f(w)\, dw,$$

and for r ≥ 2 and s − r ≥ 2 that

$$J(x) = (s-r-1)\int_0^x w^a \left[F(w)\right]^{r-1}\left[F(x) - F(w)\right]^{s-r-2} f(w)\, dw$$

$$- (r-1)\int_0^x w^a \left[F(w)\right]^{r-2}\left[F(x) - F(w)\right]^{s-r-1} f(w)\, dw$$

$$+ r\int_0^x w^a \left[F(w)\right]^{r-1}\left[F(x) - F(w)\right]^{s-r-1} f(w)\, dw$$

$$- (s-r-1)\int_0^x w^a \left[F(w)\right]^{r}\left[F(x) - F(w)\right]^{s-r-2} f(w)\, dw.$$

Upon substituting the above expressions of J(x) in Eq. (3.10) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (3.6) – (3.9).

Either the recurrence relations presented in Theorem 3 or those presented in Theorem 4 may be used in a simple systematic recursive way to compute the double moments of order statistics (of all order) for all sample sizes. The relations in Theorem 3 have been proved by Joshi (1982) for the special case when a = b = 1.

4. Relations for Triple Moments

The joint density function of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is given by

$$f_{r,s,t:n}(w,x,y) = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-1}$$
$$\left[F(y) - F(x)\right]^{t-s-1} \left[1 - F(y)\right]^{n-t} f(w)\, f(x)\, f(y),$$
$$0 \le w < x < y < \infty,\ 1 \le r < s < t \le n, \tag{4.1}$$

using which we may compute the triple moments as

$$\mu_{r,s,t:n}^{(a,b,c)} = \underset{0\le w<x<y<\infty}{\int\int\int} w^a x^b y^c f_{r,s,t:n}(w,x,y)\, dw\, dx\, dy,$$
$$1 \le r < s < t \le n,\ a,b,c \ge 1. \tag{4.2}$$

Then, the triple moments $\mu_{r,s,t:n}^{(a,b,c)}$ satisfy the recurrence relations presented in the following three theorems.

__Theorem 5__: For $n \ge 3$, $1 \le r < s \le n-1$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,s,s+1:n}^{(a,b,c)} = \mu_{r,s:n}^{(a,b+c)} + c\, \mu_{r,s,s+1:n}^{(a,b,c-1)}/(n-s); \tag{4.3}$$

and for $n \ge 4$, $1 \le r < s < t \le n$, $t - s \ge 2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,s,t:n}^{(a,b,c)} = \mu_{r,s,t-1:n}^{(a,b,c)} + c\, \mu_{r,s,t:n}^{(a,b,c-1)}/(n-t+1), \tag{4.4}$$

where $\mu_{r,s,t:n}^{(a,b,0)} \equiv \mu_{r,s:n}^{(a,b)}$.

__Proof__: From Eqs. (4.1) and (4.2), for $1 \le r < s < t \le n$ and $a,b,c \ge 1$ let us consider

$$\mu_{r,s,t:n}^{(a,b,c-1)} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!c} \int_0^\infty \int_w^\infty w^a x^b \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-1}$$
$$I(x)\, f(w) f(x) dx\, dw, \tag{4.5}$$

where

$$I(x) = \int_x^\infty \left[F(y) - F(x)\right]^{t-s-1} \left[1 - F(y)\right]^{n-t} f(y)\, d(y^c)$$
$$= \int_x^\infty \left[F(y) - F(x)\right]^{t-s-1} \left[1 - F(y)\right]^{n-t+1} d(y^c)$$

upon using the relation in (1.3). Integrating by parts, we obtain for $t = s + 1$ that

$$I(x) = (n-s) \int_x^\infty y^c \left\{1 - F(y)\right\}^{n-s-1} f(y) dy - y^c \left\{1 - F(y)\right\}^{n-s},$$

and for $t - s \ge 2$ that

$$I(x) = (n-t+1)\int_x^\infty y^c\left\{F(y)-F(x)\right\}^{t-s-1}\left\{1-F(y)\right\}^{n-t} f(y)\,dy$$

$$-\,(t-s-1)\int_x^\infty y^c\left\{F(y)-F(x)\right\}^{t-s-2}\left\{1-F(y)\right\}^{n-t+1} f(y)dy.$$

Upon substituting the above expressions of I(x) in Eq. (4.5) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (4.3) and (4.4).

Theorem 6: For $n \geq 3$, $3 \leq t \leq n$ and $a,b,c = 1,2,\ldots,$

$$\mu^{(a,b,c)}_{1,2,t:n} = \mu^{(a+b,c)}_{2,t:n} + a\,\mu^{(a-1,b,c)}_{1,2,t:n} - n\mu^{(a+b,c)}_{1,t-1:n-1}; \tag{4.6}$$

for $n \geq 4$, $2 \leq r < t \leq n$, $t - r \geq 2$ and $a,b,c = 1,2,\ldots,$

$$\mu^{(a,b,c)}_{r,r+1,t:n} = \mu^{(a+b,c)}_{r+1,t:n} + \frac{1}{r}\left[a\mu^{(a-1,b,c)}_{r,r+1,t:n} - n\left\{\mu^{(a+b,c)}_{r,t-1:n-1} - \mu^{(a,b,c)}_{r-1,r,t-1:n-1}\right\}\right]; \tag{4.7}$$

for $n \geq 4$, $3 \leq s < t \leq n$ and $a,b,c = 1,2,\ldots,$

$$\mu^{(a,b,c)}_{1,s,t:n} = \mu^{(a,b,c)}_{2,s,t:n} + a\,\mu^{(a-1,b,c)}_{1,s,t:n} - n\,\mu^{(a,b,c)}_{1,s-1,t-1:n-1}; \tag{4.8}$$

and for $n \geq 5$, $2 \leq r < s < t \leq n$, $s - r \geq 2$ and $a,b,c = 1,2,\ldots,$

$$\mu^{(a,b,c)}_{r,s,t:n} = \mu^{(a,b,c)}_{r+1,s,t:n} + \frac{1}{r}\left[a\,\mu^{(a-1,b,c)}_{r,s,t:n} - n\left\{\mu^{(a,b,c)}_{r,s-1,t-1:n-1} - \mu^{(a,b,c)}_{r-1,s-1,t-1:n-1}\right\}\right], \tag{4.9}$$

where $\mu^{(0,b,c)}_{r,s,t:n} \equiv \mu^{(b,c)}_{s,t:n}$.

Proof: From Eqs. (4.1) and (4.2), for $1 \leq r < s < t \leq n$ and $a,b,c \geq 1$ let us write

$$\mu^{(a-1,b,c)}_{r,s,t:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!a}\int_0^\infty\int_0^y x^b y^c\left[F(y)-F(x)\right]^{t-s-1}\left[1-F(y)\right]^{n-t} J(x)\,f(x)\,f(y)\,dx\,dy, \tag{4.10}$$

where

$$J(x) = \int_0^x \left[F(w)\right]^{r-1}\left[F(x)-F(w)\right]^{s-r-1} f(w)\,d(w^a)$$

$$= \int_0^x \left[F(w)\right]^{r-1}\left[F(x)-F(w)\right]^{s-r-1} d(w^a) - \int_0^x \left[F(w)\right]^{r}\left[F(x)-F(w)\right]^{s-r-1} d(w^a)$$

upon using the relation in (1.3). Integrating by parts, we obtain for r = 1 and s = 2 that

$$J(x) = x^a - x^a F(x) + \int_0^x w^a f(w)\, dw,$$

for r ≥ 2 and s = r + 1 that

$$J(x) = x^a\left[F(x)\right]^{r-1} - (r-1)\int_0^x w^a\left[F(w)\right]^{r-2} f(w)\, dw$$
$$- x^a\left[F(x)\right]^{r} + r\int_0^x w^a\left[F(w)\right]^{r-1} f(w)\, dw,$$

for r = 1 and s ≥ 3 that

$$J(x) = (s-2)\int_0^x w^a\left[F(x) - F(w)\right]^{s-3} f(w)\, dw + \int_0^x w^a\left[F(x) - F(w)\right]^{s-2} f(w) dw$$
$$- (s-2)\int_0^x w^a\, F(w)\left[F(x) - F(w)\right]^{s-3} f(w)\, dw,$$

and for r ≥ 2 and s − r ≥ 2 that

$$J(x) = (s-r-1)\int_0^x w^a\left[F(w)\right]^{r-1}\left[F(x) - F(w)\right]^{s-r-2} f(w)\, dw$$
$$- (r-1)\int_0^x w^a\left[F(w)\right]^{r-2}\left[F(x) - F(w)\right]^{s-r-1} f(w)\, dw$$
$$+ r\int_0^x w^a\left[F(w)\right]^{r-1}\left[F(x) - F(w)\right]^{s-r-1} f(w)\, dw$$
$$- (s-r-1)\int_0^x w^a\left[F(w)\right]^{r}\left[F(x) - F(w)\right]^{s-r-2} f(w)\, dw.$$

Upon substituting the above expressions of J(x) in Eq. (4.10) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (4.6) – (4.9).

<u>Theorem 7</u>: For n ≥ 3, 1 ≤ r ≤ n − 2 and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,r+1,r+2:n} = \mu^{(a+b,c)}_{r,r+2:n} + b\mu^{(a,b-1,c)}_{r,r+1,r+2:n} - (n-r-1)\left\{\mu^{(a,b+c)}_{r,r+1:n} - \mu^{(a+b,c)}_{r,r+1:n}\right\}; \tag{4.11}$$

for $n \geq 4$, $1 \leq r < t \leq n$, $t - r \geq 3$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,r+1,t:n} = \mu^{(a+b,c)}_{r,t:n} + \frac{1}{t-r-1}\left[b\mu^{(a,b-1,c)}_{r,r+1,t:n} - (n-t+1)\left\{\mu^{(a,b,c)}_{r,r+1,t-1:n} - \mu^{(a+b,c)}_{r,t-1:n}\right\}\right]; \tag{4.12}$$

for $n \geq 4$, $1 \leq r < s \leq n-1$, $s - r \geq 2$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,s,s+1:n} = \mu^{(a,b,c)}_{r,s-1,s+1:n} + b\mu^{(a,b-1,c)}_{r,s,s+1:n} - (n-s)\left\{\mu^{(a,b+c)}_{r,s:n} - \mu^{(a,b,c)}_{r,s-1,s:n}\right\}; \tag{4.13}$$

and for $n \geq 5$, $1 \leq r < s < t \leq n$, $s - r \geq 2$, $t - s \geq 2$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,s,t:n} = \mu^{(a,b,c)}_{r,s-1,t:n} + \frac{1}{t-s}\left[b\,\mu^{(a,b-1,c)}_{r,s,t:n} - (n-t+1)\left\{\mu^{(a,b,c)}_{r,s,t-1:n} - \mu^{(a,b,c)}_{r,s-1,t-1:n}\right\}\right], \tag{4.14}$$

where $\mu^{(a,0,c)}_{r,s,t:n} \equiv \mu^{(a,c)}_{r,t:n}$.

<u>Proof</u>: From Eqs. (4.1) and (4.2), for $1 \leq r < s < t \leq n$ and $a,b,c \geq 1$ let us write

$$\mu^{(a,b-1,c)}_{r,s,t:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!b}\int_0^\infty\int_0^y w^a y^c \left[F(w)\right]^{r-1}\left[1-F(y)\right]^{n-t} K(w,y)\, f(w)\, f(y)\, dw\, dy, \tag{4.15}$$

where

$$\begin{aligned} K(w,y) &= \int_w^y \left[F(x)-F(w)\right]^{s-r-1}\left[F(y)-F(x)\right]^{t-s-1} f(x)\, d(x^b) \\ &= \int_w^y \left[F(x)-F(w)\right]^{s-r-1}\left[F(y)-F(x)\right]^{t-s} d(x^b) \\ &\quad + \int_w^y \left[F(x)-F(w)\right]^{s-r-1}\left[F(y)-F(x)\right]^{t-s-1}\left[1-F(y)\right] d(x^b) \end{aligned}$$

upon using the relation in (1.3) and then writing $1 - F(x)$ as $\left[F(y)-F(x)\right] + \left[1-F(y)\right]$.

Integrating by parts now, we obtain for $s = r + 1$ and $t = r + 2$ that

$$K(w,y) = -w^b\left[F(y)-F(w)\right] + \int_w^y x^b f(x)\, dx + y^b\left[1-F(y)\right] - w^b\left[1-F(y)\right],$$

for $s = r + 1$ and $t - r \geq 3$ that

$$K(w,y) = -w^b\left[F(y) - F(w)\right]^{t-r-1} + (t-r-1)\int_w^y x^b\left[F(y) - F(x)\right]^{t-r-2} f(x)dx$$

$$-w^b\left[F(y) - F(w)\right]^{t-r-2}\left[1 - F(y)\right] + (t-r-2)\int_w^y x^b\left[F(y) - F(x)\right]^{t-r-3}\left[1 - F(y)\right]f(x)dx,$$

for $s - r \geq 2$ and $t = s + 1$ that

$$K(w,y) = \int_w^y x^b\left[F(x) - F(w)\right]^{s-r-1} f(x)dx - (s-r-1)\int_w^y x^b\left[F(x) - F(w)\right]^{s-r-2}\left[F(y) - F(x)\right]f(x)dx$$

$$+ y^b\left[F(y) - F(w)\right]^{s-r-1}\left[1-F(y)\right] - (s-r-1)\int_w^y x^b\left[F(x) - F(w)\right]^{s-r-2}\left[1 - F(y)\right] f(x)dx,$$

and for $s - r \geq 2$ and $t - s \geq 2$ that

$$K(w,y) = (t-s)\int_w^y x^b\left[F(x) - F(w)\right]^{s-r-1}\left[F(y) - F(x)\right]^{t-s-1} f(x)dx$$

$$- (s-r-1)\int_w^y x^b\left[F(x) - F(w)\right]^{s-r-2}\left[F(y) - F(x)\right]^{t-s} f(x)dx$$

$$+ (t-s-1)\int_w^y x^b\left[F(x) - F(w)\right]^{s-r-1}\left[F(y) - F(x)\right]^{t-s-2}\left[1 - F(y)\right] f(x)dx$$

$$- (s-r-1)\int_w^y x^b\left[F(x) - F(w)\right]^{s-r-2}\left[F(y) - F(x)\right]^{t-s-1}\left[1 - F(y)\right] f(x)dx.$$

Upon substituting the above expressions of $K(w,y)$ in Eq. (4.15) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (4.11) – (4.14).

The recurrence relations presented in any one of Theorems 5, 6 and 7 may be employed in a simple systematic recursive manner to compute the triple moments of order statistics (of all order) for all sample sizes.

5. Relations for Quadruple Moments

The joint density function of $X_{r:n}$, $X_{s:n}$, $X_{t:n}$ and $X_{u:n}$ is given by

$$f_{r,s,t,u:n}(w,x,y,z) = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!} \left[F(w)\right]^{r-1} \left[F(x)-F(w)\right]^{s-r-1} \left[F(y)-F(x)\right]^{t-s-1} \left[F(z)-F(y)\right]^{u-t-1} \left[1-F(z)\right]^{n-u} f(w)\, f(x) f(y) f(z),$$
$$0 \le w < x < y < z < \infty,\ 1 \le r < s < t < u \le n, \tag{5.1}$$

using which we may compute the quadruple moments as

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \iiiint\limits_{0 \le w < x < y < y < z < \infty} w^a x^b y^c z^d\, f_{r,s,t,u:n}(w,x,y,z)\, dw\, dx\, dy\, dz,$$
$$1 \le r < s < t < u \le n,\ a,b,c,d \ge 1. \tag{5.2}$$

Then, the quadruple moments $\mu^{(a,b,c,d)}_{r,s,t,u:n}$ satisfy the recurrence relations presented in the following four theorems.

Theorem 8: For $n \ge 4$, $1 \le r < s < t \le n-1$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,t,t+1:n} = \mu^{(a,b,c+d)}_{r,s,t:n} + d\, \mu^{(a,b,c,d-1)}_{r,s,t,t+1:n}/(n-t); \tag{5.3}$$

and for $n \ge 5$, $1 \le r < s < t < u \le n$, $u - t \ge 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r,s,t,u-1:n} + d\mu^{(a,b,c,d-1)}_{r,s,t,u:n}/(n-u+1), \tag{5.4}$$

where $\mu^{(a,b,c,0)}_{r,s,t,u:n} \equiv \mu^{(a,b,c)}_{r,s,t:n}$.

Proof: From Eqs. (5.1) and (5.2), for $1 \le r < s < t < u \le n$ and $a,b,c,d \ge 1$ let us write

$$\mu^{(a,b,c,d-1)}_{r,s,t,u:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!d} \int_0^\infty \int_w^\infty \int_x^\infty w^a\, x^b\, y^c \left[F(w)\right]^{r-1} \left[F(x)-F(w)\right]^{s-r-1} \left[F(y)-F(x)\right]^{t-s-1} I(y) f(w) f(x) f(y) dy\, dx\, dw, \tag{5.5}$$

where

$$I(y) = \int_y^\infty \left[F(z) - F(y)\right]^{u-t-1} \left[1 - F(z)\right]^{n-u} f(z)\, d(z^d)$$

$$= \int_y^\infty \left[F(z) - F(y)\right]^{u-t-1} \left[1 - F(z)\right]^{n-u+1} d(z^d)$$

upon using the relation in (1.3). Integrating by parts, we obtain for $u = t + 1$ that

$$I(y) = -y^d\left[1 - F(y)\right]^{n-t} + (n - t) \int_y^\infty z^d\left[1 - F(z)\right]^{n-t-1} f(z)dz,$$

and for $u - t \geq 2$ that

$$I(y) = (n - u + 1) \int_y^\infty z^d\left[F(z) - F(y)\right]^{u-t-1} \left[1 - F(z)\right]^{n-u} f(z)dz$$

$$-(u - t - 1) \int_y^\infty z^d\left[F(z) - F(y)\right]^{u-t-2} \left[1 - F(z)\right]^{n-u+1} f(z)dz.$$

Upon substituting the above expressions of $I(y)$ in Eq. (5.5) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (5.3) and (5.4).

<u>Theorem 9</u>: For $n \geq 4$, $3 \leq t < u \leq n$ and $a,b,c,d = 1,2,\ldots$,

$$\mu_{1,2,t,u:n}^{(a,b,c,d)} = \mu_{2,t,u:n}^{(a+b,c,d)} + a\, \mu_{1,2,t,u:n}^{(a-1,b,c,d)} - n\, \mu_{1,t-1,u-1:n-1}^{(a+b,c,d)}; \tag{5.6}$$

for $n \geq 5$, $2 \leq r < t < u \leq n$, $t - r \geq 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu_{r,r+1,t,u:n}^{(a,b,c,d)} = \mu_{r+1,t,u:n}^{(a+b,c,d)} + \frac{1}{r}\left[a\, \mu_{r,r+1,t,u:n}^{(a-1,b,c,d)} - n\left\{\mu_{r,t-1,u-1:n-1}^{(a+b,c,d)} - \mu_{r-1,r,t-1,u-1:n-1}^{(a,b,c,d)}\right\}\right] \tag{5.7}$$

for $n \geq 5$, $3 \leq s < t < u \leq n$ and $a,b,c,d = 1,2,\ldots$,

$$\mu_{1,s,t,u:n}^{(a,b,c,d)} = \mu_{2,s,t,u:n}^{(a,b,c,d)} + a\, \mu_{1,s,t,u:n}^{(a-1,b,c,d)} - n\, \mu_{1,s-1,t-1,u-1:n-1}^{(a,b,c,d)}; \tag{5.8}$$

and for $n \geq 6$, $2 \leq r < s < t < u \leq n$, $s - r \geq 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu_{r,s,t,u:n}^{(a,b,c,d)} = \mu_{r+1,s,t,u:n}^{(a,b,c,d)} + \frac{1}{r}\left[a\, \mu_{r,s,t,u:n}^{(a-1,b,c,d)} - n\left\{\mu_{r,s-1,t-1,u-1:n-1}^{(a,b,c,d)} - \mu_{r-1,s-1,t-1,u-1:n-1}^{(a,b,c,d)}\right\}\right], \tag{5.9}$$

where $\mu_{r,s,t,u:n}^{(0,b,c,d)} \equiv \mu_{s,t,u:n}^{(b,c,d)}$.

<u>Proof</u>: From Eqs. (5.1) and (5.2), for $1 \leq r < s < t < u \leq n$ and $a,b,c,d \geq 1$ let us write

$$\mu_{r,s,t,u:n}^{(a,b,c,d)} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!\ a} \int_0^{\infty}\int_0^{z}\int_0^{y} x^b y^c z^d \left[F(y) - F(x)\right]^{t-s-1}$$
$$\left[F(z) - F(y)\right]^{u-t-1} \left[1 - F(z)\right]^{n-u} J(x)\ f(x)f(y)f(z)\ dx\ dy\ dz, \tag{5.10}$$

where

$$J(x) = \int_0^x \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-1} d(w^a)$$
$$= \int_0^x \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-1} d(w^a) - \int_0^x \left[F(w)\right]^{r} \left[F(x) - F(w)\right]^{s-r-1} d(w^a)$$

upon using the relation in (1.3). Integrating by parts now, we obtain for $r = 1$ and $s = 2$ that

$$J(x) = x^a - x^a\ F(x) + \int_0^x w^a f(w)\ dw,$$

for $r \geq 2$ and $s = r + 1$ that

$$J(x) = x^a \left[F(x)\right]^{r-1} - (r-1) \int_0^x w^a \left[F(w)\right]^{r-2} f(w)\ dw$$
$$- x^a \left[F(x)\right]^{r} + r \int_0^x w^a \left[F(w)\right]^{r-1} f(w)\ dw,$$

for $r = 1$ and $s \geq 3$ that

$$J(x) = (s-2) \int_0^x w^a \left[F(x) - F(w)\right]^{s-3} f(w)dw + \int_0^x w^a \left[F(x) - F(w)\right]^{s-2} f(w)dw$$
$$- (s-2) \int_0^x w^a\ F(w) \left[F(x) - F(w)\right]^{s-3} f(w)\ dw,$$

and for $r \geq 2$ and $s - r \geq 2$ that

$$J(x) = (s-r-1) \int_0^x w^a \left[F(w)\right]^{r-1} \left[F(x) - F(w)\right]^{s-r-2} f(w)\ dw$$

$$- (r-1) \int_0^x w^a \Big[F(w)\Big]^{r-2} \Big[F(x) - F(w)\Big]^{s-r-1} f(w)\, dw$$

$$+ r \int_0^x w^a \Big[F(w)\Big]^{r-1} \Big[F(x) - F(w)\Big]^{s-r-1} f(w)\, dw$$

$$- (s-r-1) \int_0^x w^a \Big[F(w)\Big]^{r} \Big[F(x) - F(w)\Big]^{s-r-2} f(w)\, dw.$$

Upon substituting the above expressions of J(x) in Eq. (5.10) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (5.6) – (5.9).

__Theorem 10__: For $n \geq 4$, $1 \leq r < u \leq n$, $u - r \geq 3$ and $a,b,c,d = 1,2,3,\ldots$,

$$\mu^{(a,b,c,d)}_{r,r+1,r+2,u:n} = \mu^{(a+b,c,d)}_{r,r+2,u:n} + b\, \mu^{(a,b-1,c,d)}_{r,r+1,r+2,u:n} - (u-r-2) \Big\{\mu^{(a,b+c,d)}_{r,r+1,u:n} - \mu^{(a+b,c,d)}_{r,r+1,u:n}\Big\} - (n-u+1) \Big\{\mu^{(a,b+c,d)}_{r,r+1,u-1:n} - \mu^{(a+b,c,d)}_{r,r+1,u-1:n}\Big\}; \tag{5.11}$$

for $n \geq 5$, $1 \leq r < t < u \leq n$, $t - r \geq 3$ and $a,b,c,d = 1,2,3,\ldots$,

$$\mu^{(a,b,c,d)}_{r,r+1,t,u:n} = \mu^{(a+b,c,d)}_{r,t,u:n} + \frac{1}{t-r-1} \Big[b\, \mu^{(a,b-1,c,d)}_{r,r+1,t,u:n} - (u-t) \Big\{\mu^{(a,b,c,d)}_{r,r+1,t-1,u:n} - \mu^{(a+b,c,d)}_{r,t-1,u:n}\Big\} - (n-u+1) \Big\{\mu^{(a,b,c,d)}_{r,r+1,t-1,u-1:n} - \mu^{(a+b,c,d)}_{r,t-1,u-1:n}\Big\}\Big]; \tag{5.12}$$

for $1 \leq r < s < u \leq n$, $s - r \geq 2$ and $a,b,c,d = 1,2,3,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,s+1,u:n} = \mu^{(a,b,c,d)}_{r,s-1,s+1,u:n} + b\, \mu^{(a,b-1,c,d)}_{r,s,s+1,u:n} - (u-s-1) \Big\{\mu^{(a,b+c,d)}_{r,s,u:n} - \mu^{(a,b,c,d)}_{r,s-1,s,u:n}\Big\} - (n-u+1) \Big\{\mu^{(a,b+c,d)}_{r,s,u-1:n} - \mu^{(a,b,c,d)}_{r,s-1,s,u-1:n}\Big\}; \tag{5.13}$$

for $1 \leq r < s < t < u \leq n$, $s - r \geq 2$, $t - s \geq 2$ and $a,b,c,d = 1,2,3,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r,s-1,t,u:n} + \frac{1}{t-s} \Big[b\, \mu^{(a,b-1,c,d)}_{r,s,t,u:n} - (u-t) \Big\{\mu^{(a,b,c,d)}_{r,s,t-1,u:n} - \mu^{(a,b,c,d)}_{r,s-1,t-1,u:n}\Big\} - (n-u+1) \Big\{\mu^{(a,b,c,d)}_{r,s,t-1,u-1:n} - \mu^{(a,b,c,d)}_{r,s-1,t-1,u-1:n}\Big\}\Big], \tag{5.14}$$

where $\mu^{(a,0,c,d)}_{r,s,t,u:n} \equiv \mu^{(a,c,d)}_{r,t,u:n}$

__Proof__: From Eqs. (5.1) and (5.2), for $1 \leq r < s < t < u \leq n$ and $a,b,c,d \geq 1$ we can write

$$\mu^{(a,b-1,c,d)}_{r,s,t,u:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!b} \int_0^{\infty}\int_0^{z}\int_0^{y} w^a\, y^c z^d \Big[F(w)\Big]^{r-1}$$

$$\Big[F(z) - F(y)\Big]^{u-t-1} \Big[1 - F(z)\Big]^{n-u} K(w,y)\, f(w)\, f(y)\, f(z)\, dw\, dy\, dz, \tag{5.15}$$

where

$$K(w,y) = \int_w^y \Big[F(x) - F(w)\Big]^{s-r-1} \Big[F(y) - F(x)\Big]^{t-s-1} f(x)\, d(x^b)$$

$$= \int_w^y \Big[F(x) - F(w)\Big]^{s-r-1} \Big[F(y) - F(x)\Big]^{t-s} d(x^b)$$

$$+ \int_w^y \Big[F(x) - F(w)\Big]^{s-r-1} \Big[F(y) - F(x)\Big]^{t-s-1} \Big[F(z) - F(y)\Big]\, d(x^b)$$

$$+ \int_w^y \Big[F(x) - F(w)\Big]^{s-r-1} \Big[F(y) - F(x)\Big]^{t-s-1} \Big[1 - F(z)\Big]\, d(x^b)$$

upon using the relation in (1.3) and then writing $1 - F(x)$ as $\Big[F(y) - F(x)\Big]$ + $\Big[F(z) - F(y)\Big]$ + $\Big[1 - F(z)\Big]$. Integrating by parts now, we obtain for $s = r + 1$ and $t = r + 2$ that

$$K(w,y) = -w^b\Big[F(y) - F(w)\Big] + \int_w^y x^b f(x)\, dx + y^b\Big[F(z) - F(y)\Big]$$

$$-w^b\Big[F(z) - F(y)\Big] + y^b\Big[1 - F(z)\Big] - w^b\Big[1 - F(z)\Big],$$

for $s = r + 1$ and $t - r \geq 3$ that

$$K(w,y) = -\, w^b\Big[F(y) - F(w)\Big]^{t-r-1} + (t - r - 1) \int_w^y x^b\Big[F(y) - F(x)\Big]^{t-r-2} f(x)\, dx$$

$$-\, w^b\Big[F(y) - F(w)\Big]^{t-r-2} \Big[F(z) - F(y)\Big] + (t - r - 2) \int_w^y x^b\Big[F(y) - F(x)\Big]^{t-r-3}$$

$$\Big[F(z) - F(y)\Big]\, f(x)\, dx$$

$$-\, w^b\Big[F(y) - F(w)\Big]^{t-r-2} \Big[1 - F(z)\Big] + (t - r - 2) \int_w^y x^b\Big[F(y) - F(x)\Big]^{t-r-3}$$

$$\left[1 - F(z)\right] f(x)dx,$$

for $s - r \geq 2$ and $t = s + 1$ that

$$K(w,y) = -(s - r - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-2} \left[F(y) - F(x)\right] f(x)\, dx$$

$$+ \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-1} f(x)dx$$

$$+ y^b \left[F(y) - F(w)\right]^{s-r-1} \left[F(z) - F(y)\right]$$

$$- (s - r - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-2} \left[F(z) - F(y)\right] f(x)\, dx$$

$$+ y^b \left[F(y) - F(w)\right]^{s-r-1} \left[1 - F(z)\right] - (s - r - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-2}$$

$$\left[1 - F(z)\right] f(x)\, dx,$$

and for $s - r \geq 2$ and $t - s \geq 2$ that

$$K(w,y) = -(s - r - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-2} \left[F(y) - F(x)\right]^{t-s} f(x)dx$$

$$+ (t - s) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-1} \left[F(y) - F(x)\right]^{t-s-1} f(x)\, dx$$

$$- (s - r - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-2} \left[F(y) - F(x)\right]^{t-s-1} \left[F(z) - F(y)\right] f(x)\, dx$$

$$+ (t - s - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-1} \left[F(y) - F(x)\right]^{t-s-2} \left[F(z) - F(y)\right] f(x)\, dx$$

$$- (s - r - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-2} \left[F(y) - F(x)\right]^{t-s-1} \left[1 - F(z)\right] f(x)\, dx$$

$$+ (t - s - 1) \int_w^y x^b \left[F(x) - F(w)\right]^{s-r-1} \left[F(y) - F(x)\right]^{t-s-2} \left[1 - F(z)\right] f(x)\, dx.$$

Upon substituting the above expressions of $K(w,y)$ in Eq. (5.15) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (5.11) – (5.14).

<u>Theorem 11</u>: For $n \geq 4$, $1 \leq r < s \leq n-2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,s+1,s+2:n} = \mu^{(a,b+c,d)}_{r,s,s+2:n} + c\,\mu^{(a,b,c-1,d)}_{r,s,s+1,s+2:n} - (n-s-1)\left\{\mu^{(a,b,c+d)}_{r,s,s+1:n} - \mu^{(a,b+c,d)}_{r,s,s+1:n}\right\}; \tag{5.16}$$

for $n \geq 5$, $1 \leq r < s < u \leq n$, $u-s \geq 3$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,s+1,u:n} = \mu^{(a,b+c,d)}_{r,s,u:n} + \frac{1}{u-s-1}\left[c\,\mu^{(a,b,c-1,d)}_{r,s,s+1,u:n} - (n-u+1)\left\{\mu^{(a,b,c,d)}_{r,s,s+1,u-1:n} - \mu^{(a,b+c,d)}_{r,s,u-1:n}\right\}\right]; \tag{5.17}$$

for $n \geq 5$, $1 \leq r < s < t \leq n-1$, $t-s \geq 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,t,t+1:n} = \mu^{(a,b,c,d)}_{r,s,t-1,t+1:n} + c\,\mu^{(a,b,c-1,d)}_{r,s,t,t+1:n} - (n-t)\left\{\mu^{(a,b,c+d)}_{r,s,t:n} - \mu^{(a,b,c,d)}_{r,s,t-1,t:n}\right\}; \tag{5.18}$$

and for $n \geq 6$, $1 \leq r < s < t < u \leq n$, $t-s \geq 2$, $u-t \geq 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r,s,t-1,u:n} + \frac{1}{u-t}\left[c\,\mu^{(a,b,c-1,d)}_{r,s,t,u:n} - (n-u+1)\left\{\mu^{(a,b,c,d)}_{r,s,t,u-1:n} - \mu^{(a,b,c,d)}_{r,s,t-1,u-1:n}\right\}\right], \tag{5.19}$$

where $\mu^{(a,b,0,d)}_{r,s,t,u:n} \equiv \mu^{(a,b,d)}_{r,s,u:n}$.

<u>Proof</u>: From Eqs. (5.1) and (5.2), for $1 \leq r < s < t < u \leq n$ and $a,b,c,d \geq 1$ we can write

$$\mu^{(a,b,c-1,d)}_{r,s,t,u:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!c} \int_0^{\infty}\int_0^{z}\int_0^{x} w^a x^b z^d \left[F(w)\right]^{r-1} \left[F(x)-F(w)\right]^{s-r-1}\left[1-F(z)\right]^{n-u} L(x,z)\, f(w)\, f(x)\, f(z)\, dw\, dx\, dz, \tag{5.20}$$

where

$$L(x,z) = \int_x^z \left[F(y)-F(x)\right]^{t-s-1}\left[F(z)-F(y)\right]^{u-t-1} f(y)\, d(y^c)$$

$$= \int_x^z \left[F(y)-F(x)\right]^{t-s-1}\left[F(z)-F(y)\right]^{u-t} d(y^c)$$

$$+ \int_x^z \left[F(y)-F(x)\right]^{t-s-1}\left[F(z)-F(y)\right]^{u-t-1}\left[1-F(z)\right] d(y^c)$$

upon using the relation in (1.3) and then writing $1 - F(y)$ as $\left[F(z) - F(y)\right] + \left[1 - F(z)\right]$.
Integrating by parts now, we obtain for $t = s + 1$ and $u = s + 2$ that

$$L(x,z) = -x^c\left[F(z) - F(x)\right] + \int_x^z y^c f(y)dy + z^c\left[1 - F(z)\right] - x^c\left[1 - F(z)\right],$$

for $t = s + 1$ and $u - s \geq 3$ that

$$\begin{aligned} L(x,z) = &-x^c\left[F(z) - F(x)\right]^{u-s-1} + (u - s - 1)\int_x^z y^c\left[F(z) - F(y)\right]^{u-s-2} f(y)\, dy \\ &-x^c\left[F(z) - F(x)\right]^{u-s-2}\left[1 - F(z)\right] + (u - s - 2)\int_x^z y^c\left[F(z) - F(y)\right]^{u-s-3} \\ &\qquad\left[1 - F(z)\right] f(y)\, dy, \end{aligned}$$

for $t - s \geq 2$ and $u = t + 1$ that

$$\begin{aligned} L(x,z) = &-(t - s - 1)\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-2}\left[F(z) - F(y)\right] f(y)dy \\ &+\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-1} f(y)dy + z^c\left[F(z) - F(x)\right]^{t-s-1}\left[1 - F(z)\right] \\ &-(t - s - 1)\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-2}\left[1 - F(z)\right] f(y)\, dy, \end{aligned}$$

and for $t - s \geq 2$ and $u - t \geq 2$ that

$$\begin{aligned} L(x,z) = &-(t - s - 1)\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-2}\left[F(z) - F(y)\right]^{u-t} f(y)\, dy \\ &+ (u - t)\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-1}\left[F(z) - F(y)\right]^{u-t-1} f(y)dy \\ &-(t - s - 1)\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-2}\left[F(z) - F(y)\right]^{u-t-1}\left[1 - F(z)\right] f(y)\, dy \\ &+ (u - t - 1)\int_x^z y^c\left[F(y) - F(x)\right]^{t-s-1}\left[F(z) - F(y)\right]^{u-t-2}\left[1 - F(z)\right] f(y)\, dy. \end{aligned}$$

Upon substituting the above expressions of $L(x,z)$ in Eq. (5.20) and simplifying the resulting equations, we derive the recurrence relations given in Eqs. (5.16) – (5.19).

The recurrence relations presented in any one of Theorems 8, 9, 10 and 11 may be employed in a simple systematic recursive way to compute the quadruple moments of order statistics (of all order) for all sample sizes.

6. Applications to Inference for the One–parameter Exponential Distribution

By using the results presented in Sections 2 – 5, we computed the single, double, triple and quadruple moments of order statistics (of order up to 4) for sample sizes up to 15. For example, we have presented in Tables 1 – 4 the values of the single, double, triple and quadruple moments of order up to 4 for sample sizes up to 8. As will be displayed in this section, these quantities may be successfully used in order to develop a chi–square approximation for the distribution of the best linear unbiased estimator of the scale parameter of an exponential distribution based on doubly Type–II censored samples.

In this section we assume that

$$Y_{r+1:n} \le Y_{r+2:n} \le \cdots \le Y_{n-s:n} \tag{6.1}$$

is a doubly Type–II censored sample from the one–parameter exponential population with probability density function

$$g(y;\sigma) = \frac{1}{\sigma} e^{-y/\sigma}, \; y \ge 0, \; \sigma > 0. \tag{6.2}$$

The estimation of the parameter σ based on the doubly Type–II censored sample in (6.1) has been considered by several authors including Epstein (1956, 1962), Epstein and Sobel (1953, 1954), Sarhan and Greenberg (1957, 1958, 1962), Saleh (1966), Cohen and Helm (1973), Mann et al. (1974), Kambo (1978), Lawless (1982), and Balakrishnan and Cohen (1990).

By using the facts that

$$E\left[Y_{i:n}\right] = \sigma \sum_{\ell=n-i+1}^{n} 1/\ell$$

and

$$\mathrm{Var}\left[Y_{i:n}\right] = \mathrm{Cov}\left[Y_{i:n}, Y_{j:n}\right] = \sigma^2 \sum_{\ell=n-i+1}^{n} 1/\ell^2,$$

it can be shown that the best linear unbiased estimator of σ based on the doubly Type–II censored sample in (6.1) is given by

$$\sigma^* = \sum_{i=r+1}^{n-s} a_i\, Y_{i:n}, \tag{6.3}$$

where

$$a_i = a_i^*/K$$

with

$$K = (n - r - s - 1) + \left\{\sum_{\ell=n-r}^{n} \frac{1}{\ell}\right\}^2 / \left[\sum_{\ell=n-r}^{n} 1/\ell^2\right]$$

and

$$\alpha_i^* = \begin{cases} \left\{\left[\sum_{\ell=n-r}^{n} 1/\ell\right] / \left[\sum_{\ell=n-r}^{n} 1/\ell^2\right]\right\} - (n-r-1) & \text{for } i = r+1 \\ 1 & \text{for } r+2 \le i \le n-s-1 \\ s+1 & \text{for } i = n-s, \end{cases}$$

and the variance of this estimator is

$$\mathrm{Var}(\sigma^*) = \sigma^2/K; \tag{6.4}$$

for example, see Balakrishnan and Cohen (1990, pp. 76 – 80).

In the special case of the right–censored sample (that is, $r = 0$), the estimator σ^* in (6.3) simplifies to

$$\sigma^* = \frac{1}{n-s}\left\{\sum_{i=1}^{n-s} Y_{i:n} + s\, Y_{n-s:n}\right\} \tag{6.5}$$

and the variance of the estimator in (6.4) simply becomes

$$\mathrm{Var}(\sigma^*) = \sigma^2/(n-s). \tag{6.6}$$

In this special case of right–censoring, by writing $Y_{i:n}$ as $\sigma\, X_{i:n}$ we may immediately write from (6.5) that

$$\frac{\sigma^*}{\sigma} = \frac{1}{n-s} \left\{ \sum_{i=1}^{n-s-1} X_{i:n} + (s+1) X_{n-s:n} \right\}$$

$$= \frac{1}{n-s} \sum_{i=1}^{n-s} S_i \qquad (6.7)$$

where $S_i = (n - i + 1) \left[X_{i:n} - X_{i-1:n}\right]$ are the normalized spacings as defined earlier in Section 1. By using the property that these S_i's are independent standard exponential random variables, we immediately observe from Eq. (6.7) that $2(n - s)\ \sigma^*/\sigma$ has a chi–square distribution with $2(n - s)$ degrees of freedom. This result can be used to obtain confidence intervals for σ and carry out tests of hypotheses concerning σ. However, such an elegant distributional result does not extend to the best linear unbiased estimator of σ in (6.3) based on a doubly Type–II censored sample.

For the doubly Type–II censored sample case, by considering the best linear unbiased estimator of σ in (6.3) and writing

$$\frac{\sigma^*}{\sigma} = \sum_{i=r+1}^{n-s} a_i X_{i:n}, \qquad (6.8)$$

we may determine the mean, variance and the coefficients of skewness and kurtosis of the linear function in (6.8) by making use of the tabulated values of the single, double, triple and quadruple moments of order statistics (of order up to 4). For example, we computed the values of the mean, variance, coefficient of skewness ($\sqrt{\beta_1}$) and coefficient of kurtosis (β_2) of σ^*/σ for sample size n = 8(1)12 and various levels of censoring both on left and right sides of the sample. These values are presented in Table 5. Also given in the last column of this table are the values of $\gamma = \beta_2 - 1.5\ \beta_1 - 3$. It is observed that $\gamma = 0$ whenever the sample is just right–censored and this is expected because of the fact that $2(n - s)\ \sigma^*/\sigma$ has exactly a chi–square distribution with $2(n - s)$ degrees of freedom and that $\gamma = 0$ represents the chi–square line in the Pearson (β_1, β_2)–plane. But interestingly enough, we also observe that γ is very nearly 0 even in the case of doubly censored samples even for a sample of size as small as 8. Thus, Table 5 suggests strongly that the distribution of σ^*/σ in the case of doubly Type–II censored samples may be very closely approximated by a chi–square distribution with degrees of freedom determined approximately (for example, by looking at the variance). This chi–square approximation for the distribution of σ^*/σ may be utilized to construct approximate confidence intervals for σ and carry out tests of hypotheses concerning σ. This application is well illustrated in Section 8 through two examples involving life–time data given by Lawless (1982).

7. Generalized Results for the Right–truncated Exponential Distribution

In this section, we generalize the results established in Sections 2 – 5 to the case when the population distribution is exponential truncated on the right. As pointed out by Cohen (1959), in life–testing, dosage–response studies, target analyses, biological assays, and in other related investigations, sample selection or observation is often restricted on the right side of the range of possible population values. With this in mind, Joshi (1978, 1979, 1982), Saleh et al. (1975), and Balakrishnan and Joshi (1984) have all studied order statistics from truncated exponential distributions. Kjelsberg (1962), Tarter (1966), Balakrishnan and Kocherlakota (1986), and Gupta and Balakrishnan (1991) have all examined order statistics from truncated logistic distributions.

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics obtained from a random sample of size n from a right–truncated exponential population with probability density function

$$f(x) = \frac{1}{P} e^{-x}, \qquad 0 \le x \le P_1 \tag{7.1}$$
$$= 0 \quad , \qquad \text{elsewhere}$$

and cumulative distribution function

$$F(x) = \left[1 - e^{-x}\right]/P, \qquad 0 \le x \le P_1$$
$$= 1 \quad , \qquad x > P_1, \tag{7.2}$$

where $1 - P$ is the proportion of truncation on the right of the standard exponential distribution in (1.2), and the truncation point $P_1 = -\ln(1-P)$. It is easily observed from Eqs. (7.1) and (7.2) that

$$f(x) = \frac{1}{P} - F(x), \qquad 0 \le x \le P_1, \tag{7.3}$$

and

$$f(x) = \left[1 - F(x)\right] + \left[\frac{1-P}{P}\right], \qquad 0 \le x \le P_1. \tag{7.4}$$

These two relations will be used successfully in this section in order to generalize all the results derived in Sections 2 – 5 to the right–truncated exponential distribution in (7.2).

(i) Relations for Single Moments

The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(w) = \frac{n!}{(r-1)!(n-r)!}\left[F(w)\right]^{r-1}\left[1-F(w)\right]^{n-r} f(w), \qquad 0 \le w \le P_1,\ 1 \le r \le n, \tag{7.5}$$

using which the single moments may be computed as

$$\mu_{r:n}^{(a)} = \int_0^{P_1} w^a f_{r:n}(w)\, dw, \quad 1 \le r \le n,\ a = 1,2,\ldots \tag{7.6}$$

Then by proceeding on lines very similar to those used in proving Theorems 1 and 2, we may prove the following two theorems.

Theorem 12: For $n \ge 2$ and $a = 1,2,\ldots,$

$$\mu_{1:n}^{(a)} = \frac{a}{n}\mu_{1:n}^{(a-1)} - \left[\frac{1-P}{P}\right]\mu_{1:n-1}^{(a)} \tag{7.7}$$

with $\mu_{1:n}^{(0)} \equiv 1$.

Theorem 13: For $n \ge 3$, $2 \le r \le n-1$ and $a = 1,2,\ldots,$

$$\mu_{r:n}^{(a)} = \mu_{r-1:n}^{(a)} + \frac{1}{n-r+1}\left[a\,\mu_{r:n}^{(a-1)} - n\left[\frac{1-P}{P}\right]\left\{\mu_{r:n-1}^{(a)} - \mu_{r-1:n-1}^{(a)}\right\}\right]; \tag{7.8}$$

and for $n \ge 2$ and $a = 1,2,\ldots,$

$$\mu_{n:n}^{(a)} = \mu_{n-1:n}^{(a)} + a\,\mu_{n:n}^{(a-1)} - n\left[\frac{1-P}{P}\right]\left\{P_1^a - \mu_{n-1:n-1}^{(a)}\right\}, \tag{7.9}$$

with $\mu_{r:n}^{(0)} \equiv 1$.

It should be mentioned here that Theorems 12 and 13 have been proved by Joshi (1978) and are presented here for the sake of completeness. The recurrence relations presented in Theorems 12 and 13 will enable one to compute the single moments (of all order) of order statistics for all sample sizes in a simple recursive way.

(ii) Relations for Double Moments

The joint density function of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r,s:n}(w,x) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left[F(w)\right]^{r-1}\left[F(x)-F(w)\right]^{s-r-1}\left[1-F(x)\right]^{n-s} f(w)\,f(x),$$
$$0 \le w < x \le P_1,\ 1 \le r < s \le n, \qquad (7.10)$$

using which we may compute the double moments as

$$\mu_{r,s:n}^{(a,b)} = \iint_{0\le w<x\le P_1} w^a x^b f_{r,s:n}(w,x)\,dw\,dx,\ 1 \le r < s \le n,\ a,\ b \ge 1. \qquad (7.11)$$

Then by proceeding on lines very similar to those used in proving Theorems 3 and 4, we may prove the following two theorems.

Theorem 14: For $n \ge 3$, $1 \le r \le n-2$ and $a,b = 1,2,\ldots,$

$$\mu_{r,r+1:n}^{(a,b)} = \mu_{r:n}^{(a+b)} + \frac{1}{n-r}\left[b\,\mu_{r,r+1:n}^{(a,b-1)} - n\left[\frac{1-P}{P}\right]\left\{\mu_{r,r+1:n-1}^{(a,b)} - \mu_{r:n-1}^{(a+b)}\right\}\right]; \qquad (7.12)$$

for $n \ge 2$ and $a,b = 1,2,\ldots,$

$$\mu_{n-1,n:n}^{(a,b)} = \mu_{n-1:n}^{(a+b)} + b\,\mu_{n-1,n:n}^{(a,b-1)} - n\left[\frac{1-P}{P}\right]\left\{P_1^b\,\mu_{n-1:n-1}^{(a)} - \mu_{n-1:n-1}^{(a+b)}\right\}; \qquad (7.13)$$

for $n \ge 4$, $1 \le r < s \le n-1$, $s-r \ge 2$ and $a,b = 1,2,\ldots,$

$$\mu_{r,s:n}^{(a,b)} = \mu_{r,s-1:n}^{(a,b)} + \frac{1}{n-s+1}\left[b\,\mu_{r,s:n}^{(a,b-1)} - n\left[\frac{1-P}{P}\right]\left\{\mu_{r,s:n-1}^{(a,b)} - \mu_{r,s-1:n-1}^{(a,b)}\right\}\right]; \qquad (7.14)$$

and for $n \ge 3$, $1 \le r \le n-2$ and $a,b = 1,2,\ldots,$

$$\mu_{r,n:n}^{(a,b)} = \mu_{r,n-1:n}^{(a,b)} + b\,\mu_{r,n:n}^{(a,b-1)} - n\left[\frac{1-P}{P}\right]\left\{P_1^b\,\mu_{r:n-1}^{(a)} - \mu_{r,n-1:n-1}^{(a,b)}\right\}, \qquad (7.15)$$

where $\mu_{r,s:n}^{(a,0)} \equiv \mu_{r:n}^{(a)}$.

Theorem 15: For $n \geq 2$ and $a,b = 1,2,\ldots,$

$$\mu_{1,2:n}^{(a,b)} = \mu_{2:n}^{(a+b)} + a\,\mu_{1,2:n}^{(a-1,b)} - \frac{n}{P}\,\mu_{1:n-1}^{(a+b)}; \tag{7.16}$$

for $n \geq 3$, $2 \leq r \leq n-1$ and $a,b = 1,2,\ldots,$

$$\mu_{r,r+1:n}^{(a,b)} = \mu_{r+1:n}^{(a+b)} + \frac{1}{r}\left[a\,\mu_{r,r+1:n}^{(a-1,b)} - \frac{n}{P}\left\{\mu_{r:n-1}^{(a+b)} - \mu_{r-1,r:n-1}^{(a,b)}\right\}\right]; \tag{7.17}$$

for $n \geq 3$, $3 \leq s \leq n$ and $a,b = 1,2,\ldots,$

$$\mu_{1,s:n}^{(a,b)} = \mu_{2,s:n}^{(a,b)} + a\,\mu_{1,s:n}^{(a-1,b)} - \frac{n}{P}\,\mu_{1,s-1:n-1}^{(a,b)}; \tag{7.18}$$

and for $n \geq 4$, $2 \leq r < s \leq n$, $s - r \geq 2$ and $a,b = 1,2,\ldots,$

$$\mu_{r,s:n}^{(a,b)} = \mu_{r+1,s:n}^{(a,b)} + \frac{1}{r}\left[a\,\mu_{r,s:n}^{(a-1,b)} - \frac{n}{P}\left\{\mu_{r,s-1:n-1}^{(a,b)} - \mu_{r-1,s-1:n-1}^{(a,b)}\right\}\right], \tag{7.19}$$

where $\mu_{r,s:n}^{(0,b)} \equiv \mu_{s:n}^{(b)}$.

The recurrence relations presented in either Theorem 14 or Theorem 15 may be used in a simple systematic recursive manner in order to compute the double moments (of all order) of order statistics for all sample sizes. The relations presented in Theorem 14 have been proved by Joshi (1982) for the special case when $a = b = 1$.

(iii) Relations for Triple Moments

The joint density function of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is given by

$$f_{r,s,t:n}(w,x,y) = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!}\left[F(w)\right]^{r-1}\left[F(x) - F(w)\right]^{s-r-1}$$
$$\left[F(y) - F(x)\right]^{t-s-1}\left[1 - F(y)\right]^{n-t} f(w)\, f(x)\, f(y),$$
$$0 \leq w < x < y \leq P_1,\ 1 \leq r < s < t \leq n, \tag{7.20}$$

using which we may compute the triple moments as

$$\mu_{r,s,t:n}^{(a,b,c)} = \iiint_{0 \leq w < x < y \leq P_1} w^a x^b y^c\, f_{r,s,t:n}(w,x,y)\, dw\, dx\, dy,$$
$$1 \leq r < s < t \leq n,\ a,b,c \geq 1. \tag{7.21}$$

Then by proceeding on lines very similar to those used in proving Theorems 5 – 7, we may prove the following three theorems.

Theorem 16: For $n \geq 4$, $1 \leq r < s \leq n-2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,s,s+1:n}^{(a,b,c)} = \mu_{r,s:n}^{(a,b+c)} + \frac{1}{n-s}\left[c\,\mu_{r,s,s+1:n}^{(a,b,c-1)} - n\left[\frac{1-P}{P}\right]\left\{\mu_{r,s,s+1:n-1}^{(a,b,c)} - \mu_{r,s:n-1}^{(a,b+c)}\right\}\right]; \tag{7.22}$$

for $n \geq 3$, $1 \leq r \leq n-2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,n-1,n:n}^{(a,b,c)} = \mu_{r,n-1:n}^{(a,b+c)} + c\,\mu_{r,n-1,n:n}^{(a,b,c-1)} - n\left[\frac{1-P}{P}\right]\left\{P_1^c\,\mu_{r,n-1:n-1}^{(a,b)} - \mu_{r,n-1:n-1}^{(a,b+c)}\right\}; \tag{7.23}$$

for $n \geq 5$, $1 \leq r < s < t \leq n-1$, $t-s \geq 2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,s,t:n}^{(a,b,c)} = \mu_{r,s,t-1:n}^{(a,b,c)} + \frac{1}{n-t+1}\left[c\,\mu_{r,s,t:n}^{(a,b,c-1)} - n\left[\frac{1-P}{P}\right]\left\{\mu_{r,s,t:n-1}^{(a,b,c)} - \mu_{r,s,t-1:n-1}^{(a,b,c)}\right\}\right]; \tag{7.24}$$

and for $n \geq 4$, $1 \leq r < s \leq n-2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,s,n:n}^{(a,b,c)} = \mu_{r,s,n-1:n}^{(a,b,c)} + c\,\mu_{r,s,n:n}^{(a,b,c-1)} - n\left[\frac{1-P}{P}\right]\left\{P_1^c\,\mu_{r,s:n-1}^{(a,b)} - \mu_{r,s,n-1:n-1}^{(a,b,c)}\right\}, \tag{7.25}$$

where $\mu_{r,s,t:n}^{(a,b,0)} \equiv \mu_{r,s:n}^{(a,b)}$.

Theorem 17: For $n \geq 3$, $3 \leq t \leq n$ and $a,b,c = 1,2,\ldots$,

$$\mu_{1,2,t:n}^{(a,b,c)} = \mu_{2,t:n}^{(a+b,c)} + a\,\mu_{1,2,t:n}^{(a-1,b,c)} - \frac{n}{P}\,\mu_{1,t-1:n-1}^{(a+b,c)}; \tag{7.26}$$

for $n \geq 4$, $2 \leq r < t \leq n$, $t-r \geq 2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,r+1,t:n}^{(a,b,c)} = \mu_{r+1,t:n}^{(a+b,c)} + \frac{1}{r}\left[a\,\mu_{r,r+1,t:n}^{(a-1,b,c)} - \frac{n}{P}\left\{\mu_{r,t-1:n-1}^{(a+b,c)} - \mu_{r-1,r,t-1:n-1}^{(a,b,c)}\right\}\right]; \tag{7.27}$$

for $n \geq 4$, $3 \leq s < t \leq n$ and $a,b,c = 1,2,\ldots$,

$$\mu_{1,s,t:n}^{(a,b,c)} = \mu_{2,s,t:n}^{(a,b,c)} + a\,\mu_{1,s,t:n}^{(a-1,b,c)} - \frac{n}{P}\,\mu_{1,s-1,t-1:n-1}^{(a,b,c)}; \tag{7.28}$$

and for $n \geq 5$, $2 \leq r < s < t \leq n$, $s-r \geq 2$ and $a,b,c = 1,2,\ldots$,

$$\mu_{r,s,t:n}^{(a,b,c)} = \mu_{r+1,s,t:n}^{(a,b,c)} + \frac{1}{r}\left[a\,\mu_{r,s,t:n}^{(a-1,b,c)} - \frac{n}{P}\left\{\mu_{r,s-1,t-1:n-1}^{(a,b,c)} - \mu_{r-1,s-1,t-1:n-1}^{(a,b,c)}\right\}\right], \tag{7.29}$$

where $\mu_{r,s,t:n}^{(0,b,c)} \equiv \mu_{s,t:n}^{(b,c)}$.

Theorem 18: For $n \geq 3$, $1 \leq r \leq n-2$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,r+1,r+2:n} = \mu^{(a+b,c)}_{r,r+2:n} + b\mu^{(a,b-1,c)}_{r,r+1,r+2:n} - (n-r-1)\left\{\mu^{(a,b+c)}_{r,r+1:n} - \mu^{(a+b,c)}_{r,r+1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b+c)}_{r,r+1:n-1} - \mu^{(a+b,c)}_{r,r+1:n-1}\right\}; \tag{7.30}$$

for $n \geq 4$, $1 \leq r < t \leq n$, $t - r \geq 3$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,r+1,t:n} = \mu^{(a+b,c)}_{r,t:n} + \frac{1}{t-r-1}\left[b\mu^{(a,b-1,c)}_{r,r+1,t:n} - (n-t+1)\left\{\mu^{(a,b,c)}_{r,r+1,t-1:n} - \mu^{(a+b,c)}_{r,t-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c)}_{r,r+1,t-1:n-1} - \mu^{(a+b,c)}_{r,t-1:n-1}\right\}\right]; \tag{7.31}$$

for $n \geq 4$, $1 \leq r < s \leq n-1$, $s - r \geq 2$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,s,s+1:n} = \mu^{(a,b,c)}_{r,s-1,s+1:n} + b\mu^{(a,b-1,c)}_{r,s,s+1:n} - (n-s)\left\{\mu^{(a,b+c)}_{r,s:n} - \mu^{(a,b,c)}_{r,s-1,s:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b+c)}_{r,s:n-1} - \mu^{(a,b,c)}_{r,s-1,s:n-1}\right\}; \tag{7.32}$$

and for $n \geq 5$, $1 \leq r < s < t \leq n$, $s - r \geq 2$, $t - s \geq 2$ and a,b,c = 1,2,...,

$$\mu^{(a,b,c)}_{r,s,t:n} = \mu^{(a,b,c)}_{r,s-1,t:n} + \frac{1}{t-s}\left[b\,\mu^{(a,b-1,c)}_{r,s,t:n} - (n-t+1)\left\{\mu^{(a,b,c)}_{r,s,t-1:n} - \mu^{(a,b,c)}_{r,s-1,t-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c)}_{r,s,t-1:n-1} - \mu^{(a,b,c)}_{r,s-1,t-1:n-1}\right\}\right], \tag{7.33}$$

where $\mu^{(a,0,c)}_{r,s,t:n} \equiv \mu^{(a,c)}_{r,t:n}$.

The recurrence relations presented in any one of Theorems 16, 17 and 18 may be used in a simple systematic recursive fashion in order to compute the triple moments (of all order) of order statistics for all sample sizes.

(iv) Relations for Quadruple Moments

The joint density function of $X_{r:n}$, $X_{s:n}$, $X_{t:n}$ and $X_{u:n}$ is given by

$$f_{r,s,t,u:n}(w,x,y,z) = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!}\left[F(w)\right]^{r-1}\left[F(x)-F(w)\right]^{s-r-1} \left[F(y)-F(x)\right]^{t-s-1}\left[F(z)-F(y)\right]^{u-t-1}\left[1-F(z)\right]^{n-u} f(w)f(x)f(y)f(z),$$

$$0 \leq w < x < y < z \leq P_1,\ 1 \leq r < s < t < u \leq n, \tag{7.34}$$

using which we may compute the quadruple moments as

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \iiiint\limits_{0\leq w<x<y<z\leq P_1} w^a x^b y^c z^d\, f_{r,s,t,u:n}(w,x,y,z)\, dw\, dx\, dy\, dz,$$

$$1 \leq r < s < t < u \leq n,\ a,b,c,d \geq 1. \tag{7.35}$$

Then by proceeding on lines very similar to those used in proving Theorems 8 – 11, we may prove the following four theorems.

<u>Theorem 19</u>: For $n \geq 5$, $1 \leq r < s < t \leq n-2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,t,t+1:n} = \mu^{(a,b,c+d)}_{r,s,t:n} + \frac{1}{n-t}\left[d\,\mu^{(a,b,c,d-1)}_{r,s,t,t+1:n} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c,d)}_{r,s,t,t+1:n-1} - \mu^{(a,b,c+d)}_{r,s,t:n-1}\right\}\right]; \tag{7.36}$$

for $n \geq 4$, $1 \leq r < s \leq n-2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,n-1,n:n} = \mu^{(a,b,c+d)}_{r,s,n-1:n} + d\,\mu^{(a,b,c,d-1)}_{r,s,n-1,n:n} - n\left[\frac{1-P}{P}\right]\left\{P_1^d\,\mu^{(a,b,c)}_{r,s,n-1:n-1} - \mu^{(a,b,c+d)}_{r,s,n-1:n-1}\right\}; \tag{7.37}$$

for $n \geq 6$, $1 \leq r < s < t < u \leq n-1$, $u-t \geq 2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r,s,t,u-1:n} + \frac{1}{n-u+1}\left[d\,\mu^{(a,b,c,d-1)}_{r,s,t,u:n} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c,d)}_{r,s,t,u:n-1} - \mu^{(a,b,c,d)}_{r,s,t,u-1:n-1}\right\}\right]; \tag{7.38}$$

and for $n \geq 5$, $1 \leq r < s < t \leq n-2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,t,n:n} = \mu^{(a,b,c,d)}_{r,s,t,n-1:n} + d\,\mu^{(a,b,c,d-1)}_{r,s,t,n:n} - n\left[\frac{1-P}{P}\right]\left\{P_1^d\,\mu^{(a,b,c)}_{r,s,t:n-1} - \mu^{(a,b,c,d)}_{r,s,t,n-1:n-1}\right\}, \tag{7.39}$$

where $\mu^{(a,b,c,0)}_{r,s,t,u:n} \equiv \mu^{(a,b,c)}_{r,s,t:n}$.

<u>Theorem 20</u>: For $n \geq 4$, $3 \leq t < u \leq n$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{1,2,t,u:n} = \mu^{(a+b,c,d)}_{2,t,u:n} + a\mu^{(a-1,b,c,d)}_{1,2,t,u:n} - \frac{n}{P}\,\mu^{(a+b,c,d)}_{1,t-1,u-1:n-1}; \tag{7.40}$$

for $n \geq 5$, $2 \leq r < t < u \leq n$, $t-r \geq 2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,r+1,t,u:n} = \mu^{(a+b,c,d)}_{r+1,t,u:n} + \frac{1}{r}\left[a\,\mu^{(a-1,b,c,d)}_{r,r+1,t,u:n} - \frac{n}{P}\left\{\mu^{(a+b,c,d)}_{r,t-1,u-1:n-1} - \mu^{(a,b,c,d)}_{r-1,r,t-1,u-1:n-1}\right\}\right]; \tag{7.41}$$

for $n \geq 5$, $3 \leq s < t < u \leq n$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{1,s,t,u:n} = \mu^{(a,b,c,d)}_{2,s,t,u:n} + a\,\mu^{(a-1,b,c,d)}_{1,s,t,u:n} - \frac{n}{P}\,\mu^{(a,b,c,d)}_{1,s-1,t-1,u-1:n-1}; \tag{7.42}$$

and for $n \geq 6$, $2 \leq r < s < t < u \leq n$, $s-r \geq 2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r+1,s,t,u:n} + \frac{1}{r}\left[a\,\mu^{(a-1,b,c,d)}_{r,s,t,u:n}\right.$$

$$-\frac{n}{P}\left\{\mu^{(a,b,c,d)}_{r,s-1,t-1,u-1:n-1}-\mu^{(a,b,c,d)}_{r-1,s-1,t-1,u-1:n-1}\right\}\Bigg], \qquad (7.43)$$

where $\mu^{(0,b,c,d)}_{r,s,t,u:n} \equiv \mu^{(b,c,d)}_{s,t,u:n}$.

<u>Theorem 21</u>: For $n \geq 4$, $1 \leq r < u \leq n$, $u - r \geq 3$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,r+1,r+2,u:n} = \mu^{(a+b,c,d)}_{r,r+2,u:n} + b\,\mu^{(a,b-1,c,d)}_{r,r+1,r+2,u:n} - (u-r-2)\left\{\mu^{(a,b+c,d)}_{r,r+1,u:n} - \mu^{(a+b,c,d)}_{r,r+1,u:n}\right\} - (n-u+1)\left\{\mu^{(a,b+c,d)}_{r,r+1,u-1:n} - \mu^{(a+b,c,d)}_{r,r+1,u-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b+c,d)}_{r,r+1,u-1:n-1} - \mu^{(a+b,c,d)}_{r,r+1,u-1:n-1}\right\}; \qquad (7.44)$$

for $n \geq 5$, $1 \leq r < t < u \leq n$, $t - r \geq 3$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,r+1,t,u:n} = \mu^{(a+b,c,d)}_{r,t,u:n} + \frac{1}{t-r-1}\Bigg[b\,\mu^{(a,b-1,c,d)}_{r,r+1,t,u:n} - (u-t)\left\{\mu^{(a,b,c,d)}_{r,r+1,t-1,u:n} - \mu^{(a+b,c,d)}_{r,t-1,u:n}\right\} - (n-u+1)\left\{\mu^{(a,b,c,d)}_{r,r+1,t-1,u-1:n} - \mu^{(a+b,c,d)}_{r,t-1,u-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c,d)}_{r,r+1,t-1,u-1:n-1} - \mu^{(a+b,c,d)}_{r,t-1,u-1:n-1}\right\}\Bigg]; \qquad (7.45)$$

for $n \geq 5$, $1 \leq r < s < u \leq n$, $s - r \geq 2$, $u - s \geq 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,s+1,u:n} = \mu^{(a,b,c,d)}_{r,s-1,s+1,u:n} + b\mu^{(a,b-1,c,d)}_{r,s,s+1,u:n} - (u-s-1)\left\{\mu^{(a,b+c,d)}_{r,s,u:n} - \mu^{(a,b,c,d)}_{r,s-1,s,u:n}\right\} - (n-u+1)\left\{\mu^{(a,b+c,d)}_{r,s,u-1:n} - \mu^{(a,b,c,d)}_{r,s-1,s,u-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b+c,d)}_{r,s,u-1:n-1} - \mu^{(a,b,c,d)}_{r,s-1,s,u-1:n-1}\right\}; \qquad (7.46)$$

for $n \geq 6$, $1 \leq r < s < t < u \leq n$, $s - r \geq 2$, $t - s \geq 2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r,s-1,t,u:n} + \frac{1}{t-s}\Bigg[b\,\mu^{(a,b-1,c,d)}_{r,s,t,u:n} - (u-t)\left\{\mu^{(a,b,c,d)}_{r,s,t-1,u:n} - \mu^{(a,b,c,d)}_{r,s-1,t-1,u:n}\right\} - (n-u+1)\left\{\mu^{(a,b,c,d)}_{r,s,t-1,u-1:n} - \mu^{(a,b,c,d)}_{r,s-1,t-1,u-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c,d)}_{r,s,t-1,u-1:n-1} - \mu^{(a,b,c,d)}_{r,s-1,t-1,u-1:n-1}\right\}\Bigg], \qquad (7.47)$$

where $\mu^{(a,0,c,d)}_{r,s,t,u:n} \equiv \mu^{(a,c,d)}_{r,t,u:n}$.

<u>Theorem 22</u>: For $n \geq 4$, $1 \leq r < s \leq n-2$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,s+1,s+2:n} = \mu^{(a,b+c,d)}_{r,s,s+2:n} + c\,\mu^{(a,b,c-1,d)}_{r,s,s+1,s+2:n} - (n-s-1)\left\{\mu^{(a,b,c+d)}_{r,s,s+1:n} - \mu^{(a,b+c,d)}_{r,s,s+1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c+d)}_{r,s,s+1:n-1} - \mu^{(a,b+c,d)}_{r,s,s+1:n-1}\right\}; \qquad (7.48)$$

for $n \geq 5$, $1 \leq r < s < u \leq n$, $u - s \geq 3$ and $a,b,c,d = 1,2,\ldots$,

$$\mu^{(a,b,c,d)}_{r,s,s+1,u:n} = \mu^{(a,b+c,d)}_{r,s,u:n} + \frac{1}{u-s-1}\Bigg[c\,\mu^{(a,b,c-1,d)}_{r,s,s+1,u:n} - (n-u+1)\left\{\mu^{(a,b,c,d)}_{r,s,s+1,u-1:n}\right.$$

$$- \mu^{(a,b+c,d)}_{r,s,u-1:n}\Big\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c,d)}_{r,s,s+1,u-1:n-1} - \mu^{(a,b+c,d)}_{r,s,u-1:n-1}\right\}\Big]; \tag{7.49}$$

for $n \geq 5$, $1 \leq r < s < t \leq n-1$, $t-s \geq 2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,t,t+1:n} = \mu^{(a,b,c,d)}_{r,s,t-1,t+1:n} + c\mu^{(a,b,c-1,d)}_{r,s,t,t+1:n} - (n-t)\left\{\mu^{(a,b,c+d)}_{r,s,t:n} - \mu^{(a,b,c,d)}_{r,s,t-1,t:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c+d)}_{r,s,t:n-1} - \mu^{(a,b,c,d)}_{r,s,t-1,t:n-1}\right\}; \tag{7.50}$$

and for $n \geq 6$, $1 \leq r < s < t < u \leq n$, $t-s \geq 2$, $u-t \geq 2$ and a,b,c,d = 1,2,...,

$$\mu^{(a,b,c,d)}_{r,s,t,u:n} = \mu^{(a,b,c,d)}_{r,s,t-1,u:n} + \frac{1}{u-t}\left[c\,\mu^{(a,b,c-1,d)}_{r,s,t,u:n} - (n-u+1)\left\{\mu^{(a,b,c,d)}_{r,s,t,u-1:n} - \mu^{(a,b,c,d)}_{r,s,t-1,u-1:n}\right\} - n\left[\frac{1-P}{P}\right]\left\{\mu^{(a,b,c,d)}_{r,s,t,u-1:n-1} - \mu^{(a,b,c,d)}_{r,s,t-1,u-1:n-1}\right\}\right], \tag{7.51}$$

where $\mu^{(a,b,0,d)}_{r,s,t,u:n} \equiv \mu^{(a,b,d)}_{r,s,u:n}$.

The recurrence relations presented in any one of Theorems 19, 20, 21 and 22 may be used in a simple systematic recursive manner in order to compute the quadruple moments (of all order) of order statistics for all sample sizes. In particular, as shown earlier in Section 6 for the exponential case, the results presented in Theorems 12 – 21 may be used to compute the mean, variance and the coefficients of skewness and kurtosis for any linear function of order statistics from the right–truncated exponential distribution in (7.2) which may then be used to approximate the distribution of that linear function of order statistics. It is of interest to mention here that Theorems 1 – 11 may be deduced from Theorems 12 – 22 by letting $P \longrightarrow 1$.

8. Illustrative Examples

In this section, we consider two examples involving life–test data and illustrate how we may use the results presented in Section 6 in order to develop approximate confidence intervals and carry out tests of hypotheses approximately for the scale parameter σ of the exponential distribution based on doubly Type–II censored samples.

Example 1: Twelve components were placed on a life–test and the time–to–fail of the first eight components that failed were recorded and the experiment was terminated as soon as the eighth failure occurred. The Type–II right–censored sample that resulted from this experiment, as reported by Lawless (1982), is given below:

31, 58, 157, 185, 300, 470, 497, 673, —, —, —, —

In this case, we have n = 12, r = 0 and s = 4. The best linear unbiased estimate of σ is obtained from (6.3) to be

$$\sigma^* = \frac{1}{8}\left\{\sum_{i=1}^{8} Y_{i:12} + 4\, Y_{8:12}\right\} = 632.875.$$

Furthermore, since the censoring in this data is only on the right, it is known that $2(n - s)\, \sigma^*/\sigma = 16\sigma^*/\sigma$ has exactly a chi–square distribution with 16 degrees of freedom. By using this distributional result, we obtain the 95% lower confidence limit for the true expected life–time σ to be $16\sigma^*/26.296 = 385.078$. Similarly, we obtain the 95% confidence interval for σ to be

$$\left[\frac{16\,\sigma^*}{28.845}, \frac{16\,\sigma^*}{6.908}\right] = \left[351.049,\ 1465.837\right].$$

Example 2: Let us consider the following data which represent failure times, in minutes, for a specific type of electrical insulation in an experiment in which the insulation was subjected to a continuously increasing voltage stress (see Lawless, 1982, pp. 138):

—, —, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6, —

Here, the smallest two observations are censored as their failure times were not recorded due to experimental difficulty and the largest observation is censored because the experiment was stopped as soon as the eleventh failure occurred.

In this case, we have n = 12, r = 2, s = 1, and

$$K = 8 + \frac{\left[\frac{1}{10} + \frac{1}{11} + \frac{1}{12}\right]^2}{\left[\frac{1}{100} + \frac{1}{121} + \frac{1}{144}\right]} = 10.9834.$$

The best linear unbiased estimate of σ is obtained from (6.3) to be

$$\sigma^* = \frac{1}{10.9834}\left\{1.8788\ Y_{3:12} + \sum_{i=4}^{10} Y_{i:12} + 2\ Y_{11:12}\right\}$$

$= 71.1385$ minutes.

From Table 5, we observe that the value of γ in this case is 0.000815 indicating that the distribution of σ^*/σ can be approximated very closely by a chi–square distribution. Specifically, we may approximate the distribution of $22\ \sigma^*/\sigma$ by a chi–square distribution with 22 degrees of freedom. By making use of this approximate distributional result, we obtain the 95% lower confidence limit for the true expected life–time σ to be $22\sigma^*/33.924$ $= 46.1339$ mins. Further, we obtain an approximate 95% confidence interval for σ to be

$$\left[\frac{22\ \sigma^*}{36.781}, \frac{22\ \sigma^*}{10.982}\right] = \left[42.5504,\ 142.5102\right].$$

Proceeding similarly, we obtain the approximate 90% lower confidence limit for the true expected life–time σ to be $22\sigma^*/30.813 = 50.7918$ mins. We also obtain an approximate 90% confidence interval for σ to be

$$\left[\frac{22\ \sigma^*}{33.924}, \frac{22\ \sigma^*}{12.338}\right] = \left[46.1339,\ 126.8477\right].$$

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Table 1. Single moments $\mu_{r:n}^{(a)}$ for $1 \le r \le n \le 8$

n	r	(a) (1)	(2)	(3)	(4)
1	1	1.000000	2.000000	6.000000	24.000000
2	1	0.500000	0.500000	0.750000	1.500000
2	2	1.500000	3.500000	11.250000	46.500000
3	1	0.333333	0.222222	0.222222	0.296296
3	2	0.833333	1.055556	1.805556	3.907407
3	3	1.833333	4.722222	15.972222	67.796296
4	1	0.250000	0.125000	0.093750	0.093750
4	2	0.583333	0.513889	0.607639	0.903935
4	3	1.083333	1.597222	3.003472	6.910880
4	4	2.083333	5.763889	20.295139	88.091435
5	1	0.200000	0.080000	0.048000	0.038400
5	2	0.450000	0.305000	0.276750	0.315150
5	3	0.783333	0.827222	1.103972	1.787113
5	4	1.283333	2.110556	4.269806	10.326724
5	5	2.283333	6.677222	24.301472	107.532613
6	1	0.166667	0.055556	0.027778	0.018519
6	2	0.366667	0.202222	0.149111	0.137807
6	3	0.616667	0.510556	0.532028	0.669835
6	4	0.950000	1.143889	1.675917	2.904391
6	5	1.450000	2.593889	5.566750	14.037891
6	6	2.450000	7.493889	28.048417	126.231557
7	1	0.142857	0.040816	0.017493	0.009996
7	2	0.309524	0.143991	0.089488	0.069655
7	3	0.509524	0.347800	0.298168	0.308189
7	4	0.759524	0.727562	0.843840	1.152030
7	5	1.092857	1.456134	2.299974	4.218662
7	6	1.592857	3.048991	6.873460	17.965582
7	7	2.592857	8.234705	31.577576	144.275887
8	1	0.125000	0.031250	0.011719	0.005859
8	2	0.267857	0.107781	0.057910	0.038951
8	3	0.434524	0.252622	0.184221	0.161765
8	4	0.634524	0.506431	0.488080	0.552229
8	5	0.884524	0.948693	1.199600	1.751830
8	6	1.217857	1.760598	2.960198	5.698761
8	7	1.717857	3.478455	8.177881	22.054523
8	8	2.717857	8.914170	34.920390	161.736081

Table 2. Double moments $\mu_{r,s:n}^{(a,b)}$ for $1 \leq r < s \leq n \leq 8$

n	r	s	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(1,3)
			(a,b)					
2	1	2	1.000000	1.250000	2.250000	2.750000	4.000000	9.750000
3	1	2	0.388889	0.333333	0.407407	0.611111	0.629630	1.212963
3	1	3	0.722222	0.555556	0.629630	2.055556	1.740741	7.379630
3	2	3	1.888889	2.861111	5.712963	5.583333	9.629630	20.657407
4	1	2	0.208333	0.135417	0.125000	0.232639	0.184028	0.326389
4	1	3	0.333333	0.197917	0.171875	0.565972	0.381944	1.175347
4	1	4	0.583333	0.322917	0.265625	1.732639	1.027778	6.373264
4	2	3	0.805556	0.864583	1.207755	1.413194	1.768519	3.023727
4	2	4	1.388889	1.378472	1.815394	4.190972	4.525463	15.596644
4	3	4	2.680556	4.600694	9.914352	8.364583	16.112269	32.004630
5	1	2	0.130000	0.068000	0.050400	0.113000	0.072400	0.123150
5	1	3	0.196667	0.094667	0.066400	0.244111	0.135511	0.367261
5	1	4	0.296667	0.134667	0.090400	0.540778	0.270178	1.178428
5	1	5	0.496667	0.214667	0.138400	1.534111	0.699511	5.780761
5	2	3	0.455000	0.378417	0.407400	0.580083	0.567428	0.895233
5	2	4	0.680000	0.530917	0.545775	1.260083	1.098344	2.785358
5	2	5	1.130000	0.835917	0.822525	3.520083	2.770178	13.345608
5	3	4	1.218889	1.517583	2.339099	2.322861	3.304696	5.271405
5	3	5	2.002222	2.344806	3.443071	6.327306	7.994307	24.253321
5	4	5	3.393889	6.380361	14.596530	11.057583	23.087446	43.499474
6	1	2	0.088889	0.038889	0.024074	0.063333	0.034074	0.056519
6	1	3	0.130556	0.052778	0.031019	0.128611	0.060463	0.152977
6	1	4	0.186111	0.071296	0.040278	0.252685	0.107994	0.405662
6	1	5	0.269444	0.099074	0.054167	0.522130	0.207068	1.188856
6	1	6	0.436111	0.154630	0.081944	1.394352	0.516327	5.371912
6	2	3	0.293889	0.199667	0.175085	0.296056	0.237641	0.359849
6	2	4	0.416111	0.267074	0.224789	0.573463	0.415690	0.933312
6	2	5	0.599444	0.368185	0.299344	1.172907	0.783875	2.692673
6	2	6	0.966111	0.570407	0.448456	3.105130	1.924690	12.008062
6	3	4	0.716111	0.702213	0.847178	1.009435	1.137977	1.679270
6	3	5	1.024444	0.957491	1.113192	2.033880	2.095468	4.730090
6	3	6	1.641111	1.468046	1.645219	5.316102	5.031560	20.678395
6	4	5	1.618889	2.247861	3.742349	3.294806	5.152252	7.846599
6	4	6	2.568889	3.391750	5.418266	8.432583	11.935752	33.144349
6	5	6	4.043889	8.160639	19.604641	13.654528	30.359169	55.001474
7	1	2	0.064626	0.024295	0.012911	0.039035	0.018094	0.029513
7	1	3	0.093197	0.032459	0.016410	0.076314	0.031078	0.075301
7	1	4	0.128912	0.042663	0.020783	0.140769	0.052409	0.180878
7	1	5	0.176531	0.056268	0.026614	0.258456	0.089921	0.439335
7	1	6	0.247959	0.076676	0.035360	0.506416	0.166598	1.198958
7	1	7	0.390816	0.117493	0.052853	1.288048	0.401583	5.063103
7	2	3	0.205896	0.118286	0.087552	0.171846	0.116969	0.172762
7	2	4	0.283277	0.154284	0.109924	0.313485	0.194111	0.407876
7	2	5	0.386451	0.202281	0.139754	0.571119	0.328965	0.978995
7	2	6	0.541213	0.274277	0.184498	1.112332	0.603242	2.647493

n	r	s	(a,b) (1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(1,3)
7	2	7	0.850737	0.418267	0.273986	2.813806	1.439777	11.088911
7	3	4	0.475181	0.385119	0.382731	0.535759	0.500749	0.710009
7	3	5	0.645023	0.501052	0.482121	0.965774	0.834783	1.675783
7	3	6	0.899785	0.674952	0.631205	1.865559	1.509736	4.474121
7	3	7	1.409308	1.022753	0.929374	4.684176	3.555241	18.526648
7	4	5	0.980737	1.086361	1.433310	1.497665	1.876270	2.649694
7	4	6	1.360499	1.450142	1.855230	2.858164	3.326412	6.936940
7	4	7	2.120023	2.177705	2.699070	7.098209	7.681822	28.231567
7	5	6	2.002562	3.028041	5.368649	4.302536	7.246702	10.672466
7	5	7	3.095420	4.484175	7.668623	10.493375	16.215052	42.152592
7	6	7	4.641848	9.922451	24.839043	16.157157	37.810485	66.437052
8	1	2	0.049107	0.016183	0.007533	0.025749	0.010483	0.016895
8	1	3	0.069940	0.021391	0.009487	0.049063	0.017614	0.041426
8	1	4	0.094940	0.027641	0.011830	0.087039	0.028670	0.093650
8	1	5	0.126190	0.035454	0.014760	0.150134	0.046397	0.206250
8	1	6	0.167857	0.045871	0.018666	0.262039	0.076977	0.468289
8	1	7	0.230357	0.061496	0.024526	0.492396	0.138473	1.206884
8	1	8	0.355357	0.092746	0.036244	1.203110	0.323964	4.816215
8	2	3	0.152423	0.075874	0.048603	0.108718	0.064242	0.093310
8	2	4	0.205995	0.097430	0.060185	0.191116	0.103214	0.207980
8	2	5	0.272959	0.124375	0.074662	0.327596	0.165402	0.453677
8	2	6	0.362245	0.160302	0.093966	0.569092	0.272270	1.022769
8	2	7	0.496173	0.214192	0.122921	1.065266	0.486462	2.620668
8	2	8	0.764031	0.321973	0.180832	2.593327	1.130408	10.400649
8	3	4	0.339527	0.234746	0.198610	0.320032	0.255664	0.353785
8	3	5	0.448158	0.297901	0.244665	0.544111	0.404614	0.761868
8	3	6	0.592999	0.382109	0.306072	0.939443	0.659353	1.701311
8	3	7	0.810261	0.508419	0.398183	1.749704	1.167773	4.325867
8	3	8	1.244785	0.761041	0.582404	4.239273	2.689855	17.043687
8	4	5	0.665062	0.614688	0.674250	0.820611	0.859574	1.167688
8	4	6	0.876570	0.783499	0.836943	1.404992	1.381906	2.572680
8	4	7	1.193832	1.036714	1.080983	2.598824	2.418620	6.470915
8	4	8	1.828356	1.543146	1.569063	6.255536	5.504911	25.237523
8	5	6	1.243535	1.515831	2.151696	2.028623	2.762384	3.780453
8	5	7	1.685796	1.990178	2.751497	3.714420	4.752562	9.352083
8	5	8	2.570320	2.938871	3.951097	8.855060	10.630304	35.917264
8	6	7	2.369527	3.840497	7.178860	5.329725	9.539258	13.693348
8	6	8	3.587384	5.601095	10.139058	12.504492	20.741449	51.206826
8	7	8	5.196312	11.656336	30.232404	18.570506	45.367196	77.766040

Table 3. Triple moments $\mu_{r,s,t:n}^{(a,b,c)}$ for $1 \le r < s < t \le n \le 8$

n	r	s	t	(a,b,c) (1,1,1)	(2,1,1)	(1,2,1)	(1,1,2)
3	1	2	3	1.000000	0.962963	1.824074	3.212963
4	1	2	3	0.336806	0.251736	0.442708	0.663194
4	1	2	4	0.545139	0.387153	0.675347	1.753472
4	1	3	4	0.899306	0.579861	1.741319	2.973958
4	2	3	4	2.218750	2.633102	4.436921	7.461227
5	1	2	3	0.156333	0.095067	0.160817	0.227372
5	1	2	4	0.221333	0.129067	0.217317	0.448706
5	1	2	5	0.351333	0.197067	0.330317	1.151372
5	1	3	4	0.342444	0.182844	0.489317	0.709706
5	1	3	5	0.539111	0.277511	0.733428	1.787928
5	1	4	5	0.837444	0.404844	1.719206	2.853317
5	2	3	4	0.807583	0.756636	1.185275	1.702817
5	2	3	5	1.262583	1.135053	1.765358	4.227983
5	2	4	5	1.940083	1.629261	4.045442	6.665525
5	3	4	5	3.541750	4.822280	7.594266	12.354905
6	1	2	3	0.085556	0.043796	0.072352	0.099296
6	1	2	4	0.115185	0.056759	0.093463	0.176086
6	1	2	5	0.159630	0.076204	0.125130	0.335716
6	1	2	6	0.248519	0.115093	0.188463	0.832753
6	1	3	4	0.172130	0.078056	0.195847	0.267730
6	1	3	5	0.237407	0.104444	0.260153	0.505137
6	1	3	6	0.367963	0.157222	0.388764	1.241063
6	1	4	5	0.345741	0.143642	0.532005	0.751403
6	1	4	6	0.531852	0.214938	0.784690	1.815106
6	1	5	6	0.791574	0.306142	1.710986	2.772005
6	2	3	4	0.394019	0.304196	0.458534	0.622528
6	2	3	5	0.540963	0.404030	0.606562	1.163491
6	2	3	6	0.834852	0.603696	0.902618	2.833195
6	2	4	5	0.781519	0.549227	1.220044	1.714831
6	2	4	6	1.197630	0.816301	1.793506	4.110090
6	2	5	6	1.772352	1.152060	3.865581	6.237377
6	3	4	5	1.367491	1.489084	2.183988	3.046761
6	3	4	6	2.083602	2.191297	3.193423	7.213965
6	3	5	6	3.058324	3.052959	6.763969	10.846738
6	4	5	6	4.913694	7.400113	11.141405	17.673988
7	1	2	3	0.051960	0.022953	0.037320	0.050297
7	1	2	4	0.068116	0.029027	0.047079	0.084355
7	1	2	5	0.089658	0.037126	0.060090	0.144127
7	1	2	6	0.121971	0.049273	0.079608	0.266099
7	1	2	7	0.186597	0.073569	0.118642	0.639293
7	1	3	4	0.099613	0.039192	0.094380	0.125108
7	1	3	5	0.130679	0.050012	0.119818	0.212227
7	1	3	6	0.177277	0.066241	0.157974	0.389504
7	1	3	7	0.270475	0.098700	0.234288	0.930453
7	1	4	5	0.183740	0.066630	0.227801	0.303372
7	1	4	6	0.248196	0.087961	0.298186	0.551567
7	1	4	7	0.377107	0.130624	0.438955	1.305782
7	1	5	6	0.346722	0.118055	0.568563	0.786056
7	1	5	7	0.523252	0.174324	0.827019	1.832561

n	r	s	t	(a,b,c) (1,1,1)	(2,1,1)	(1,2,1)	(1,1,2)
7	1	6	7	0.754375	0.243274	1.705374	2.707708
7	2	3	4	0.223320	0.146541	0.215724	0.284423
7	2	3	5	0.291952	0.185970	0.273006	0.479058
7	2	3	6	0.394900	0.245113	0.358929	0.873958
7	2	3	7	0.600796	0.363399	0.530776	2.075549
7	2	4	5	0.407910	0.245539	0.512371	0.679816
7	2	4	6	0.549549	0.322681	0.669113	1.229365
7	2	4	7	0.832825	0.476965	0.982598	2.895016
7	2	5	6	0.764345	0.430106	1.264554	1.743340
7	2	5	7	1.150796	0.632387	1.835673	4.044931
7	2	6	7	1.653345	0.877518	3.759825	5.954584
7	3	4	5	0.694153	0.629122	0.888595	1.172777
7	3	4	6	0.931744	0.821681	1.156475	2.104521
7	3	4	7	1.406925	1.206799	1.692234	4.918371
7	3	5	6	1.288286	1.085309	2.158670	2.964069
7	3	5	7	1.933308	1.586361	3.124444	6.830685
7	3	6	7	2.765343	2.184688	6.339680	10.004808
7	4	5	6	1.988033	2.419451	3.398527	4.637728
7	4	5	7	2.968770	3.505812	4.896192	10.575268
7	4	6	7	4.218663	4.776555	9.795104	15.374265
7	5	6	7	6.305099	10.274743	14.975003	23.282664
8	1	2	3	0.033934	0.013180	0.021186	0.028206
8	1	2	4	0.043755	0.016417	0.026336	0.045708
8	1	2	5	0.056032	0.020463	0.032774	0.073724
8	1	2	6	0.072401	0.025857	0.041357	0.121992
8	1	2	7	0.096955	0.033949	0.054231	0.218946
8	1	2	8	0.146062	0.050132	0.079981	0.511070
8	1	3	4	0.063051	0.021892	0.051239	0.066647
8	1	3	5	0.080536	0.027240	0.063505	0.106915
8	1	3	6	0.103850	0.034370	0.079859	0.176148
8	1	3	7	0.138820	0.045066	0.104390	0.314967
8	1	3	8	0.208760	0.066457	0.153453	0.732488
8	1	4	5	0.110774	0.035580	0.115409	0.149037
8	1	4	6	0.142421	0.044794	0.144422	0.243984
8	1	4	7	0.189891	0.058615	0.187942	0.433875
8	1	4	8	0.284832	0.086256	0.274981	1.003539
8	1	5	6	0.192198	0.058215	0.256295	0.334382
8	1	5	7	0.255293	0.075942	0.331362	0.589675
8	1	5	8	0.381483	0.111396	0.481497	1.352642
8	1	6	7	0.345968	0.099913	0.599309	0.814257
8	1	6	8	0.513825	0.145783	0.861348	1.841907
8	1	7	8	0.722753	0.199968	1.699280	2.652390
8	2	3	4	0.139203	0.079417	0.115054	0.148991
8	2	3	5	0.177309	0.098386	0.142233	0.237646
8	2	3	6	0.228117	0.123677	0.178473	0.389724
8	2	3	7	0.304328	0.161614	0.232832	0.694052
8	2	3	8	0.456752	0.237488	0.341550	1.607556
8	2	4	5	0.242615	0.127572	0.255759	0.329287
8	2	4	6	0.311280	0.160049	0.319464	0.536807
8	2	4	7	0.414277	0.208764	0.415022	0.951085

n	r	s	t	(a,b,c) (1,1,1)	(2,1,1)	(1,2,1)	(1,1,2)
8	2	4	8	0.620272	0.306194	0.606139	2.191629
8	2	5	6	0.418582	0.206860	0.562875	0.732732
8	2	5	7	0.555062	0.269048	0.726673	1.287793
8	2	5	8	0.828021	0.393423	1.054269	2.943835
8	2	6	7	0.750215	0.352421	1.307315	1.772984
8	2	6	8	1.112460	0.512723	1.876408	3.997904
8	2	7	8	1.561439	0.700655	3.685934	5.743547
8	3	4	5	0.404914	0.314350	0.433793	0.556241
8	3	4	6	0.518089	0.392599	0.540470	0.901634
8	3	4	7	0.687853	0.509972	0.700486	1.589487
8	3	4	8	1.027379	0.744717	1.020518	3.644245
8	3	5	6	0.693497	0.503915	0.943238	1.224199
8	3	5	7	0.917576	0.652865	1.215293	2.141774
8	3	5	8	1.365733	0.950766	1.759404	4.873240
8	3	6	7	1.235943	0.850408	2.171033	2.937254
8	3	6	8	1.828942	1.232516	3.110476	6.595137
8	3	7	8	2.559965	1.676192	6.075572	9.445797
8	4	5	6	1.042299	1.064470	1.441225	1.862554
8	4	5	7	1.374830	1.371814	1.851531	3.237384
8	4	5	8	2.039892	1.986502	2.672142	7.317169
8	4	6	7	1.843277	1.773655	3.275175	4.415956
8	4	6	8	2.719847	2.557154	4.680167	9.855650
8	4	7	8	3.792656	3.455334	9.069739	14.056227
8	5	6	7	2.650391	3.520300	4.794765	6.430844
8	5	6	8	3.893925	5.036131	6.823388	14.218694
8	5	7	8	5.400216	6.742740	13.066502	20.152515
8	6	7	8	7.699252	13.379755	19.023073	29.091851

Table 4. Quadruple moments $\mu^{(a,b,c,d)}_{r,s,t,u:n}$ for $1 \leq r < s < t < u \leq n \leq 8$

n	r	s	t	u	(a,b,c,d) = (1,1,1,1)	n	r	s	t	u	(a,b,c,d) = (1,1,1,1)
4	1	2	3	4	1.000000	7	1	4	6	7	0.799763
5	1	2	3	4	0.305539	7	1	5	6	7	1.132778
5	1	2	3	5	0.461872	7	2	3	4	5	0.358863
5	1	2	4	5	0.670039	7	2	3	4	6	0.470523
5	1	3	4	5	1.052150	7	2	3	4	7	0.693843
5	2	3	4	5	2.510400	7	2	3	5	6	0.625034
6	1	2	3	4	0.127815	7	2	3	5	7	0.916986
6	1	2	3	5	0.170593	7	2	3	6	7	1.268858
6	1	2	3	6	0.256148	7	2	4	5	6	0.883771
6	1	2	4	5	0.233679	7	2	4	5	7	1.291682
6	1	2	4	6	0.348864	7	2	4	6	7	1.778914
6	1	2	5	6	0.495346	7	2	5	6	7	2.507684
6	1	3	4	5	0.353795	7	3	4	5	6	1.519854
6	1	3	4	6	0.525924	7	3	4	5	7	2.214007
6	1	3	5	6	0.742545	7	3	4	6	7	3.036265
6	1	4	5	6	1.097144	7	3	5	6	7	4.252354
6	2	3	4	5	0.819537	7	4	5	6	7	6.625761
6	2	3	4	6	1.213556	8	1	2	3	4	0.034993
6	2	3	5	6	1.704454	8	1	2	3	5	0.043476
6	2	4	5	6	2.496349	8	1	2	3	6	0.054788
6	3	4	5	6	4.414252	8	1	2	3	7	0.071755
7	1	2	3	4	0.063287	8	1	2	3	8	0.105688
7	1	2	3	5	0.080607	8	1	2	4	5	0.056647
7	1	2	3	6	0.106587	8	1	2	4	6	0.071232
7	1	2	3	7	0.158547	8	1	2	4	7	0.093110
7	1	2	4	5	0.107061	8	1	2	4	8	0.136865
7	1	2	4	6	0.141119	8	1	2	5	6	0.092402
7	1	2	4	7	0.209235	8	1	2	5	7	0.120418
7	1	2	5	6	0.188957	8	1	2	5	8	0.176450
7	1	2	5	7	0.278615	8	1	2	6	7	0.158192
7	1	2	6	7	0.388070	8	1	2	6	8	0.230593
7	1	3	4	5	0.158312	8	1	2	7	8	0.315901
7	1	3	4	6	0.208119	8	1	3	4	5	0.082409
7	1	3	4	7	0.307731	8	1	3	4	6	0.103426
7	1	3	5	6	0.277566	8	1	3	4	7	0.134952
7	1	3	5	7	0.408245	8	1	3	4	8	0.198003
7	1	3	6	7	0.566781	8	1	3	5	6	0.133760
7	1	4	5	6	0.395242	8	1	3	5	7	0.174028
7	1	4	5	7	0.578981	8	1	3	5	8	0.254564

n	r	s	t	u	(a,b,c,d) = (1,1,1,1)	n	r	s	t	u	(a,b,c,d) = (1,1,1,1)
8	1	3	6	7	0.228072	8	2	4	5	7	0.531467
8	1	3	6	8	0.331922	8	2	4	5	8	0.774081
8	1	3	7	8	0.453787	8	2	4	6	7	0.692447
8	1	4	5	6	0.185961	8	2	4	6	8	1.003727
8	1	4	5	7	0.241349	8	2	4	7	8	1.365362
8	1	4	5	8	0.352123	8	2	5	6	7	0.942023
8	1	4	6	7	0.315195	8	2	5	6	8	1.360605
8	1	4	6	8	0.457616	8	2	5	7	8	1.842855
8	1	4	7	8	0.623767	8	2	6	7	8	2.523199
8	1	5	6	7	0.430481	8	3	4	5	6	0.691213
8	1	5	6	8	0.622679	8	3	4	5	7	0.893669
8	1	5	7	8	0.844968	8	3	4	5	8	1.298583
8	1	6	7	8	1.160225	8	3	4	6	7	1.160679
8	2	3	4	5	0.183792	8	3	4	6	8	1.678768
8	2	3	4	6	0.230193	8	3	4	7	8	2.277339
8	2	3	4	7	0.299795	8	3	5	6	7	1.570947
8	2	3	4	8	0.438998	8	3	5	6	8	2.264444
8	2	3	5	6	0.296749	8	3	5	7	8	3.059350
8	2	3	5	7	0.385403	8	3	6	7	8	4.173197
8	2	3	5	8	0.562712	8	4	5	6	7	2.383703
8	2	3	6	7	0.503782	8	4	5	6	8	3.426002
8	2	3	6	8	0.731899	8	4	5	7	8	4.612214
8	2	3	7	8	0.998380	8	4	6	7	8	6.259233
8	2	4	5	6	0.410159	8	5	6	7	8	9.081234

Table 5. Mean, variance, $\sqrt{\beta_1}$, β_2 and $\gamma = \beta_2 - 1.5\ \beta_1 - 3$ values of σ^* / σ for doubly Type – II censored samples

n	r	s	Mean	Variance	$\sqrt{\beta_1}$	β_2	γ
8	0	0	1.000000	0.125000	0.707107	3.750000	0.000000
8	0	1	1.000000	0.142857	0.755929	3.857143	0.000000
8	0	2	1.000000	0.166667	0.816497	4.000000	0.000000
8	1	0	1.000000	0.125138	0.708271	3.753285	0.000813
8	1	1	1.000000	0.143038	0.757352	3.861434	0.001061
8	1	2	1.000000	0.166913	0.818290	4.005843	0.001445
8	2	0	1.000000	0.125645	0.712513	3.765300	0.003788
8	2	1	1.000000	0.143701	0.762540	3.877152	0.004951
8	2	2	1.000000	0.167816	0.824838	4.027282	0.006745
8	3	0	1.000000	0.126929	0.723164	3.795792	0.011343
8	3	1	1.000000	0.145383	0.775601	3.917184	0.014849
8	3	2	1.000000	0.170114	0.841378	4.082145	0.020270
9	0	0	1.000000	0.111111	0.666667	3.666667	0.000000
9	0	1	1.000000	0.125000	0.707107	3.750000	0.000000
9	0	2	1.000000	0.142857	0.755929	3.857143	0.000000
9	1	0	1.000000	0.111196	0.667428	3.668693	0.000503
9	1	1	1.000000	0.125108	0.708016	3.752566	0.000636
9	1	2	1.000000	0.142998	0.757040	3.860495	0.000831
9	2	0	1.000000	0.111500	0.670135	3.675917	0.002296
9	2	1	1.000000	0.125493	0.711247	3.761717	0.002909
9	2	2	1.000000	0.143501	0.760991	3.872462	0.003801
9	3	0	1.000000	0.112243	0.676703	3.693582	0.006692
9	3	1	1.000000	0.126434	0.719101	3.784140	0.008481
9	3	2	1.000000	0.144733	0.770614	3.901863	0.011094
10	0	0	1.000000	0.100000	0.632456	3.600000	0.000000
10	0	1	1.000000	0.111111	0.666667	3.666667	0.000000
10	0	2	1.000000	0.125000	0.707107	3.750000	0.000000
10	1	0	1.000000	0.100055	0.632977	3.601317	0.000327
10	1	1	1.000000	0.111179	0.667278	3.668293	0.000403
10	1	2	1.000000	0.125086	0.707836	3.752059	0.000511
10	2	0	1.000000	0.100249	0.634795	3.605919	0.001472
10	2	1	1.000000	0.111418	0.669408	3.673978	0.001817
10	2	2	1.000000	0.125389	0.710379	3.759259	0.002302
10	3	0	1.000000	0.100708	0.639098	3.616872	0.004203
10	3	1	1.000000	0.111986	0.674454	3.687525	0.005193
10	3	2	1.000000	0.126108	0.716410	3.776445	0.006580
11	0	0	1.000000	0.090909	0.603023	3.545455	0.000000
11	0	1	1,000000	0.100000	0.632456	3.600000	0.000000
11	0	2	1.000000	0.111111	0.666667	3.666667	0.000000
11	0	3	1.000000	0.125000	0.707107	3.750000	0.000000
11	1	0	1.000000	0.090947	0.603393	3.546347	0.000222
11	1	1	1.000000	0.100045	0.632883	3.601080	0.000269
11	1	2	1.000000	0.111167	0.667167	3.668000	0.000332

n	r	s	Mean	Variance	$\sqrt{\beta_1}$	β_2	γ
11	1	3	1.000000	0.125071	0.707704	3.751688	0.000421
11	2	0	1.000000	0.091075	0.604665	3.549417	0.000987
11	2	1	1.000000	0.100201	0.634351	3.604796	0.001194
11	2	2	1.000000	0.111359	0.668887	3.672590	0.001475
11	2	3	1.000000	0.125314	0.709758	3.757500	0.001865
11	3	0	1.000000	0.091375	0.607618	3.556574	0.002775
11	3	1	1.000000	0.100564	0.637761	3.613466	0.003357
11	3	2	1.000000	0.111808	0.672885	3.683310	0.004149
11	3	3	1.000000	0.125883	0.714534	3.771093	0.005255
11	4	0	1.000000	0.091970	0.613443	3.570803	0.006335
11	4	1	1.000000	0.101285	0.644494	3.630734	0.007675
11	4	2	1.000000	0.112700	0.680792	3.704705	0.009488
11	4	3	1.000000	0.127015	0.723999	3.798294	0.012032
12	0	0	1.000000	0.083333	0.577350	3.500000	0.000000
12	0	1	1.000000	0.090909	0.603023	3.545455	0.000000
12	0	2	1.000000	0.100000	0.632456	3.600000	0.000000
12	0	3	1.000000	0.111111	0.666667	3.666667	0.000000
12	1	0	1.000000	0.083360	0.577622	3.500626	0.000155
12	1	1	1.000000	0.090940	0.603332	3.546200	0.000186
12	1	2	1.000000	0.100038	0.632812	3.600902	0.000226
12	1	3	1.000000	0.111158	0.667085	3.667780	0.000276
12	2	0	1.000000	0.083449	0.578542	3.502752	0.000686
12	2	1	1.000000	0.091046	0.604381	3.548730	0.000815
12	2	2	1.000000	0.100166	0.634022	3.603964	0.000988
12	2	3	1.000000	0.111316	0.668502	3.671562	0.001220
12	3	0	1.000000	0.083653	0.580645	3.507628	0.001905
12	3	1	1.000000	0.091289	0.606779	3.554538	0.002267
12	3	2	1.000000	0.100460	0.636791	3.610999	0.002745
12	3	3	1.000000	0.111680	0.671748	3.680258	0.003390
12	4	0	1.000000	0.084049	0.584714	3.517118	0.004282
12	4	1	1.000000	0.091762	0.611422	3.565854	0.005099
12	4	2	1.000000	0.101033	0.642157	3.624724	0.006176
12	4	3	1.000000	0.112388	0.678045	3.697252	0.007635