A GEOMETRICAL CHARACTERIZATION OF INTRINSIC ULTRA-CONTRACTIVITY FOR PLANAR DOMAINS WITH BOUNDARIES GIVEN BY THE GRAPHS OF FUNCTIONS

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Abstract

We give a simple geometric characterization for planar domains with boundaries given by the graphs of a finite number of functions, in perhaps different orthonormal coordinate systems, to have the property that the semigroup associated with the heat kernel for the Dirichlet Laplacian is intrinsically ultracontractive.

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§0. Introduction.

Let D be a domain in $\mathbb{R}^d, d \geq 2$, and let $P_t^D(z,w),\ t>0, z,w \in D$, be the heat kernel for one half the Dirichlet Laplacian in D. We assume that D has a positive eigenfunction φ in $L^2(dz)$ with eigenvalue λ, an assumption which holds for all domains considered in this paper. Since

$$P_t^D(z,w) \leq \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|z-w|^2}{2t}},$$

the Markovian semigroup associated with $P_t^D(z,w)$ is ultracontractive. That is, it maps $L^2(D,dz)$ into $L^\infty(D,dz)$ for all $t>0$. Following Davies and Simon [10] we shall say that D is *intrinsically ultracontractive*, which henceforth we write as IU, if the new Markovian semigroup in $L^2(\varphi^2 dz)$ with kernel

$$\tilde{P}_t(z,w) = \frac{e^{\lambda t} P_t^D(z,w)}{\varphi(z)\varphi(w)}$$

is ultracontractive. That is, if it maps $L^2(\varphi^2 dz)$ into $L^\infty(\varphi^2 dz)$ for all $t>0$. Davies and Simon [10], (Theorem 3.1), gave several other equivalent formulations of IU including the following: There exist constants a_t and b_t depending only on t such that

$$a_t\varphi(z)\varphi(w) \leq P_t^D(z,w) \leq b_t\varphi(z)\varphi(w), \tag{0.1}$$

for all $t>0, z,w \in D$. In this paper we shall also say that D is IU for $t>t_0$ if (0.1) holds for all $t>t_0$. IU is closely related to estimates on the expected lifetime of certain conditioned Brownian motions (h–processes) in D, and naturally, to estimates on eigenfunctions.

In [10], Davies and Simon also introduced a weaker notion of intrinsic contractivity. Following their definition we shall also say that D is *intrinsically supercontractive*, which we shall write as ISC, if the semigroup of $\tilde{P}_t$ maps $L^2(\varphi^2 dz)$ into $L^p(\varphi^2 dz)$ for any $2<p<\infty$ and all t. As with IU, Simon and Davis [10], (Theorem 3.1), have several equivalent formulations of ISC including the following: Let $Q_t(z) = \sqrt{\tilde{P}_t(z,z)}$. Then D is ISC if and only if

$$||Q_t(z)||_{L^p(\varphi^2 dz)} < \infty \text{ for all } t>0. \tag{0.2}$$

Our main result is a simple geometrical characterization for IU, for planar domains with boundaries given by the graphs of a finite number of functions. We also determine for which of these domains the expected lifetimes of all h processes are bounded. We give a geometrical characterization for ISC, in terms of the Whitney distance, for domains in $\mathbb{R}^d, d \geq 2$, which satisfy a capacity boundary condition, a class which includes all simply connected planar domains, (see Theorem 5). Theorem 5 leads to a result on IU, (see Theorem 6), for domains in $\mathbb{R}^d$ with the capacity boundary condition which even though not sharp, is likely to be the best that can be done in terms of the quantities used in Theorem 5.

To simplify our presentation, we first state special cases of some of our results when the domain is "above the graph of a function".

Let f be an uppersemicontinuous function on (0,1), taking values in $[-\infty, 0)$ which is not identically $-\infty$. Define

$$D_f = \{z = (x, y) : 0 < x < 1, f(x) < y < 1\}.$$

A maximal horizontal line segment, MHLS, of D_f, is a subset of D_f of the form $\{(x, y) : a < x < b\}$ which is not strictly contained in another set of the same form. We call y the height of this segment. Let $\mathcal{A}$ be the collection of all connected subsets of D_f which are unions of MHLSs, no two of which have the same height. Let $\mathcal{A}_r$ be those sets in $\mathcal{A}$ which contain no points with y coordinate larger than r. If $A \subset \mathbb{R}^2, |A|$ will denote its area. We have

Theorem 1. D_f is IU if and only if $\lim_{r \to -\infty} \sup_{A \in \mathcal{A}_r} |A| = 0$.

If f is increasing on (0, 1), Theorem 1 says that D_f is IU if and only if it has finite area. Davies and Simon [10] prove this under certain additional conditions on f, but even for general increasing f, Theorem 1 is new. When f is a bounded function, it was proved in B. Davis [13] that D_f is IU. This result was extended to the case when f belongs to $L^p[0,1]$ for $p > 1$ by R. Bass and K. Burdzy [7]. Theorem 1 as stated was conjectured in Davis [13] and parts of our proof are modifications of the arguments used there. All our theorems are proved in ways which translate geometric information about a domain into estimates for the a_t and b_t of (0.1), and in this sense they give information even for C^∞ domains. For example, for the domains D_f, it can be shown that given $\varepsilon > 0$ there are numbers c and C which can be explicitly given, such that if $t > c^{-1}$ and $\sup_{A \in \mathcal{A}} |A| < ct$, then a_t and b_t in (0.1) may be chosen so that $1 \leq b_t/a_t \leq 1+\varepsilon$ (we note that always $a_t \leq 1 \leq b_t$), while if $\sup_{A \in \mathcal{A}} |A| > Ct$, it must hold that $b_t/a_t > 1/\varepsilon$.

IU has also been proved for several other types of domains and we refer the reader to Bañuelos [3], where a survey of recent results is given. As is well known by now, if D is IU for $t > t_0$ for some t_0 then the expected lifetimes of h-processes in D are bounded, (a subject which also has been widely investigated in recent years), but the converse is false. For this connection we refer the reader to R. Bañuelos and B. Davis [6]. Our second result, which is a corollary of the proof of Theorem 1, gives a geometrical characterization for the boundedness of the expected lifetimes in D_f.

Let $H(D_f)$ be the collection of all positive superharmonic functions in D_f. For $z \in D_f$ and $h \in H(D_f)$, we write $E_z^h(\tau_{D_f})$ for the expected lifetime of the Brownian motion in D starting at z and conditioned by h (the Doob-h process).

Theorem 2. $\sup_{\substack{z \in D_f \\ h \in H(D_f)}} E_z^h(\tau_{D_f}) < \infty$ if and only if D_f is IU for $t > t_0$ for some t_0 and this in turn holds if and only if $\sup_{A \in \mathcal{A}} |A| < \infty$.

The equivalence of the finiteness of the two suprema appearing in the statement of Theorem 2, for a special class of functions f, has been proved by Xu [17], and we use some of his methods. Our main result, Theorem 3, states that if the domain $D = \bigcup_{i=1}^{n} V_i$, where V_i is the image under an analytic map $z \to a_i z + b_i$, with a_i, b_i constants, $a_i \neq 0$, of a domain D_{f_i}, then D is *IU* if and only if each D_{f_i} is *IU*, that is, satisfies the conditions of Theorem 1. We also prove an analogous extension of Theorem 2. The formal statements appear at the beginning of Section 4. We note that it is not in general true that a domain, which is the union of two *IU* domains, is itself *IU*.

The paper is organized as follows. In §1, we set up some notation and give a new proof of a lemma due to Davies and Simon [11] and Bass and Burdzy [7] which provides the probabilistic connection to IU. In §2, we prove the "only if" part of Theorem 1. In §3, we prove the "if" part of Theorem 1 and explain how the proof of Theorem 1 implies Theorem 2. In §4, we state and prove our results for domains given locally by the graph of a function. In this section we also present the characterization of ISC in terms of the Whitney distance, (Theorem 5), and its consequences for IU, (Theorem 6). We can also give an analytical (that is, non probabilistic) proof of the special case of Theorem 1 in the case that f is increasing (see Theorem 6 and the comments at the end of Theorem 6).

Throughout the paper, the letters c, C, c', C', will be used to denote constants which may change from line to line but which do not depend on the variable points x, y, z, w, etc. $C(r), C_1, C_2, \ldots$ are also constants but they will not change. Constants depending only on t, or on t, ε, and which may also change from line to line, will be denoted by $a_t, b_t, C_t, C_{t,\varepsilon}$, etc. We will sometimes use $\wedge$ and $\vee$ to denote the minimum and maximum respectively.

§1. Notation and Preliminaries.

If f is negative and uppersemicontinuous, we set $\Omega_f = \{z = (x, y) : 0 < x < 1, f(x) < y < 0\}$. The MHLS for Ω_f are defined as were those for D_f. If L is a MHLS we denote by $h(L)$ and $\ell(L)$ its height and length, respectively. If L_1 and L_2 are MHLS we shall say that L_1 is above L_2 if $h(L_1) > h(L_2)$ and the vertical line through any point in L_2 intersects L_1. We also let $T_0 = (0,1) \times \{0\}$ and $T_1 = (0,1) \times \{1\}$. Notice that if L is a MHLS of Ω_f then the union of all MHLS of Ω_f below L is also an Ω_g, (after scaling and translating), for some g, and furthermore note that each D_f is also, after translation, an Ω_g.

Points in $\mathbb{R}^2$ will be written as $z = (x, y)$ or $w = (u, v)$. In Sections 2 and 3, D will always stand for a domain of the form D_f and Ω will always stand for a domain of the form Ω_f. We will use Θ for the generic domain in $\mathbb{R}^2$. If h is a positive superharmonic function in Θ, we will use $P_z^{h,\Theta} = P_z^h$ and $E_z^{h,\Theta} = E_z^h$ to denote the probability and expectation associated with the Doob h-process in Θ started at z. In the case h is the Green function for Θ, $G_\Theta(z, w)$, (which gives Brownian motion conditioned to go from z to w in Θ), we simply write P_z^w and E_z^w. Similarly, if $h(z) = K(z, \xi), z \in \Theta$, $\xi \in \partial\Theta$, K the Martin kernel, we will write P_z^ξ and E_z^ξ. We refer the reader to Doob [14] for more information on h processes. We just recall here that if τ_Θ denotes the lifetime of this process in Θ, then up to time τ_Θ the h process is a strong Markov process with transition functions

$$P_t^h(z, w) = \frac{1}{h(z)} P_t^\Theta(z, w) h(w),$$

where $P_t^\Theta(z, w)$ is as in the introduction. The following result is due to Cranston and McConnell [8].

Lemma 1.1. There is a constant C such that for any $\Theta \subset \mathbb{R}^2$,

$$E_z^h \tau_\Theta \le C|\Theta|.$$

By a square Q we shall always mean a closed square with sides parallel to the coordinate axes. A Whitney decomposition of Θ, denoted by $W(\Theta) = \{Q_j\}$, is a collection of squares in Θ with disjoint interiors whose union is Θ and which satisfy $1 \le d(Q_j, \partial\Theta)/\ell(Q_j) \le 4\sqrt{2}$ for all j. This can be easily seen to imply $\frac{1}{10} \le \ell(Q_j)/\ell(Q_k) \le 10$ if $Q_i \cap Q_k \ne \phi$. Here $\ell(Q_j)$ is the side length of Q_j and $d(Q_j, \partial\Theta)$ is the Euclidean distance from Q_j to $\partial\Theta$. The Whitney decomposition gives rise to the quasi-hyperbolic distance in the following way. Fix $Q_0, Q_k \in W(\Theta)$. We say that $Q_0 \to Q_1 \to \dots \to Q_m = Q_k$ is a Whitney chain connecting Q_0 to Q_k of length m if $Q_i \in W(\Theta)$ for all i and if $Q_i \cap Q_{i+1} \ne \phi$, $0 \le i < m$. We define the Whitney distance $d_W(Q_0, Q_k)$ to be the length of the shortest Whitney chain connecting Q_0 to Q_k. If z_1 and $z_2 \in \Theta$, we let $\rho_\Theta(z_1, z_2) = d_W(Q_1, Q_2)$ where $z_1 \in Q_1$ and $z_2 \in Q_2$. This is the quasi-hyperbolic distance between z_1 and z_2. From these definitions it follows by the Harnack inequality that if h is a positive harmonic function in Θ then

$$h(z_2) \ge c e^{-C\rho_\Theta(z_1,z_2)} h(z_1) \tag{1.1}$$

where c and C are absolute constants.

If Θ is a simply connected domain in $\mathbb{R}^2$ we let $d_\Theta(z_1, z_2)$ be the hyperbolic distance in Θ. It is well known, (it follows easily from the Koebe distortion theorem and the Schwarz lemma), that $c\rho_\Theta(z_1, z_2) - C \le d_\Theta(z_1, z_2) \le c\rho_\Theta(z_1, z_2)$. We recall that if Θ is simply connected then the curve Γ is a hyperbolic geodesic if it is the image of the segment (-1, 1) in the unit disc under a conformal map from the disc to D. The hyperbolic geodesic Γ splits the boundary of Θ into two pieces F_1 and F_2 with the property that if $z \in \Gamma$ then the harmonic measures of F_1 and F_2 with respect to z are both $1/2$. This follows from the disc case by the conformal invariance of harmonic measure.

The next lemma is from Bañuelos and Carroll [5]. A weaker and somewhat different form of this lemma , which will be enough for our applications in this paper, will also follow from some of the arguments in Xu [17].

Lemma 1.2. Let Θ be a simply connected planar domain and let Γ be a hyperbolic geodesic ending at the Martin boundary point ξ. Let $z \in \Gamma$ and let γ be the part of Γ from z to ξ. Then if $Q \in W(\Theta)$ and z_Q denotes the center of Q, we have

$$(1.2) \qquad P_z^{\xi}\{B_t \in Q \text{ for some } t < \tau_\Theta\} \geq ce^{-Cd_\Theta(z_Q,\gamma)}$$

for some constants c and C. In particular, if Q intersects the curve Γ at some point (or points) between x and ξ, the probability in (1.2) is larger than c.

Next we need a well known estimate for harmonic measure in simply connected domains. (See Tsuji [16], p. 112 for a proof which even gives information on constants.)

Lemma 1.3. Let Θ be a simply connected domain. Given $\delta > 0$ and $\rho < 1$ there exists a constant $C(\rho)$ such that for all $x \in \Theta$ with $d(x, \partial\Theta) < \delta C(\rho)$,

$$(1.3) \qquad P_x\{\tau_\Theta < \tau_{B(x,\delta)}\} > \rho,$$

where $\tau_{B(x,\delta)}$ is the exit time from the ball centered at x and radius δ.

We now present a lemma which provides the probabilistic connection with IU. The lemma was stated without proof in Davies and Simon [11] and independently discovered later and proved by Bass and Burdzy [7]. Here we provide a different proof. Let us assume that Θ is a domain in $\mathbb{R}^d, d \geq 2$, for which the Dirichlet Laplacian has discrete spectrum in $L^2(dz)$. Notice that this condition is clearly satisfied for our domains in Theorems 1 and 2 by Theorem 1.6.8 in Davies [9].

Lemma 1.4. Suppose that for each $t > 0$ there exists a compact set $K_t \subset \Theta$, such that for all $z \in \Theta$,

$$(1.4) \qquad P_z\{\tau_\Theta > t\} \leq a_t P_z\{\tau_\Theta > t, B_t \in K_t\}$$

where a_t does not depend on z. Then Θ is IU.

Proof: Let $\varphi_0 = \varphi, \varphi_1, \varphi_2, \ldots$ be the eigenfunctions with corresponding eigenvalues $\lambda_0 = \lambda, \lambda_1, \lambda_2, \ldots$ and normalized to have L^2 norm 1. Since the semigroup of $P_t^\Theta(z,w)$ is ultracontractive, (independent of the domain Θ), we have that for all $z \in \Theta$,

$$(1.5) \qquad e^{-\lambda_n t}|\varphi_n(z)| = \left|\int_\Theta P_t^\Theta(z,w)\varphi_n(w)dw\right| \leq a_t||\varphi_n||_2 = a_t.$$

Thus we have $|\varphi_n(z)| \le a_t e^{\lambda_n t}$. On the other hand, since φ is strictly positive and continuous, $\varphi(z) \ge C_t$ for all $z \in K_t$. Using our convention that a_t may change from line to line we have,

$$\begin{aligned} e^{-\lambda_n t/2}|\varphi_n(z)| &= |E_z(\varphi_n(B_{t/2}); \tau_\Theta > t/2)| \\ &\le a_t e^{\lambda_n t/2} P_z\{\tau_\Theta > t/2\} \\ &\le a_t e^{\lambda_n t/2} P_z\{\tau_\Theta > t/2, B_{t/2} \in K_{t/2}\} \\ &\le a_t e^{\lambda_n t/2} E_z(\varphi(B_t); \tau_\Theta > t/2) \\ &= a_t e^{\lambda_n t/2} e^{-\lambda t/2}\varphi(z). \end{aligned}$$

Thus (1.4) implies that for all $n = 0, 1, 2, \ldots$,

$$|\varphi_n(z)| \le a_t e^{\lambda_n t}\varphi(z), \tag{1.6}$$

where a_t depends only on t. It is proved in Davies and Simon [10], Theorem 3.1, (and it follows very easily from the expansion of the heat kernel in terms of eigenfunctions), that (1.6) is equivalent to IU. □

Remark 1. Notice that our proof shows that if (1.4) holds for $t > t_0$ then (1.6) holds for $t > 2t_0$ and this gives IU, (0.1), for $t > Ct_0$, C an absolute constant, (4 will work). We shall use this in the proofs of Theorems 1 and 2 below.

Remark 2. IU may be defined without reference to eigenfunctions purely in terms of $P_t^\Theta(z, w)$ as in Davis [13]. However, as shown by Davies and Simon [10], [11], IU implies discrete spectrum in L^2.

§2. Proof of the "only if" part of Theorem 1.

We start with some lemmas.

Lemma 2.1. Let $\varepsilon > 0$. Let Θ be a simply connected domain, and let $Q_1, Q_2, \ldots, Q_n$ be squares in Θ with $\ell(Q_j) = \varepsilon$, $d(Q_j, \partial\Theta) \ge \varepsilon/4$ for all j and such that Q_j and Q_{j+1} have a common side for all $j = 1, 2, \ldots, n-1$. There exist constants c and C such that for all $z \in Q_1$ and S_n any one of the four sides of Q_n,

$$P_z\{B_t \in S_n \text{ for some } t < cn\varepsilon^2, t < \tau_\Theta\} > C^n. \tag{2.1}$$

Proof: Let $\Theta_\varepsilon = \{z \in \Theta : d(z, \bigcup_{j=1}^{n} Q_j) < \varepsilon/4\}$. Let z_0 be the center of Q_n. Then $P_{z_0}\{\tau_{S_n} < \tau_{\Theta_\varepsilon}\} = C > 0$ and by the Harnack inequality, (1.1),

$$P_z\{\tau_{S_n} < \tau_{\Theta_\varepsilon}\} > C^n \tag{2.2}$$

for all $z \in Q_1$.

Next, by Lemma 1.1 we have

$$E_z(\tau_{S_n} | \tau_{S_n} < \tau_{\Theta_\varepsilon}) \leq c \text{ area } (\Theta_\varepsilon) \leq cn\varepsilon^2.$$

This inequality together with the Chebychev inequality gives that

$$P_z\{\tau_{S_n} < 2cn\varepsilon^2 | \tau_{S_n} < \tau_{D_\varepsilon}\} \geq 1/2, \tag{2.3}$$

which together with (2.2) gives

$$P_z\{\tau_{S_n} < 2cn\varepsilon^2, \tau_{S_n} < \tau_{D_\varepsilon}\} > \frac{1}{2}C^n,$$

and (2.1) follows. □

Lemma 2.2. Let L_1 be either T_0 or a MHLS of Ω and let L_2 be a MHLS of Ω. Suppose L_2 is below L_1, $\ell(L_2) \geq \frac{1}{2}\ell(L_1)$, and $h(L_1) - h(L_2) \geq \frac{1}{2}\ell(L_1)$. Let Γ be a hyperbolic geodesic in Ω_f which has one end point $\xi_0 \in T_0$ and which has a point $z \in L_2$. Then there exists a constant C_1 such that $\text{dist}(\tilde{z}, \partial\Omega \cup L_1 \cup L_2) \geq C_1\ell(L_1)$ for some $\tilde{z} \in \Gamma, \tilde{z}$ below L_1 and above L_2.

Proof: Let L be the MHLS midway between L_1 and L_2. Let D_3 be the points of Ω which are below L_2 and let D_2 be those that are below L but not below L_2 and let $D_1 = \Omega\backslash(D_2 \cup D_3)$. Let $\xi_1 \in \partial\Omega$ be the other end point of Γ. We shall consider three cases, namely (i) $\xi_1 \in \partial D_1$, (ii) $\xi_1 \in \partial D_2$, and (iii) $\xi_1 \in \partial D_3$. All cases are similar so we just discuss (i).

Let F_1 and F_2 be the two sets into which the end points of Γ divide the boundary of Ω. As explained earlier, $\omega_z(F_1) = \omega_z(F_2) = 1/2$, for any $z \in \Gamma$ where $\omega_z(\cdot)$ is the harmonic measure for Ω at z. If $\xi_1 \in \partial D_1$, then $\partial D_2\backslash(L \cup L_2)$ is either completely contained in F_1

or completely contained in F_2. If L' is the MHLS midway between L and L_2, Γ cannot intersect L' at a point $\tilde{z}$ with

$$d(\tilde{z}, \partial\Omega) < \min\left(C(\tfrac{1}{2})\,\ell(L_1)/4,\ \ell(L_1)/4\right) \tag{2.4}$$

where $C(\frac{1}{2})$ is the constant of Lemma 1.3 corresponding to $\rho = 1/2$. If this were to happen then by Lemma 1.3, there would be a subset $F \subset \partial D_2$ with $F \subset F_1$ or $F \subset F_2$ such that $\omega_{\tilde{z}}(F) > 1/2$, which is impossible. $\square$

Lemma 2.3. Let L be a MHLS of Ω such that $\ell(L) \geq \frac{1}{2}$ and $h(L) \leq -\frac{1}{2}$. There is a constant C_2 such that if z is on or below L, then

$$P_z\{B_{\tau_\Omega} \in \hat{T}_0 | B_{\tau_\Omega} \in T_0\} > C_2, \tag{2.5}$$

where $\hat{T}_0 = \{z \in T_0 : d(z, \partial\backslash\Omega T_0) > C_1\}$ and C_1 is the constant of Lemma 2.2.

Proof: We may and do assume that $z \in L$, since otherwise we just use the strong Markov property at τ_L. We may also assume that $h(L) = -\frac{1}{2}$, since if $h(L) < -\frac{1}{2}$ we can just replace L by the MHLS of height $-\frac{1}{2}$ and use the strong Markov property again. Set $F = \{z = (x, y) \in \Omega : y > -\frac{1}{2},\ d(z, \partial\Omega) > C_1\}$. If $\xi \in T_0$ we have by Lemma 1.2 that

$$P_z^\xi\left\{B_t \in \tilde{Q} \text{ for some } t < \tau_\Omega\right\} > c$$

where $\tilde{Q}$ is the Whitney cube containing the $\tilde{z} \in F$ given by Lemma 2.2. By the Harnack inequality, we have

$$P_z^\xi(\tau_F < \tau_\Omega) > c,$$

which implies

$$P_z(\tau_F < \tau_\Omega) > cP_z(B_{\tau_\Omega} \in T_0). \tag{2.6}$$

Next, if $w \in F$, we can construct squares $Q_1, Q_2, \ldots, Q_n$ with $w \in Q_1$, n depending on C, but not on $w, \ell(Q_j) = C$ and $d(Q_j, \partial\Omega) > C$, C also depending on C_1 but not on w. Furthermore we can construct this chain in such a way that Q_j and Q_{j+1} have a common side, and such that any curve in $G = \{w : d(w, \cup Q_i) < C\}$ which connects w to the top side S_n of Q_n must intersect $\hat{T}_0$. By Lemma 2.1, there is c' not depending on w such that

$$P_w(\tau_{S_n} < \tau_G) > c',$$

implying

$$P_w(\tau_\Omega = \tau_{\hat{T}_0}) > c'. \tag{2.7}$$

The lemma now follows from (2.6), (2.7) and the strong Markov property at τ_F. □

In general, if L is a MHLS of Ω we define $\hat{L} = \{z \in L: d(z, \partial\Omega \backslash T_0) > C_1 \ell(L)\}$, where C_1 is the constant of Lemma 2.2 and if L is a MHLS of D we define $\hat{L}$ analogously.

Lemma 2.4. Let Γ be a set of the form $\bigcup_{\gamma \le r \le 0}((a(r), b(r)) \times \{r\})$ where $\gamma \le -2$ and $(a(r_2), b(r_2)) \subseteq (a(r_1), b(r_1)) \subset (0,1)$ if $\gamma \le r_2 \le r_1 \le 0$. There exist numbers $\gamma = a_0 \le a_1' < a_1 \le a_2' < \ldots < a_M$ such that

(i) $-2 \le a_M \le 0$

(ii) $a_i - a_i' = \lambda(a_i') \ge \frac{1}{2}\lambda(a_i),\ 1 \le i \le M$

and

(iii) $a_i - a_{i-1} \le 2\lambda(a_i),\ 1 \le i \le M$.

Proof: With $a_0 = \gamma$ let $z_0 = a_0, z_1 = z_0 + \lambda(z_0), \ldots, z_j = z_{j-1} + \lambda(z_{j-1})$ and let $N = \inf\{j: \lambda(z_j) \le 2\lambda(z_{j-1})\}$. Set $a_1' = z_{N-1}$ and $a_1 = z_N$. Let a_1 now play the role of a_0 and define a_2' and a_2 in the same way. Continuing this way we get $a_0 \le a_1' < a_1 \le a_2' < \ldots$. Let us now define M. First, we claim that $a_1' < -1$. To see this observe that by definition, $\lambda(z_{j-1}) < \frac{1}{2}\lambda(z_j)$ for $1 \le j \le N-1$. Also, since $\lambda(\gamma) \le 1$, we must have $\lambda(z_j) < \frac{1}{2}$ for $0 \le j \le N-2$. Thus,

$$\begin{aligned} a_1' - a_0 &= (\lambda(z_0) + \ldots + \lambda(z_{N-3})) + \lambda(z_{N-2}) \\ &< (\tfrac{1}{2}\lambda(z_1) + \ldots + \tfrac{1}{2}\lambda(z_{N-2})) + \lambda(z_{N-2}) \\ &= \tfrac{1}{2}(\lambda(z_1) + \ldots + \lambda(z_{N-3})) + (1 + \tfrac{1}{2})\lambda(z_{N-2}). \end{aligned} \tag{2.8}$$

Continuing this way and using the fact that $a_0 \le -2$ we find that

$$a_1' + 2 \le a_1' - a_0 < (1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots)\lambda(z_{N-2}) < 2 \cdot \tfrac{1}{2} = 1$$

and so $a_1' < -1$. Thus $a_1 < 0$. We define $M = \min\{j: a_j \ge -2\}$. The properties (i) and (ii) are clear from the definition of N and M. For (iii), let $z_0, z_1, \ldots, z_N$ be the z_j's constructed by starting with $z_0 = a_{i-1}$. Then $a_i - a_{i-1} = (\lambda(z_0)+\ldots+\lambda(z_{N-2}))+\lambda(z_{N-1})$ and continuing as in (2.8), (iii) follows. □

We now complete the proof of the "only if" part of Theorem 1. Let us first note that if $\lim_{r\to-\infty} \sup_{A\subset\mathcal{A}_r} |A| > 0$ then one or both of the following hold:

(i) $\sup_{A\in\mathcal{A}} |A| = \infty$.

(ii) There exists a $\delta > 0$ such that for any $\varepsilon > 0$ there is $A_\varepsilon \in \mathcal{A}$ such that each MHLS contained in A_ε has length smaller than ε and $|A_\varepsilon| > \delta$.

Suppose (i) holds. Let $N > 0$ and $A \subset \mathcal{A}$ be such that $|A \cap \{y \le -2\}| > N$. By Lemma 2.4 and scaling so that the top line of A is scaled to have length 1 and assumes the role of T_0, we may find disjoint intervals (d_k, e_k), $1 \le k \le n$, such that $(d_k, e_k) \subset \{y: (x,y) \in A\}$ and the MHLS of A of height e_k has length at most twice the length of the MHLS in A of height d_k which equals $e_k - d_k$ and such that if $\Gamma_k = \{(x,y) \in A: d_k < y < e_k\}$ then $\Sigma|\Gamma_k| \ge C|A \cap \{y \le -2\}| - c$. Let z be a point below the MHLS in A of height d_1. Let Γ be a geodesic with one end point at $\xi = (1/2, 1)$ and which contains z. By Lemma 2.2 there exists a Whitney cube Q_k in each Γ_k which touches this geodesic and such that $|\Gamma_k| \le C|Q_k|$. By Lemma 1.2,

$$P_z^\xi\{\tau_{Q_k} < \tau_D\} > C.$$

By Theorem 1.1 in Davis [12] (or Corollary 2.2 in Bañuelos and Carroll [5]),

$$E_z^\xi(\tau_D) \ge C\sum|Q_k| \ge C\sum|\Gamma_k| \ge C(|A \cap \{y \le -2\}| - c) \ge CN - c.$$

Now, N may be arbitrarily large. Thus if (i) holds, the domain D does not have the expected lifetime property and hence it is not IU; (see Bañuelos and Davis [6]).

Before we deal with case (ii) we make some observations concerning IU. The left equality of (1.5) with $n = 0$ implies that in (0.1) with $D = \Theta$ we may always choose $a_t = a_t$ nondecreasing as t increases and b_t nonincreasing as t increases, if Θ is IU. From this it follows that there is a C_t, depending only on t, such that

$$\int_t^{2t} P_s^\Theta(z,w)ds \ge C_t \int_{2t}^\infty P_s^\Theta(z,w)ds. \tag{2.9}$$

By definition,

$$P_z^w\{\tau_\Theta > t\} = \frac{1}{G_\Theta(z,w)} \int_\Theta P_t^\Theta(z,\tilde{z})G_\Theta(\tilde{z},w)d\tilde{z}, \tag{2.10}$$

for any $z, w \in \Theta$. Differentiating both sides of (2.10) in t and integrating by parts we find that the density of τ_Θ under P_z^w is given by $P_t^\Theta(z,w)/G_\Theta(z,w)$. Thus dividing both sides of (2.9) by $G_\Theta(z,w)$ we obtain

$$P_z^w\{t < \tau_\Theta < 2t\} \geq C_t P_z^w\{\tau_\Theta > 2t\} \tag{2.11}$$

for all $z, w \in \Theta$. Since C_t depends only on t, we also have that for any $z \in \Theta$ and $\xi \in \partial\Theta$,

$$P_z^\xi\{t < \tau_\Theta < 2t\} \geq C_t P_z^\xi\{\tau_\Theta > 2t\}. \tag{2.12}$$

Let us now assume that (ii) holds. Then as in case (i) for large enough n there exists a hyperbolic geodesic Γ_n ending at $\xi_n \in T_1$ and a collection of Whitney cubes $Q_1^n, Q_2^n, \ldots, Q_{n'}^n$ in $A_{\frac{1}{n}}$ such that $\ell(Q_j^n) < \frac{1}{n}$ for all j, $|\bigcup_{j=1}^{n'} Q_j| > C\delta$, and with Q_j^n touching the geodesic Γ_n for every j. Let $R_n = \bigcup_{j=1}^{n'} Q_j^n$ and let z_n be a point in D such that each of the Q_j^n, $1 \leq j \leq n'$, touches that part of Γ_n between z_n and ξ_n. Let T_{R_n} be the total time B_t spends in R_n. That is,

$$T_{R_n} = \int_0^{\tau_D} 1(B_t \in R_n)dt.$$

Then by Lemma 1.2 and Theorem 1.1 in Davis [12] (or Corollary 2.2 in Bañuelos and Carroll [5]),

$$E_{z_n}^{\xi_n} T_{R_n} \geq C|R_n| \geq C\delta. \tag{2.13}$$

By the argument used to prove (5.1) in Davis [12] there exists h_n which goes to zero as n goes to infinity such that

$$\text{var}\,(T_{R_n}) \leq h_n |R_n|^2, \tag{2.14}$$

where by $\text{var}(T_{R_n})$ we mean the variance of T_{R_n} with respect to the measure $P_{z_n}^{\xi_n}$. This implies there is a constant $k = k(\delta)$ such that $\lim_{n\to\infty} P_{z_n}^{\xi_n}\{T_{R_n} > k\} = 1$ and hence

$$\lim_{n\to\infty} P_{z_n}^{\xi_n}\{\tau_D > k\} = 1. \tag{2.15}$$

Then with $t = k/2$, $P_{z_n}^{\xi_n}\{\tau_D > 2t\} \to 1$ and $P_{z_n}^{\xi_n}\{t < \tau_D < 2t\} \to 0$, contradicting (2.12) with $D = \Theta$. Thus the proof of the "only if" part is complete. □

§3. Proof of the "if" part of Theorem 1.

Lemma 3.1. Let $z_0 = (x_0, y_0) \in \Omega$. There exists a constant C_3 such that

$$E_{z_0}(\tau_\Omega | B_{\tau_\Omega} \in T_0) \le C_3 + C_3|y_0|. \tag{3.1}$$

Proof: Let $h(z) = P_z\{B_{\tau_\Omega} \in T_0\}$, the harmonic measure of T_0 with respect to Ω. Let $u(z) = P_z\{B_{\tau_S} \in T_0\}$ where S is the half strip $\{(x,y): 0 < x < 1, y < 0\}$. Clearly $h(z) \le u(z)$ for all $z \in \Omega$. Let $v(z) = 1 - u(z)$. Then $v(z)$ is a positive harmonic function in S which vanishes on T_0. As is well known, such functions cannot vanish faster than the distance to the boundary. That is, for $y > -C$, where C is small enough,

$$v(z) \ge c|y| \tag{3.2}$$

where $z = (x, y)$. From (3.2) we have

$$u(z) \le 1 - c|y|, \; -C < y < 0. \tag{3.3}$$

Now let $\varepsilon < \min(C, |y_0|)$. Let L be the MHLS above z_0 and of height $-\varepsilon$. Let $\tilde{L}$ be the set of those points of T_0 directly above L and let $z_\varepsilon = (x_0, y_0 + \varepsilon)$. If we take a path which starts at z_0 and terminates at L and which has not exited Ω and translate it up to start at z_ε, it will terminate at $\tilde{L}$ and hence

$$P_z\{\tau_L < \tau_\Omega\} \le P_{z_\varepsilon}\{\tau_{\tilde{L}} = \tau_\Omega\} \le h(z_\varepsilon). \tag{3.4}$$

From the martingale property of $h(B_t); t < \tau_\Omega$, we obtain

$$h(z) = E_z(h(B_{\tau_L}); \tau_L < \tau_\Omega). \tag{3.5}$$

Since $h(z) \le u(z) \le (1 - c|y|) = (1 - c\varepsilon), z \in L$,

$$h(z) \le (1 - c\varepsilon) P_z\{\tau_L < \tau_\Omega\}. \tag{3.6}$$

Thus

$$h(z_\varepsilon) - h(z) \ge P_z\{\tau_L < \tau_D\} c\varepsilon \ge c\varepsilon h(z),$$

and we conclude that

$$\frac{\partial h}{\partial y}(z) \geq ch(z), z \in \Omega. \tag{3.7}$$

Next we recall that h-processes satisfies the stochastic differential equation

$$dX_t = dB_t + \frac{\nabla h}{h}(X_t)dt$$

and hence (3.7) implies that the vertical component of the drift of the associated h-process is larger than or equal to ct everywhere in Ω. Thus if $\eta < y_0$,

$$\begin{aligned} P^h_{z_0}\{B_t \text{ ever gets below the line } y = \eta\} \\ \leq P_0\{w_t + ct \leq \eta - y \text{ for any } t\} \\ \leq e^{2c(y_0-\eta)}, \end{aligned} \tag{3.8}$$

where w_t is standard one dimensional Brownian motion, the second inequality since if w_t is one dimensional Brownian motion, then $-\inf_t\{w_t + ct\}$ has an exponential distribution with parameter 2c (this follows from the exponential martingale).

In particular we conclude from above that the probability that our h processes ever hits a Whitney square below $y = \eta$, for $\eta < y_0$, is bounded by $e^{2c(\eta-y_0)}$. Now

$$E_{z_0}(\tau_\Omega | B_{\tau_\Omega} \in T_0) \leq C \sum_{Q\in W(D)} P_Q|Q| \tag{3.9}$$

where $P_Q = P_{y_0}(\tau_Q < \tau_{T_0} | \tau_\Omega = \tau_{T_0})$, which follows from Theorem 1.1 of Davis [12], upon integrating over the points of T_0. Together with (3.8), (3.9) proves the lemma. □

Remark. One can also use Corollary 2.2 in Bañuelos and Carroll [5] to prove Lemma 2.3, by using the fact that hyperbolic distance decreases with increasing domains, and the easily computable distance in S.

Lemma 3.2. There is a positive constant $C(r), -\infty < r < 0$, which is bounded below on bounded subsets of $(-\infty, 0)$ such that if $z = (x, y) \in \Omega$,

$$P_z\{\tau_\Omega > 2|y|\} \leq C(y) P_z\{B_{\tau_\Omega} \in T_0, \tau_\Omega < 2|y|\}. \tag{3.10}$$

Proof: Let L_{2y} be the horizontal line at level $2y$. Let τ_{2y} be the hitting time of L_{2y}. Set

$$A = \{\tau_{2y} < \tau_\Omega; \tau_{2y} \leq 2|y|\}$$
$$B = \{B_{\tau_\Omega} \in T_0; \tau_\Omega \leq 2|y|\}$$
$$C = \{\tau_\Omega > 2|y|; \tau_{2y} > 2|y|\}.$$

Notice that if we take a path in A and reflect it about the MHLS L_y containing z after the last time before $2|y|$ it hits L_y we obtain a path in B. Since the reflected motion is still Brownian motion, (see the explanation following (3.8) in Davis [13]); we have

$$P_z\{A\} \leq P_z\{B\}. \tag{3.11}$$

Next we apply the Girsanov argument used in [13]. Under the transformation $B_t + t$ for $0 \leq t \leq 2|y|$, any path from C is transformed into a path in B. Thus if we apply (2.9) in [13] with $M = 2|y|$ we have

$$P_z\{C\} \leq C(y)P_z\{B\} \tag{3.12}$$

which together with (3.11) proves the lemma, since $\{\tau_\Omega > 2|y|\} \subset A \cup C$. □

Lemma 3.3. Let L be a MHLS of $D = D_f$ which lies below T_0. Then if $z \in \hat{L}$ and w is on or below L,

$$P_w\{\tau_D > t\} \leq C_t P_z\{\tau_D > t\}. \tag{3.13}$$

Proof: Let $K = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$. Let $\tilde{z}$ be any point directly above w. That is, if $w = (u, v)$, then $\tilde{z} = (\tilde{x}, \tilde{y})$ with $\tilde{x} = u$, $v < \tilde{y}$. By translating the path up we see that

$$\begin{aligned} P_w\{\tau_D > t\} &+ P_w\{\tau_D \leq t, B_{\tau_D} \in T_1\} \\ &\leq P_{\tilde{z}}\{\tau_D > t\} + P_{\tilde{z}}\{\tau_D \leq t, B_{\tau_D} \in T_1\}. \end{aligned} \tag{3.14}$$

By the argument used to prove inequality (3.4) in Davis [13] (the reader may read the proof of (3.4) in [13] without reading the rest of that paper; just substitute K and D for

A and Ω respectively, note that we never use $Imx \geq -1/4$, and that to hit $(0,1) \times \{1\}$ without hitting A you must hit L without hitting A) we have

$$P_{\tilde{z}}\{\tau_D \leq t, B_{\tau_D} \in T_1\} \leq C_t P_{\tilde{z}}\{\tau_K < \tau_D, \tau_K < t\}, \tag{3.15}$$

and we may even take $C_t = 1$. Since $\inf_{z \in K} P_z\{\tau_D > t\} = C_t > 0$, the right hand side of (3.15) is less than $C_t P_{\tilde{z}}\{\tau_D > t\}$. This together with (3.14) imply that

$$P_w\{\tau_D > t\} \leq C_t P_{\tilde{z}}\{\tau_D > t\}. \tag{3.16}$$

If we now let V be the vertical line through w and τ_{V^+} the hitting time of those points in V above L. The lemma then follows from the strong Markov property provided we show that

$$P_z\{\tau_{V^+} < \tau_D\} > C \tag{3.17}$$

for some constant C independent of z. To see (3.17) we may assume that V_+ is to the left of z. Let $\varepsilon = \frac{C_1}{4}\ell(L)$ where C_1 is the constant in the definition of $\hat{L}$. Let Q_1 be the square centered at z and with side length ε. Let Q_2 be the square of same size as Q_1 and on top of Q_1. That is, the bottom side of Q_2 is the top side of Q_1. Let Q_3 be the square of same size as Q_2 and on the left of Q_2. That is, the left side of Q_2 is the right side of Q_3. Continuing this way we get squares $Q_1, Q_2, \ldots, Q_n$ where n depends only on C_1 and such that $V_+ \cap Q_n \neq \phi$. Our desired inequality (3.17) now follows from the Harnack inequality, applied to the domain consisting of all points within $\varepsilon/2$ of $\bigcup_{i=1}^{n} Q_i$. Even though this domain is not necessarily contained in D, if a curve in this domain, started at z, hits V_+, it also hits V_+ before leaving D. $\square$

Lemma 3.4. Let L be a MHLS of Ω such that $-1 \leq h(L) \leq -\frac{1}{2}$ and $\ell(L) \geq \frac{1}{2}$. There are constants C_4 and C_5 such that

$$P_z\{B_{\tau_\Omega} \in \hat{T}_0, \tau_\Omega < C_4\} \;\geq C_5[P_z\{\tau_\Omega < C_4, B_{\tau_\Omega} \in T_0\} + P_z\{\tau_\Omega > C_4\}]$$

for $z = (x, y)$ on or below L and $y \geq -2$.

Proof: Let d be the distance between T_0 and L. By Lemma 3.1,

$$\begin{aligned} P_z\{\tau_\Omega \geq t | B_{\tau_\Omega} \in T_0\} &\leq \frac{1}{t} E_z(\tau_\Omega | B_{\tau_\Omega} \in T_0) \\ &\leq \frac{1}{t}(C_3 + C_3 d) \leq \frac{3C_3}{t}. \end{aligned}$$

Together with Lemma 2.3, this implies

$$P_z\{\tau_\Omega \geq t, B_{\tau_\Omega} \in T_0\} \leq \frac{3C_3}{C_2 t} P_z\{B_{\tau_\Omega} \in \hat{T}_0\}.$$

Thus

$$\begin{aligned} C_2 P_z\{B_{\tau_\Omega} \in T_0\} \leq P_z\{B_{\tau_\Omega} \in \hat{T}_0\} &= P_z\{\tau_\Omega \geq t, B_{\tau_\Omega} \in \hat{T}_0\} + P_z\{\tau_\Omega < t, B_{\tau_\Omega} \in \hat{T}_0\} \\ &\leq \frac{3C_3}{C_2 t} P_z\{B_{\tau_\Omega} \in \hat{T}_0\} + P_z\{\tau_\Omega < t, B_{\tau_\Omega} \in \hat{T}_0\} \end{aligned}$$

and if we choose t_0 large enough depending only on C_2 and C_3 we find that

$$\frac{C_2}{2} P_z\{B_{\tau_\Omega} \in T_0\} \leq P_z\{\tau_\Omega < t_0, B_{\tau_\Omega} \in \hat{T}_0\},$$

which together with Lemma 3.2 gives the result with $C_4 = t_0$. □

Lemma 3.5. Let L_1 be a MHLS of Ω such that $-1 \leq h(L_1) \leq -\frac{1}{2}$ and $\frac{1}{2} \leq \ell(L_1) \leq 1$. Let L_0 be a MHLS such that $h(L_0) \leq h(L_1)$ and $h(L_0) \geq -2$. Let m be a subprobability measure which puts all of its mass on or below L_0 and such that $m(\hat{L}_0) \geq \min\left(\frac{C_2}{2}, C_5\right) m(\Omega)$. Then there is a constant C_6 such if η is the distribution under P_m of

$$B_{C_6 \wedge \tau_\Omega} I(\{\tau_\Omega > C_6\} \cup \{C_6 > \tau_\Omega, B_{\tau_\Omega} \in T_0\}),$$

then $\eta(\hat{T}_0) \geq \frac{C_2}{2}\eta(\Omega \cup T_0)$. Here $C_2 < 1$ is the constant of Lemma 2.3.

Proof: We can and do, without loss of generality, assume m is a probability measure. Define the function

$$\begin{aligned} h(z,t) &= P_z\{B_{\tau_\Omega} \in T_0, \tau_\Omega < t\} \\ \hat{h}(z,t) &= P_z\{B_{\tau_\Omega} \in \hat{T}_0, \tau_\Omega < t\}, \\ g(z,t) &= P_z\{\tau_\Omega > t\}, \\ f(z,t) &= h(z,t) + g(z,t), \end{aligned}$$

and

$$\omega_z(A) = P_z\{B_{\tau_\Omega} \in A\}, \qquad A \subset \partial\Omega.$$

We first show some properties of these functions. Clearly,

$$h(z,s) \leq h(z,t), \qquad s \leq t \tag{3.18}$$

and

$$\omega_z(T_0) \le f(z,t), \text{ for any } t. \tag{3.19}$$

By Lemma 2.3, Lemma 3.2, and a translation of the path argument similar to that used in the proof of (3.14), we have, respectively,

$$\omega_z(\hat{T}_0) \ge C_2\omega_z(T_0), \quad z \text{ on or below } L_0 \tag{3.20}$$

$$g(z,4) \le Ch(z,4), \qquad z \in L_0 \tag{3.21}$$

and

$$f(w,t) \le f(z,t), w \text{ directly below } z \in L_0. \tag{3.22}$$

From (3.21) we obtain

$$h(z,4) \le f(z,4) \le Ch(z,4), \; z \in L_0. \tag{3.23}$$

Also, since $h(z,4) \le \omega_z(T_0)$ we have

$$\omega_z(T_0) \le f(z,4) \le C\omega_z(T_0), \; z \in L_0. \tag{3.24}$$

Next, if $z_0 \in \hat{L}_0$ and z is any other point in L_0 we have, by the strong Markov property, that

$$\omega_{z_0}(T_0) \ge \inf_{w \in V^+(z)} \omega_w(T_0) P_{z_0}\{\tau_{V^+(z)} < \tau_\Omega\}$$

where $V^+(z)$ is the vertical segment connecting z to T_0. As in (3.17), we have that $P_{z_0}\{\tau_{V^+(z)} < \tau_\Omega\} > C$ and by translating the path again we see that $\inf_{w \in V^+(z)} \omega_w(T_0) = \omega_z(T_0)$. Thus we have

$$\omega_z(T_0) \le C\omega_{z_0}(T_0), \quad z_0 \in \hat{L}_0, \quad z \in L_0. \tag{3.25}$$

This together with (3.22) and (3.24) gives

$$f(w,4) \le Cf(z,4), \; z \in \hat{L}_0, \; w \text{ on or below } L_0. \tag{3.26}$$

From (3.23), (3.26), and the hypotheses of the lemma we obtain,

$$
\begin{aligned}
(3.27) \qquad E_m(h(B_0,4)1(B_0 \in \hat{L}_0)) &\geq C E_m(f(B_0,4)1(B_0 \in \hat{L}_0)) \\
&\geq C \max\{f(z,4)\colon z \text{ on or below } L_0\} \\
&\geq C E_m f(B_0,4) \\
&\geq C E_m g(B_0,4).
\end{aligned}
$$

On the other hand if we apply the Markov property at $t = 4$ and use the fact that Ω is contained in the half strip $(0,1) \times (-\infty, 0)$ we obtain for any $z = (x,y) \in \Omega$,

$$
\begin{aligned}
(3.28) \qquad g(z,4+j) &= P_z\{\tau_D > 4+j\} \\
&\leq \sup_{x \in (0,1)} P_x\{\tau_{(0,1)} > j\} P_z\{\tau_D > 4\} \\
&\leq C e^{\frac{-\pi^2}{2} j} g(z,4),
\end{aligned}
$$

where the last inequality follows from the fact that $\frac{\pi^2}{2}$ is the lowest eigenvalue for $(0,1)$. From (3.28), (3.27) and (3.18) we see that for j large enough,

$$
\begin{aligned}
(3.29) \qquad E_m g(B_0, 4+j) &\leq C e^{-\frac{\pi^2}{2} j} E_m g(B_0,4) \\
&\leq C e^{-\frac{\pi^2}{2} j} E_m(h(B_0,4)1(B_0 \in \hat{L}_0)) \\
&\leq C e^{-\frac{\pi^2}{2} j} E_m(h(B_0,4+j)1(B_0 \in \hat{L}_0)) \\
&\leq \frac{C_2}{4} E_m(h(B_0,4+j)1(B_0 \in \hat{L}_o)) \\
&\leq \frac{C_2}{4} E_m h(B_0, 4+j).
\end{aligned}
$$

From now on we consider j fixed and large enough so that (3.29) holds.

Next, by (3.19), we have

$$
(3.30) \qquad E_m \omega_{B_0}(T_0) - E_m h(B_0, 4+j) \leq E_m g(B_0, 4+j)
$$

which together with (3.20) and (3.29) gives

$$
\begin{aligned}
E_m \hat{h}(B_0, 4=j) &\geq E_m \omega_{B_0}(\hat{T}_0) - [E_m \omega_{B_0}(T_0)) - E_m(h(B_0,4+j)] \\
&\geq E_m \omega_{B_0}(\hat{T}_0) - E_m g(B_0, 4+j)
\end{aligned}
$$

$$
\begin{aligned}
&\geq E_m\omega_{B_0}(\hat{T}_0) - \frac{C_2}{4}E_m h(B_0, 4+j) \\
&\geq E_m\omega_{B_0}(T_0) - \frac{C_2}{4}E_m h(B_0, 4+j) \\
&\geq E_m h(B_0, 4+j) - \frac{C_2}{4}E_m h(B_0, 4+j) \\
&= \frac{3}{4}C_2 E_m h(B_0, 4+j) \\
&= \frac{3}{4}C_2[\frac{4}{5}E_m h(B_0, 4+j) + \frac{1}{5}E_m h(B_0, 4+j)] \\
&\geq \frac{3}{4}C_2\frac{4}{5}[E_m h(B_0, 4+j)) + \frac{1}{4}E_m h(B_0, 4+j)] \\
&\geq \frac{3}{5}C_2[E_m h(B_0, 4+j) + \frac{C_2}{4}E_m h(B_0, 4+j)] \\
&\geq \frac{3}{5}C_2[E_m h(B_0, 4+j) + E_m g(B_0, 4+j)],
\end{aligned}
$$

and thus we may take $C_6 = 4 + j$, since $\frac{3}{5} > \frac{1}{2}$. □

Lemma 3.6. Let L be a MHLS of D and let $z \in D$ be below L and a distance at least two from L. Let V be the vertical line segment connecting z to L and let Γ be the union of all MHLS of D which intersect V. Let $\gamma = y$, $(z = (x, y))$, and let $a_0, a_1, \ldots, a_M$ be the numbers corresponding to Γ and γ guaranteed by Lemma 2.4. Let L_k be the MHLS with $h(L_k) = a_k$, let $\psi = \{z \in D : z \text{ below } L_M\}$, and set $\delta = |\Gamma|$. Then

$$
P_z\{B_{\tau_\psi} \in \hat{L}, \tau_\psi < C_7\delta\} \geq \min(\frac{C_2}{2}, C_5)[P_z\{B_{\tau_\psi} \in L, \tau_\psi < C_7\delta\} + |P_z\{\tau_\psi > C_7\delta\}],
$$

where $C_7 = \max(C_4, C_6)$.

Proof: Let $\Gamma_k = \{(x, y) \in \Gamma : a_{k-1} \leq y \leq a_k\}$ and let $\lambda_k = C_7|\Gamma_k|$. Let $v_k = \inf\{t > 0 : B_t \in L_k\}$, $k = 1, 2, \ldots, M$. We denote by θ be the usual shift transformation for Markov Processes. Let $T_1 = (\lambda_1 \wedge v_1), T_2 = (\lambda_1 \wedge v_2) \circ \theta_{T_1}, \ldots T_M = (\lambda \wedge v_M) \circ \theta_{T_{M-1}}$. Notice that $T_1 < T_2 < \ldots < T_M < \sum_{k=1}^{M} \lambda_k \leq C_7|\Gamma| = C_7\delta$. Define $A_k = \{B_{T_k} \in L_k, T_k < \tau_{\psi_k}\}$, $\hat{A}_k = \{B_{T_k} \in \hat{L}_k, T_k < \tau_{\psi_k}\}$ and $B_k = \{B_{T_k} \notin L_k, T_k < \tau_{\psi_k}\}$, where ψ_k is the region below L_k. Notice that each ψ_k is a region of the shape $\ell(L_k)\Omega_f$ for some f. By Lemma 2.4 and scaling, we may apply Lemma 3.4 to obtain that

$$
P_z\{\hat{A}_1\} \geq C_5 P_z\{A_1 \cup B_1\} \geq \min\left(\frac{C_2}{2}, C_5\right) P_z\{A_1 \cup B_1\}. \tag{3.31}
$$

Now, Lemma 3.5 with m the distribution of $B_{T_1} I(T_1 < \tau_\psi)$, and the strong Markov property, give

$$(3.32) \qquad P_z\{\hat{A}_2\} \geq \frac{C_2}{2} P_z\{A_2 \cup B_2\} \geq \min\left(\frac{C_2}{2}, C_5\right) P_z\{A_2 \cup B_2\}.$$

Continuing to apply Lemma 3.5, we obtain the analog of (3.32) with $\hat{A}_M$, A_M and B_M in place of $\hat{A}_2$, A_2, and B_2, which proves the lemma. □

We are now ready to complete the proof of the sufficiency part of Theorem 1. We retain the notation of the previous proof. Let $\varepsilon > 0$ and choose $\varphi = \varphi(\varepsilon)$ so negative that the maximal length squared of the horizontal line segments of height φ is less than ε and such that $\sup_{A \in \mathcal{A}_\varphi} |A| < \varepsilon$. The proof of (3.1) in Davis [13] gives a constant $C_{t,\varepsilon}$ depending only on t and φ, hence on t and ε, such that

$$(3.33) \qquad P_z\{B_t \in K; \tau_D > t\} \geq C_{t,\varepsilon} P_z\{\tau_D > t\}$$

for any z above the line $y = \varphi - 2$. As above, $K = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$. (Notice, however, that the constant given in Davis [13] gets worse and worse as φ decreases. Later in this paper we give an alternative proof of (3.33)).

Next, suppose z is below the line of height $\varphi - 2$. Let L be the L_M in Lemma 3.6 corresponding to z.

If $u(w,t) = P_w\{\tau_D > t\}$ then by the parabolic Harnack inequality (see [2]) for all $t < s$ and $w_1, w_2 \in \hat{L}$ we have

$$u(w_1, t) \leq C \, \exp\left(c\left(\frac{|w_1 - w_2|^2}{t-s} + \frac{t-s}{R}\right)\right) u(w_2, s)$$

where $R = \min(1, t, d^2), d = \operatorname{dist}(\hat{L}, \partial D)$. Since $d^2 \approx \ell^2(L) \approx |\Gamma| = \delta$, we have for $t > C_7\delta$,

$$P_\omega\{\tau_D > t\} \leq C P_\omega\{\tau_D > t + C_7\delta\}$$

for all $w \in \hat{L}$. By this, (3.33) and the strong Markov property applied at time τ_L,

$$\begin{aligned} &P_z\{B_{t+C_7\delta} \in K; \tau_D > t + C_7\delta\} \\ (3.34) \qquad &\geq C_{t,\varepsilon} \inf_{w \in \hat{L}} P_w\{\tau_D > t\} P_z\{B_{\tau_\psi} \in \hat{L}, \tau_L \leq C_7\delta\}. \end{aligned}$$

where the $C_{t,\varepsilon}$ here may be taken to be the minimum of the $C_{s,\varepsilon}$ from (3.33) for $C_7\delta \le s \le C_7\delta + t$, assuming, as we may, that we have chosen the constants $C_{s,\varepsilon}$ from (3.33) continuous on compact s intervals. On the other hand, we have

$$\text{(3.35)} \qquad P_z\{\tau_D > t + C_7\delta\} = P_z\{\tau_D > t + C_7\delta, \tau_L \le C_7\delta\} + P_z\{\tau_D > t + C_7\delta, \tau_L > C_7\delta\}.$$

Again, the strong Markov property and Lemma 3.3 give

$$\text{(3.36)} \qquad \begin{aligned} P_z\{\tau_D > t + C_7\delta, \tau_L > C_7\delta\} &\le C_t \sup_{w\in L} P_w\{\tau_D > t\} P_z\{\tau_L > C_7\delta\} \\ &\le C_t \inf_{w\in \hat{L}} P_z\{\tau_D > t\} P_z\{\tau_L > C_7\delta\} \end{aligned}$$

and in the same way that

$$\text{(3.37)} \qquad \begin{aligned} &P_z\{\tau_D > t + C_7\delta, \tau_L \le C_7\delta\} \\ &\le C_t \inf_{w\in \hat{L}} P_w\{\tau_D > t\} P_z\{B_{\tau_\Psi} \in L, \tau_L \le C_7\delta\}. \end{aligned}$$

From (3.34)–(3.37) and Lemma 3.6, we obtain,

$$P_z\{B_{t+C_7\delta} \in K, \tau_D > t + C_7\delta\} \ge C_{t,\varepsilon} P_z\{\tau_D > t + C_7\delta\}.$$

Since $\delta = |\Gamma| \le \varepsilon$, we have proved IU, by Lemma 1.4 and the remark following its proof, for any $t > C_7\varepsilon$. Since by our assumption on f, ε can be taken arbitrarily small, we have IU for all $t > 0$ and Theorem 1 is completely proved. □

Proof of Theorem 2. An easy modification of the proof of Theorem 1 shows that $\alpha \sup_{A\in\mathcal{A}} |A| < \infty$, then D is IU for $t > t(\alpha)$ and this gives one direction of Theorem 2. On the other hand, in the proof of Theorem 1 we also showed that if $\sup_{A\in\mathcal{A}} |A| = \infty$, (case (i)), then the lifetime estimate does not hold. Thus we also have completely proved Theorem 2.

§4. Domains with Boundaries Given by Graphs of Functions

First, we rephrase the definition of §1 of a domain with boundary given by functions. Our definition is from Bass and Burdzy [7]. A domain $D \subset \mathbb{R}^2$ is said to have boundary given by a graph of a function if there exist a finite number of orthonormal coordinate systems $CS_1, CS_2, CS_3, \ldots, CS_n$, real numbers $r_1, r_2, \ldots, r_n$, and uppersemicontinuous functions $f_k: (0, r_k) \to [-\infty, 0)$, $1 \le k \le n$, such that f_k is not everywhere $-\infty$ and if $D = \cup_{i=1}^n V_i$ where

$$V_i = \{(x,y): 0 < x < r_i,\ f_i(x) < y < r_i \text{ in } CS_i\}. \tag{4.1}$$

Theorem 3. Let the domain $D = \cup_{i=1}^n V_i$, V_i as in (4.1). D is IU if and only if each V_i satisfies the obvious analog of the condition of Theorem 1.

Theorem 4. Let the domain $D = \cup_{i=1}^n V_i$, V_i as in (4.1). Then

$$\sup_{\substack{z \in D \\ h \in H(D)}} E_z^h(\tau_D) < \infty$$

if and only if each V_i satisfies the obvious analog of the condition of Theorem 2.

The proof of the necessity is almost a carbon copy of the proof of Theorem 1, and is omitted. The proof of the sufficiency requires new arguments, which we now describe, leaving some of the details to the reader.

We let $L_r^k, r < 1$, stand for the intersection of V_k with the line parallel to the line $y = 1$ in CS_k and a distance $-r+1$ below (in CS_k) this line, and let B_r^k be that part of V_k below L_r^k. Clearly, if CS_j and CS_k have a different orientation (that is, if the rotations involved to transform to the usual coordinate systems are different) then for small enough r, B_r^j and B_r^k are disjoint.

Assume without loss of generality that $1 = m_0 \le m_1 \le m_2 < \ldots < m_{j_0} = n+1$ are such that V_i, $m_j \le i < m_{j+1}$ have the same orientation and that $\bigcup_{k=m_j}^{m_{j+1}-1} V_k = D_j$ is connected, and that V_{m_j} and $V_{m_{j+1}-1}$ contain respectively the smallest and largest x values, in terms of CS_{m_j} (or equivalently, in terms of any of the CS_k, $m_j \le k < m_{j+1}$). We also insist that the j_0 sets $\left(\bigcup_{k=m_j}^{m_{j+1}-1} B_r^k\right)$, $0 \le j < j_0$, are disjoint for small enough r. We let, for $0 \le \alpha < j_0$, $K_\alpha = \bigcup_{j=m_\alpha}^{m_{\alpha+1}-1} R_j \cup \bigcup_{j=m_\alpha}^{m_{\alpha+1}-2} \Delta_j$ where $R_j = [r_j/4, 3r_j/4] \times [r_j/4,\ 3r_j/4]$ (in coordinate CS_j), and Δ_j is a curve lying in D_α and connecting R_j and R_{j+1}. Let

P_r^j denote the line parallel to the 'top' line (in CS_{m_j}) of V_k, $m_j \le k < m_{j+1}$, and such that the maximum distance of P_r^j from these $m_{j+1} - m_j$ 'top' lines is r. Pick t_j so small that G_j, $0 \le j < j_0$, are disjoint, where G_j is that part of D_j 'below' $P_{t_j}^j$. We observe that any curve in D_j which lies between $P_{t_j}^j$ and $P_{2t_j}^j$ and which starts in $P_{t_j}^j$ and ends in $P_{2t_j}^j$, upon reflection about $P_{t_j}^j$, either connects P_{t_j} to K_j before leaving D_j, or else, before leaving D_j intersects from below either the line segment parallel to the 'top' of R_{m_j} which connects this 'top' to the 'left' boundary of D_{m_j}, or the line segment parallel to the 'top' of $R_{m_{j+1}-1}$ which connects this 'top' to the 'right' boundary of $D_{m_{j+1}-1}$.

Now let D_j^+ be that part of D_j which lies 'above' $P_{2t_j}^j$, and let $D^+ = \bigcup_{j=1}^{j_0} D_j^+$. Then D^+ is a (connected) bounded domain, and it is easily shown that D^+ is IU, using either results of Bañuelos [4] or Bass–Burdzy [7] to the effect that a domain with boundary given by the graphs of bounded functions is IU. Let $K = \bigcup_{j=1}^{j_0} K_j$.

To prove sufficiency we will show

$$P_z(\tau_K < \tau_D, \tau_K < t) > C_t P_z(\tau_D > t) \tag{4.2}$$

and apply Lemma 1.4, together with the strong Markov property at τ_K together with the fact that $P_z(\tau_D > t,\ B_t \in K) > C_t$ if $z \in K$, which holds since K is compact and has positive area.

Let D'_j be those parts of D_j lying above P_{t_j} and let $D' = \bigcup_{j=1}^{j_0} D'_j$. We first prove (4.2) for $z \in D'$. We have

$$P_z(\tau_D > t) = P_z(\tau_{D^+} > t) + P_z(\tau_{D^+} \le t,\ \tau_D > t). \tag{4.3}$$

Now since D^+ is IU,

$$P_z(\tau_{D^+} > t) \le C_t P_z(\tau_K \le t,\ \tau_K < \tau_{D^+}). \tag{4.4}$$

Thus to complete the proof of (4.2) for $z \in D'$ it suffices to show

$$P_z(\tau_{D^+} \le t,\ \tau_D > t) \le C_t P_z(\tau_K < \tau_D,\ \tau_K < t),\ z \in D', \tag{4.5}$$

which follows, using the fact that $B_{\tau_{D^+}} \in \cup P_{2t_j}$ on $\{\tau_{D^+} < \tau_D\}$, from

$$P_z(\tau_{P_{2t_j}} < t,\ \tau_D > t) \le C_t P_z(\tau_{K_j} < \tau_D,\ \tau_{K_j} < t),\ z \in D^+,\ 0 \le j < j_0. \tag{4.6}$$

To prove (4.6), reflect B_t about P_{t_j} after the last time, after hitting P_{t_j}, that it hits P_{t_j} before hitting $P_{2t_{j+1}} \cup P_0$. We see that the probability that this reflected motion, which is still standard Brownian motion, exhibits the behavior described in connection with the reflection discussion above, is at least the probability that the original motion hit P_{2t_j} before leaving D^+. An argument identical to one used in [13] to prove (3.1), now proves (4.6).

Finally, we complete the proof of (4.2). Suppose $z \in V_j$. Since V_j is IU by Theorem 1, we have

$$\begin{aligned} P_z(\tau_K < t,\ \tau_K < \tau_D) &> P_z(\tau_K < \tau_{V_j},\ \tau_K < t/2) \\ &> C_t P_z(\tau_{V_j} > t/2) \\ &> C_t P_z(\tau_D > t,\ \tau_{V_j} > t/2). \end{aligned}$$

Thus, to finish the proof of (4.2) it suffices to show

$$\begin{aligned} (4.7) \qquad P_z(\tau_K < t,\ \tau_K < \tau_D) &> C_t P_z(\tau_D > t,\ \tau_{V_i} \le t/2), \\ &= C_t P_z(\tau_D > t,\ \tau_{V_i} \le t/2,\ B_{\tau_{V_i}} \in D'), z \in V_j \end{aligned}$$

the last equality since that part of the boundary of V_i which is not in D' is also part of the boundary of D. Let $\hat{C}_t = \min_{t/2 \le s \le t} C'_s$, where C'_s is the constant which works in (4.2) for all $z \in D'$. We can and do choose these constants bounded below on compact time intervals of $(0, \infty)$. We have on $\{B_{\tau_{V_j}} \in D',\ \tau_{V_i} \le t/2\} = F_t$,

$$\begin{aligned} P_z(\tau_K < t,\ \tau_K < \tau_D | B_{\tau_{V_j}}) &\ge P_{B_{\tau_{V_j}}}(\tau_K < t - \tau_{V_j},\ \tau_K < \tau_D) \\ &\ge \hat{C}_t P_{B_{\tau_{V_j}}}(\tau_D > t - \tau_{V_j}), \end{aligned}$$

which, upon integration over F_t gives

$$P_z(\tau_K < t,\ \tau_K < \tau_D) \ge \hat{C}_t P_z(\tau_D > t,\ F_t),$$

which is (4.7). □

We now present our characterization of intrinsic supercontractivity. First, as we said earlier, IU has been proved for a wide class of domains in $\mathbb{R}^d$, $d \ge 2$. In particular, in Bañuelos [4] IU is proved for what are called "uniformly Hölder domains." More precisely, a domain D in $\mathbb{R}^d$, $d \ge 2$, is said to be in $UH(\alpha)$ for $0 < \alpha < \infty$ if

$$\rho_D(z) \le \frac{c}{d_D(z)^\alpha} + C$$

where $\rho_D(z) = \rho_D(z_0, z)$, $z_0 \in D$ is fixed and $d_D(z)$ is the euclidean distance from z to ∂D, and if D satisfies capacity condition

$$\text{Cap}(B(Q,R) \cap D^C) \geq C' R^{d-2} \tag{4.8}$$

for all $Q \in \partial D$ and all $R > 0$ with a similar definition relative to balls in the plane. Here, Cap denotes the Newtonian capacity. For simply connected planar domains the condition (4.8) is automatically satisfied. In Bañuelos [4], it is proved that if $D \in UH(\alpha)$ for any $0 < \alpha < 2$, then D is IU and that for every $\alpha \geq 2$ there exists $D \in UH(\alpha)$ which is not IU and for which even the weaker result of the expected lifetime does not hold. For $0 < \alpha < 2$, the $UH(\alpha)$ class includes the uniformly twisted L^p domains, $p > d-1$, of Bass and Burdzy [7]. In the plane they include any domain which is of the form $\bigcup_{i=1}^{n} V_i$, V_i as in (4.1) and $f_i \in L^p$ for $p > 1$ for every i. The results for $UH(\alpha)$ motivate the following question: Under the assumption (4.8), is $d_D^2(z)\rho_D(z) \to 0$ as $d_D(z) \to 0$, a necessary and sufficient condition for IU? First, if $\theta(x) = \frac{1}{x \log x}$ for $x > e$ and $\theta(x) = 1/e$ for $0 < x \leq e$, the domain $D_\theta = \{(x,y) \colon x > 0,\ -\theta(x) < y < \theta(x)\}$ is not IU and in fact, even the lifetime estimate does not hold by Theorem 4. On the other hand, it is easy to see that $d_{D_\theta}^2(x,0)\rho_{D_\theta}(x,0) \sim \frac{1}{\log x}$ and thus the condition $d_D^2(z)\rho_D(z) \to 0$ as $d_{D_\theta}(z) \to 0$ does not imply IU. In the other direction we do have an affirmative result and even a stronger result.

Theorem 5. Under the assumption (4.8) D is ISC if and only if $d_D^2(z)\rho_D(z) \to 0$ as $d_D(z) \to 0$.

We also have the following result for IU. Part (a) is a corollary of Theorem 5, (b) follows from the example D_θ discussed above. Part (c) follows exactly as the proof of Theorem 1 in Bañuelos [4] with minor changes and part (d) follows from Theorem 1 in Davis [13] or our Theorem 1 above.

Theorem 6. (a) Suppose D satisfies (4.8). If D is IU then $d_D^2(z)\rho_D(z) \to 0$ as $d_D^2(z) \to 0$.
(b) There exists a D satisfying (4.8) such that $d_D^2(z)\rho_D(z) \to 0$ but D is not IU.

(c) Suppose D satisfies (4.2) and in addition

$$\rho_D(z) \leq \frac{c\eta(d_D(z))}{d_D^2(z)} + C$$

with $\eta(r) \downarrow 0$ as $r \downarrow 0$ and such that

$$\int_0^1 \frac{\eta(r)}{r} dr < \infty. \tag{4.9}$$

Then D is IU.

(d) Let $\{a_n\}$ be any sequence of positive real numbers such that $a_n \to 0$ as $n \to \infty$. There exists a domain D satisfying (4.8) which is IU and with points $z_n \in D$ such that $ca_n \le d_D^2(z_n)\rho_D(z_n) \le Ca_n$, where c and C are constants independent of n.

Thus under the assumption (4.8), IU implies $d_D^2(z)\rho_D(z) \to 0$ but the converse is false. However, if we assume something about the rate, namely (4.9), we do have IU. In general, however, we cannot conclude anything about the rate at which $d_D^2(z)\rho_D(z) \to 0$ from IU. It is also interesting to note that for D_θ as above with any $\theta \downarrow 0$ as $x \uparrow \infty$, (4.9) is equivalent to $|D_\theta| < \infty$ which in turn in equivalent to IU by Theorem 1.

Proof of Theorem 5. The argument in Bañuelos [4] shows that if $d_D^2(z)\rho_D(z) \to 0$ as $d_D^2(z) \to 0$, then for all $\varepsilon > 0$ there exists a $g(\varepsilon)$ such that

$$\int_D |u(z)|^2 \log \frac{1}{\varphi(z)} dz \le \varepsilon \int_D |\nabla u(z)|^2 dz + g(\varepsilon) \int_D |u(z)|^2 dz \tag{4.10}$$

for all $u \in C_0^\infty(D)$, (the C^∞ functions with compact support in D). By (4.10) and Theorem 5.2(d) in Davies and Simon [10], (p. 357), D is intrinsically supercontractive and the sufficient part of Theorem 5 follows. It remains to prove that ISC implies $d^2(z)\rho(z) \to 0$ as $d_D(z) \to 0$.

Assume for the rest of this section that, in addition to (4.8), D satisfies

$$\lim_{|z| \to \infty} d_D(z) = 0. \tag{4.11}$$

We note that by Theorem A.4 in Davies and Simon [10], (p. 380), and Theorem 1.6.8 in Davies [9], (p. 39), (4.11) is always satisfied under the assumption of ISC. Under (4.8) and (4.11) we have

Lemma 4.1. There is a positive constant C such that

$$\int_D e^{C\rho_D(z)} |\varphi(z)|^2 dz < \infty. \tag{4.12}$$

Proof: We shall apply Theorem 3.1 of Evans, Harris and Kauffman [15]. First we recall that the distance function ρ_D is equivalent to the quasi–hyperbolic distance defined by

$$\tilde{\rho}_D(z) = \int_\gamma \frac{ds}{d_D(w)}, \tag{4.13}$$

where the infimum is taken over all rectifiable curves joining z to z_0 in D. It is easy to show that $\tilde{\rho}_D$ is Lipschitz continuous with

$$|\nabla \tilde{\rho}_D(z)|^2 \leq \frac{1}{d_D^2(z)}, \tag{4.14}$$

(see Agmon [1], Theorem 1.4). Thus if we set

$$w(z) = e^{\tilde{\rho}_D(z)}$$

we find that

$$\left|\frac{\nabla w(z)}{w(z)}\right| \leq \frac{1}{d_D(z)}. \tag{4.15}$$

By our assumption (4.8), there exists a constant C such that

$$\int_D \frac{|u(z)|^2}{d_D^2(z)} dz \leq C \int_D |\nabla u(z)|^2 dz \tag{4.16}$$

for all $u \in C_0^\infty(D)$; (this is a result of A. Ancona and we refer the reader to Bañuelos [4] for the exact reference to his paper). By our assumption (4.11) $S_c = \{z \in D: \frac{1}{d_D^2(z)} \leq C\}$ is compact in D for any positive constant C. With this, (4.15) and (4.16) we may apply Theorem 3.1 of [15] to conclude that there is a positive constant C such that

$$\int_D e^{C\tilde{\rho}_D(z)} |\varphi(z)|^2 dz < \infty$$

and hence our Lemma is proved. □

We are now ready to prove the "only if" part of Theorem 5. Since D is ISC, $\tilde{P}_t$ maps $L^2(\varphi^2 dz)$ into $L^4(\varphi^2 dz)$ for all $t > 0$ and by (0.2),

$$\int_D Q_t^4(z)\varphi^2(z) dz = e^{2\lambda t} \int_D \left|\frac{P_t^D(z,z)}{\varphi(z)}\right|^2 dz < C_t < \infty$$

for all $t > 0$. Therefore

$$\begin{aligned}
&\int_D e^{\frac{C}{2}\rho_D(z)} P_t^D(z,z)dz \\
&\qquad \le \left(\int_D e^{C\rho_D(z)}|\varphi(z)|^2 dz\right)^{1/2} \left(\int_D \left|\frac{P_t^D(z,z)}{\varphi(z)}\right|^2 dz\right)^{1/2} \\
&\qquad \le C \cdot C_t < \infty \qquad (4.17)
\end{aligned}$$

for all $t > 0$.

Next, let $Q_j \in W(D)$, a Whitney cube for D. By the properties of the Whitney decomposition there exists a universal constant C such that $CQ_j = \tilde{Q}_j \subset D$ where by CQ we mean the cube concentric with Q_j and $\ell(CQ_j) = C\ell(Q_j)$. Let $P_t^{\tilde{Q}_j}(z,w)$ be the Dirichlet heat kernel for $\tilde{Q}_j$. Then

$$P_t^{\tilde{Q}_j}(z,z) \ge \frac{C'}{\ell(Q_j)^d} \exp\left(-\frac{C't}{\ell^2(Q_j)}\right) \qquad (4.18)$$

for all $z \in Q_j$. This follows by first proving (4.18) for the unit cube and then scaling. From (4.17), (4.18) and the fact that $P_t^D(z,w) \ge P_t^{\tilde{Q}_j}(z,w)$ we have that

$$\begin{aligned}
\sum_{Q_j \in W(D)} \exp\left(\frac{C}{2}\rho_D(z_j) - \frac{C't}{\ell^2(Q_j)}\right) &\le C \sum_{Q_j \in W(D)} \int_{Q_j} e^{\frac{C}{2}\rho_D(z)} P_t^{\tilde{Q}_j}(z,z)dz \\
&\le C \int_D e^{\frac{C}{2}\rho_D(z)} P_t^D(z,z)dz \\
&< C_t < \infty \qquad (4.19)
\end{aligned}$$

for every $t > 0$. Here z_j is the center of Q_j. From the convergence of the sum in the left hand side of (4.19) we conclude that $\ell^2(Q_j)\rho_D(z_j) \to 0$ and the result follows from the properties of the Whitney decomposition and the definition of $\rho_D(z)$. □

Remark: Notice that in the above argument we only used that $\tilde{P}_t: L^2(\varphi^2 dz) \to L^4(\varphi^2 dz)$ for all $t > 0$. It is easy to show directly that this implies ISC.

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