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Under mild assumptions, an Edgeworth expansion with remainder $o(N^{-1/2})$ is established for a U-statistic with a kernel h of degree two using observations from an m-dependent shift.

1 Introduction

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed random variables and $f : R^{m+1} \to R$ be a measurable function. For $j \geq 1$, let $X_j = f(\xi_j, \ldots, \xi_{j+m})$. The sequence $X_1, X_2, \ldots$ is said to be an m-dependent shift and an immediate consequence is that $(X_1, \ldots, X_r)$ and $(X_s, X_{s+1}, \ldots)$ are stochastically independent whenever $s - r > m$. Next let $h : R^2 \to R$ be a measurable function symmetric in its two arguments. We shall assume throughout this paper that for some $p > 5/3$,

$$E|h(X_1, X_j)|^p < \infty, \qquad \forall 1 < j \leq m+2. \tag{1}$$

Then $Eh(X_j, X_k)$ exists for all $j < k$. We write

$$h_{j,k}(X_j, X_k) = h(X_j, X_k) - Eh(X_j, X_k), \qquad \forall j < k,$$

and for $N \geq 2$, a U-statistic of degree two is defined as

$$U_N = \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} h_{j,k}(X_j, X_k).$$

Also we define for $N > 6m + 1$,

$$\begin{aligned} g(x) &= E[h_{j,k}(X_j, X_k)|X_j = x], \qquad \forall k - j > m, \\ \psi(x,y) &= h_{j,k}(x,y) - g(x) - g(y), \qquad \forall k - j > m, \\ \hat{U}_N &= (N - 6m - 1) \sum_{j=1}^{N} g(X_j), \end{aligned}$$

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$$\begin{aligned} \Delta_N &= \sum_{j=1}^{N-3m-1} \sum_{k=3m+j+1}^{N} \psi(X_j, X_k) + \sum_{j=1}^{N-1} \sum_{k=j+1}^{(j+3m)\wedge N} h_{j,k}(X_j, X_k) \\ &\quad + \sum_{j=1}^{3m} (3m - j + 1) g(X_j) + \sum_{j=N-3m+1}^{N} (3m + j - N) g(X_j). \end{aligned} \tag{2}$$

Straightforward calculations show that $U_N = \hat{U}_N + \Delta_N$. In this paper, we suppose that

$$\sigma_g^2 = E[g^2(X_1) + 2\sum_{j=1}^{m} g(X_1)g(X_{j+1})] > 0, \tag{3}$$

and

$$E|g(X_1)|^3 < \infty. \tag{4}$$

Let $\hat{\sigma}_N^2$ denote the variance of $\hat{U}_N$. Then by the stationarity of the X_j's, we have

$$\begin{aligned} \hat{\sigma}_N^2 &= (N - 6m - 1)^2 E[N g^2(X_1) + 2\sum_{j=1}^{m} (N - j) g(X_1) g(X_{j+1})] \\ &= N^3 \sigma_g^2 + O(N^2), \end{aligned}$$

as $N \to \infty$. Next let

$$\begin{aligned} \kappa_3 &= \sigma_g^{-3} E\{g^3(X_1) + 3\sum_{j=1}^{m} [g^2(X_1) g(X_{j+1}) + g(X_1) g^2(X_{j+1})] \\ &\quad + 6 \sum_{j=2}^{m+1} \sum_{k=j+1}^{j+m} g(X_1) g(X_j) g(X_k) \\ &\quad + 3 \sum_{j=1}^{2m+1} \sum_{k=3m+2}^{5m+2} \psi(X_{m+1}, X_{4m+2}) g(X_j) g(X_k)\}. \end{aligned} \tag{5}$$

We observe that if $E|h(X_j, X_k)|^3 < \infty$ whenever $j < k$, then $\kappa_3 N^{-1/2}$ is an asymptotic approximation [with error $O(N^{-3/2})$] for the third cumulant of $\hat{\sigma}_N^{-1} U_N$. Define

$$F_N(x) = \Phi(x) - \phi(x) \frac{\kappa_3}{6} N^{-1/2} (x^2 - 1), \tag{6}$$

where ϕ and Φ denote the standard normal density and distribution function respectively.

The main aim of this paper is to establish the validity of a single term Edgeworth expansion for $\hat{\sigma}_N^{-1}U_N$ under mild conditions. In particular, we prove

Theorem 1 *Suppose (1), (3), (4) are satisfied and*

$$\limsup_{|t|\to\infty} E|E[e^{it\sum_{j=1}^{m+1} g(X_j)}|\xi_1,\ldots,\xi_m,\xi_{m+2},\ldots,\xi_{2m+1}]| < 1. \tag{7}$$

Then

$$\sup_x |P(\hat{\sigma}_N^{-1}U_N \le x) - F_N(x)| = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. Without loss of generality, we assume that $5/3 < p \le 2$. To prove Theorem 1, we shall study the characteristic function (c.f.) of $\hat{\sigma}_N^{-1}U_N$. Let ϕ_N denote the c.f. of $\hat{\sigma}_N^{-1}U_N$, that is

$$\phi_N(t) = E\exp(it\hat{\sigma}_N^{-1}U_N),$$

and for κ_3, as in (5), let

$$\phi_N^*(t) = e^{-t^2/2}(1 - \frac{i\kappa_3}{6}N^{-1/2}t^3)$$

be the Fourier transform $\int \exp(itx)dF_N(x)$ of F_N in (6). By the smoothing lemma of Esseen [see for example, Feller (1971), p. 538], it suffices to show that

$$\int_{-N^{1/2}\log N}^{N^{1/2}\log N} |\frac{\phi_N(t) - \phi_N^*(t)}{t}|dt = o(N^{-1/2}), \tag{8}$$

as $N \to \infty$. However (8) is an immediate consequence of Propositions 1, 2 and 3 whose statements and proofs are provided in Sections 2, 3 and 4 respectively. This proves Theorem 1. □

REMARK. Götze and Hipp (1983) showed that (7) holds if ξ_1 has a probability density f_{ξ_1} with respect to Lebesgue measure and $gf : R^{m+1} \to R$ is continuously differentiable such that there exist $y_1, \ldots, y_{2m+1} \in R$ and an open subset $\Omega \supset \{y_1, \ldots, y_{2m+1}\}$ satisfying $f_{\xi_1} > 0$ on Ω and

$$\sum_{j=1}^{m+1} \frac{\partial}{\partial x_j} gf(x_1, \ldots, x_{1+m})|_{(x_1,\ldots,x_{1+m})=(y_j,\ldots,y_{j+m})} \neq 0.$$

REMARK. If the observations are independent and identically distributed [that is $m = 0$], (7) reduces to the well known Cramér's condition.

In the case where $Eh^2(X_1, X_j) < \infty$ whenever $1 < j \le m + 2$, the variance σ_N^2 of U_N exists and we have

Theorem 2 *Suppose that (3), (4) are satisfied,*

$$Eh^2(X_1, X_j) < \infty, \qquad \forall 1 < j \leq m+2,$$

and

$$\limsup_{|t|\to\infty} E\big|E[e^{it\sum_{j=1}^{m+1} g(X_j)} | \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]\big| < 1.$$

Then

$$\sup_x |P(\sigma_N^{-1} U_N \leq x) - F_N(x)| = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. The proof of Theorem 2 is similar to that of Theorem 1 and hence is omitted. □

We end this section with a brief review of previous related literature. There has been a great deal of research done on U-statistics based on independent and identically distributed observations. In this paragraph, we shall assume that the observations are independent and identically distributed. U-statistics were first discussed by Hoeffding (1948) who also showed the asymptotic normality of $\hat{\sigma}_N^{-1} U_N$ under very weak conditions. The rate of convergence to normality was investigated in increasing generality and precision by Grams and Serfling (1973), Bickel (1974), Chan and Wierman (1977), Callaert and Janssen (1978) and Helmers and van Zwet (1982). In particular, Helmers and van Zwet showed that if $p > 5/3$ and (1) and (4) hold, then

$$\sup_x |P(\hat{\sigma}_N^{-1} U_N \leq x) - \Phi(x)| = O(N^{-1/2}), \tag{9}$$

as $N \to \infty$. If furthermore we have $Eh^2(X_1, X_2) < \infty$, then $\hat{\sigma}_N$ can be replaced by σ_N in (9).

If the independence assumption were relaxed, Berry-Esseen type bounds were obtained by Rhee (1988) for U-statistics based on m-dependent observations and by Zhao and Chen (1987) for finite population U-statistics.

Regarding the corresponding more involved problem of Edgeworth expansions, Callaert, Janssen and Veraverbeke (1980) and Bickel, Götze and van Zwet (1986) established for a U-statistic with independent and identically distributed observations, the validity of a one [and two] term Edgeworth expansion with remainder $o(N^{-1/2})$ [and $o(N^{-1})$] respectively.

REMARK. In their Theorem 1.2, Bickel, Götze and van Zwet stated that one of the conditions needed in obtaining a one term Edgeworth expansion is that $E|\psi(X_1, X_2)|^p < \infty$ for some $p > 2$. Since independent and identically distributed observations are a special case of an m-dependent shift, we observe from Theorem 2 that this condition can be relaxed to $E\psi^2(X_1, X_2) < \infty$.

Under dependent observations, the only result that we are aware of is by Kokic and Weber (1990) who established the validity of a one term Edgeworth expansion for U-statistics based on samples from finite populations.

2 The c.f. for small values of the argument

In this section we begin by studying $\phi_N(t)$ for small values of $|t|$, namely $|t| \le N^{\varepsilon_1}$, where $0 < \varepsilon_1 < (3p-5)/(2p)$.

Proposition 1 *Let* $5/3 < p \le 2$ *and* $0 < \varepsilon_1 < (3p-5)/(2p)$. *Then*

$$\int_{-N^{\varepsilon_1}}^{N^{\varepsilon_1}} |\frac{\phi_N(t) - \phi_N^*(t)}{t}|dt = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. It is well known that

$$|e^{ix} - \sum_{j=0}^{r} \frac{(ix)^j}{j!}| \le \min\{\frac{2}{r!}|x|^{r+\theta}, \frac{|x|^{r+1}}{(r+1)!}\}, \qquad \forall\theta \in [0,1). \tag{10}$$

Hence

$$\begin{aligned} \phi_N(t) &= Ee^{it\hat{\sigma}_N^{-1}\hat{U}_N}(1 + it\hat{\sigma}_N^{-1}\Delta_N) + O(E|t\hat{\sigma}_N^{-1}\Delta_N|^p) \\ &= Ee^{it\hat{\sigma}_N^{-1}\hat{U}_N}(1 + it\hat{\sigma}_N^{-1}\Delta_N) + O(|t|^p N^{2-3p/2}). \end{aligned} \tag{11}$$

The last equality uses the fact that $E|\Delta_N|^p = O(N^2)$ [see for example Lemma 5–1 of Rhee (1988)]. Define for $1 \le a < b \le N$,

$$\begin{aligned} S_{a,b}^{(\nu)} &= (N - 6m - 1) \sum_{|j-a|\wedge|j-b|>\nu m} g(X_j), \qquad \forall\nu \ge 1, \\ S_{a,b}^{(0)} &= \hat{U}_N. \end{aligned}$$

As $\hat{U}_N = S_{a,b}^{(0)}$, for all $a < b$, it follows from (11) and Lemma 1 [see Appendix] that

$$\begin{aligned} &\phi_N(t) - e^{-t^2/2}(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3) \\ &= Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\ &\quad + O(|t|^p N^{2-3p/2}) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}], \end{aligned} \tag{12}$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon_1}$. It remains to approximate the term $Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}\hat{U}_N}$. Following a method of Tikhomirov (1980), we write

$$\begin{aligned}
&\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j,X_k)e^{it\hat{\sigma}_N^{-1}\hat{U}_N} \\
=&\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N} E\{it\hat{\sigma}_N^{-1}\psi(X_j,X_k)e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}} \\
&+it\hat{\sigma}_N^{-1}\psi(X_j,X_k)\sum_{r=2}^{4}\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}} \\
&+it\hat{\sigma}_N^{-1}\psi(X_j,X_k)\prod_{l=1}^{4}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(4)}}\} \\
=&\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N}\sum_{r=2}^{4} it\hat{\sigma}_N^{-1}\{E\psi(X_j,X_k)\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]\} \\
&\times[Ee^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}}]+O(|t|^6N^{-2}),
\end{aligned} \tag{13}$$

as $N \to \infty$ uniformly in t. The last equality uses Lemma 3 and the independence of $S_{j,k}^{(r)}$ and $\psi(X_j,X_k)]\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]$. Furthermore using Lemmas 1, 2 and 3, we have

$$\begin{aligned}
&\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N}\sum_{r=2}^{4} it\hat{\sigma}_N^{-1}\{E\psi(X_j,X_k) \\
&\times\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}-S_{j,k}^{(l)})}-1]\}[Ee^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}}] \\
=&-\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N} it^3e^{-t^2/2}\sigma_g^{-3}N^{-5/2} \\
&\times[E\sum_{a=(j-m)\vee 1}^{j+m}\sum_{b=k-m}^{(k+m)\wedge N}\psi(X_j,X_k)g(X_a)g(X_b)] \\
&+o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}]
\end{aligned} \tag{14}$$

as $N \to \infty$ uniformly over $|t| \le N^{\varepsilon_1}$, where $\mathcal{P}(|t|)$ is a generic linear combination [not depending on N] of non-negative powers of $|t|$. Also for convenience of notation, $\mathcal{P}$ may represent different linear combinations at different occurrences. Thus it follows from (13) and (14) that

$$\sum_{j=1}^{N-3m-1}\sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j,X_k)e^{it\hat{\sigma}_N^{-1}\hat{U}_N}$$

$$= -e^{-t^2/2}\frac{i}{6}N^{-1/2}t^3E[3\sigma_g^{-3}\sum_{j=1}^{2m+1}\sum_{k=3m+2}^{5m+2}\psi(X_{m+1},X_{4m+2})g(X_j)g(X_k)]$$

$$+O(|t|^6N^{-2})+o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}], \tag{15}$$

as $N\to\infty$ uniformly over $|t|\le N^{\epsilon_1}$. In a similar though less tedious way, we have

$$Eit\hat{\sigma}_N^{-1}\sum_{j=1}^{N-1}\sum_{k=j+1}^{(j+3m)\wedge N} h_{j,k}(X_j,X_k)e^{it\hat{\sigma}_N^{-1}\hat{U}_N}$$

$$= it\hat{\sigma}_N^{-1}\sum_{j=1}^{N-1}\sum_{k=j+1}^{(j+3m)\wedge N} E\{h_{j,k}(X_j,X_k)e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}}$$

$$+h_{j,k}(X_j,X_k)[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(0)}-S_{j,k}^{(1)})}-1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}}\}$$

$$= O(|t|^2N^{-1}), \tag{16}$$

and

$$Ee^{it\hat{\sigma}_N^{-1}\hat{U}_N}it\hat{\sigma}_N^{-1}[\sum_{j=1}^{3m}(3m-j+1)g(X_j)$$

$$+\sum_{j=N-3m+1}^{N}(3m+j-N)g(X_j)] = O(|t|N^{-3/2}), \tag{17}$$

as $N\to\infty$ uniformly in $|t|$. Thus it follows from (2), (15), (16) and (17) that

$$Eit\hat{\sigma}_N^{-1}\Delta_N e^{it\hat{\sigma}_N^{-1}\hat{U}_N}$$

$$= -e^{-t^2/2}\frac{i}{6}N^{-1/2}t^3E[3\sigma_g^{-3}\sum_{j=1}^{2m+1}\sum_{k=3m+2}^{5m+2}\psi(X_{m+1},X_{4m+2})g(X_j)g(X_k)]$$

$$+O(|t|N^{-3/2}+|t|^2N^{-1}+|t|^6N^{-2})+o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}],$$

as $N\to\infty$ uniformly over $|t|\le N^{\epsilon_1}$. Hence we conclude from (12) that

$$\phi_N(t)-\phi_N^*(t) = \phi_N(t)-e^{-t^2/2}(1-\frac{i\kappa_3}{6}N^{-1/2}t^3)$$

$$= O(|t|N^{-3/2} + |t|^2N^{-1} + |t|^6N^{-2} + |t|^pN^{2-3p/2}) \\ +o[|t|\mathcal{P}(|t|)e^{-t^2/4}N^{-1/2}],$$

as $N \to \infty$ uniformly over $|t| \leq N^{\varepsilon_1}$ and hence

$$\int_{-N^{\varepsilon_1}}^{N^{\varepsilon_1}} |\frac{\phi_N(t) - \phi_N^*(t)}{t}|dt = o(N^{-1/2}),$$

as $N \to \infty$. This completes the proof of Proposition 1. □

3 The c.f. for moderate values of the argument

First define constants ε_3 and ε_4 to satisfy

$$0 < 6(m+1)\varepsilon_3\sigma_g^{-1}[E|g(X_1)|^3]^{1/3} = \varepsilon_4 < 1. \tag{18}$$

In this section we prove

Proposition 2 *Let ε_1 be as in Section 2 and ε_2 as in Lemma 4 (see Appendix). Then*

$$\int_{N^{\varepsilon_1} \leq |t| \leq \varepsilon_2 N^{1/2}} |\frac{\phi_N(t) - \phi_N^*(t)}{t}|dt = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. For $|t| \geq N^{\varepsilon_1}$, define

$$u = \lfloor -\frac{N \log N}{t^2 \log \varepsilon_4} \rfloor, \tag{19}$$

$$C_1 = 5 - \frac{5(m+1)}{\log \varepsilon_4}, \tag{20}$$

$$n = \lceil \frac{C_1 N \log N}{t^2} \rceil, \tag{21}$$

$$\Delta_N(n) = \sum_{j=1}^{n\wedge(N-3m-1)} \sum_{k=3m+j+1}^{N} \psi(X_j, X_k) \\ + \sum_{j=1}^{n\wedge(N-1)} \sum_{k=j+1}^{(j+3m)\wedge N} h_{j,k}(X_j, X_k) + \sum_{j=1}^{3m}(3m-j+1)g(X_j), \tag{22}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then it follows from (10) that

$$\begin{aligned} & |\phi_N(t)| \\ = \; & |Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}[1 + it\hat{\sigma}_N^{-1}\Delta_N(n)]| + O[E|t\hat{\sigma}_N^{-1}\Delta_N(n)|^p] \\ = \; & |Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}[1 + it\hat{\sigma}_N^{-1}\Delta_N(n)]| + O(|t|^p n N^{1-3p/2}), \end{aligned} \tag{23}$$

since $E|\Delta_N(n)|^p = O(nN)$ [see Rhee (1988)]. Next define for $1 \leq a < b \leq N$,

$$\begin{aligned} S_{a,b}^{(0)}(n) &= (N - 6m - 1)\sum_{j=1}^{n} g(X_j), \\ S_{a,b}^{(\nu)}(n) &= (N - 6m - 1)\sum_{j \in \Omega_{a,b}^{(\nu)}(n)} g(X_j), \qquad \forall \nu \geq 1, \end{aligned} \tag{24}$$

where $\Omega_{a,b}^{(\nu)}(n) = \{j : |j - a| > \nu m, |j - b| > \nu m, n + 1 - j > \nu m\}$. We shall now approximate $Eit\hat{\sigma}_N^{-1}\Delta_N(n)e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}$. As in the proof of Proposition 1, we write

$$\begin{aligned} & \sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))} \\ = & \sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} E\{it\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))}e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(1)}(n)} \\ & + it\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))} \\ & \times \sum_{r=2}^{u}\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}(n) - S_{j,k}^{(l)}(n))} - 1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(r)}(n)} \\ & + it\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))} \\ & \times \prod_{l=1}^{u}[e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}(n) - S_{j,k}^{(l)}(n))} - 1]e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(u)}(n)}\}. \end{aligned} \tag{25}$$

To bound the r.h.s. of (25), we first observe from Lemma 5 that

$$|\sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))}$$

$$\times \prod_{l=1}^{u} [e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}(n) - S_{j,k}^{(l)}(n))} - 1] e^{it\hat{\sigma}_N^{-1} S_{j,k}^{(u)}(n)}|$$

$$\leq \sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} E|t\hat{\sigma}_N^{-1}\psi(X_j, X_k) \prod_{l=1}^{u} [e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}(n) - S_{j,k}^{(l)}(n))} - 1]|$$

$$= O(|t|nN^{-3/2}), \tag{26}$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \leq |t| \leq \varepsilon_2 N^{1/2}$. Also from Lemmas 4 and 5, we get

$$|\sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)$$

$$\times e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))} e^{it\hat{\sigma}_N^{-1} S_{j,k}^{(1)}(n)}|$$

$$\leq \sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} [E|t\hat{\sigma}_N^{-1}\psi(X_j, X_k)|] |Ee^{it\hat{\sigma}_N^{-1} S_{j,k}^{(1)}(n)}|$$

$$= O(|t|nN^{-3/2}), \tag{27}$$

and

$$|\sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))}$$

$$\times \sum_{r=2}^{u} \prod_{l=1}^{r-1} [e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}(n) - S_{j,k}^{(l)}(n))} - 1] e^{it\hat{\sigma}_N^{-1} S_{j,k}^{(r)}(n)}|$$

$$\leq \sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} \sum_{r=2}^{u} \{E|t\hat{\sigma}_N^{-1}\psi(X_j, X_k)$$

$$\times \prod_{l=1}^{r-1} [e^{it\hat{\sigma}_N^{-1}(S_{j,k}^{(l-1)}(n) - S_{j,k}^{(l)}(n))} - 1]|\} |Ee^{it\hat{\sigma}_N^{-1} S_{j,k}^{(r)}(n)}|$$

$$= O(|t|nN^{-3/2}), \tag{28}$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \leq |t| \leq \varepsilon_2 N^{1/2}$. Hence it follows from (25) to (28) that

$$|\sum_{j=1}^{n} \sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(|t|nN^{-3/2}), \tag{29}$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$. Similarly, we have

$$(30)\quad |\sum_{j=1}^{n} \sum_{k=j+1}^{j+3m} Eit\hat{\sigma}_N^{-1} h_{j,k}(X_j, X_k) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(|t|nN^{-5/2}),$$

and

$$(31)\quad |\sum_{j=1}^{3m} Eit\hat{\sigma}_N^{-1}(3m - j + 1) g(X_j) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(|t|N^{-5/2}),$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$. We conclude from (29), (30) and (31) that

$$(32)\quad |Eit\hat{\sigma}_N^{-1} \Delta_N(n) e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(|t|nN^{-3/2}),$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$. In the same way, we have

$$(33)\quad |Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(N^{-1}),$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$. We now conclude from (23), (31), (32) and (33) that

$$|\phi_N(t)| = O(N^{-1} + |t|nN^{-3/2} + |t|^p nN^{1-3p/2}),$$

as $N \to \infty$ uniformly over $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$ and hence

$$(34)\quad \int_{N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}} |\phi_N(t)/t| dt = o(N^{-1/2}),$$

as $N \to \infty$. Furthermore it is clear that

$$(35)\quad \int_{|t| \ge N^{\varepsilon_1}} |\phi_N^*(t)/t| dt = o(N^{-1/2})$$

as $N \to \infty$. Proposition 2 now follows from (34) and (35). □

4 The c.f. for large values of the argument

Let ε_2 be defined as in Proposition 2. Then we observe from (7) that there exists a constant $0 < \gamma < 1$ such that

$$(36)\quad E|E[e^{it\sum_{j=1}^{m+1} g(X_j)} | \xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]| \le 1 - \gamma,$$

for all $|t| \ge \sigma_g^{-1} \varepsilon_2 / 2$.

Proposition 3 *Let ε_2 be as in Section 2. Then*

$$\int_{\varepsilon_2 N^{1/2} \le |t| \le N^{1/2}\log N} \left|\frac{\phi_N(t) - \phi_N^*(t)}{t}\right| dt = o(N^{-1/2}),$$

as $N \to \infty$.

PROOF. For sufficiently large N, let n denote a positive integer satisfying

$$\left\lfloor \frac{n-2m}{2(m+1)} \right\rfloor - 3 = \left\lceil -\frac{\log N}{\log(1-\gamma)} \right\rceil. \tag{37}$$

We observe as in the proof of Proposition 2 that

$$\begin{aligned} &|\phi_N(t)| \\ &= |Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}[1 + it\hat{\sigma}_N^{-1}\Delta_N(n)]| + O(|t|^p n N^{1-3p/2}) \end{aligned} \tag{38}$$

as $N \to \infty$ uniformly in t, where $\Delta_N(n)$ is as in (22).

We shall now approximate the first term of the r.h.s. of (38). For simplicity we let $\mathcal{A}_{j,k,n}$ denote the σ-field generated by the random variables ξ_l, $l \in [j, j+m] \cup [k, k+m] \cup [n+1, \infty)$. With $S_{j,k}^{(0)}(n)$ as in (24), we observe from Lemma 6 that

$$\begin{aligned} & |Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| \\ = \; & |Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - S_{j,k}^{(0)}(n) - \Delta_N(n))}E[e^{it\hat{\sigma}_N^{-1}S_{j,k}^{(0)}(n)}|\mathcal{A}_{j,k,n}]| \\ \le \; & |t|\hat{\sigma}_N^{-1}E|\psi(X_j, X_k)|N^{-1}, \end{aligned}$$

and hence

$$\left|\sum_{j=1}^{n}\sum_{k=3m+j+1}^{N} Eit\hat{\sigma}_N^{-1}\psi(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}\right| = O(|t|nN^{-3/2}), \tag{39}$$

as $N \to \infty$ uniformly over $|t| \ge \varepsilon_2 N^{1/2}$.

In a similar way, we have

$$|Ee^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}| = O(N^{-1}), \tag{40}$$

$$\left|\sum_{j=1}^{n}\sum_{k=j+1}^{j+3m} Eit\hat{\sigma}_N^{-1}h_{j,k}(X_j, X_k)e^{it\hat{\sigma}_N^{-1}(U_N - \Delta_N(n))}\right| = O(|t|nN^{-5/2}), \tag{41}$$

and

$$(42) \quad |\sum_{j=1}^{3m} Eit\hat{\sigma}_N^{-1}(3m-j+1)g(X_j)e^{it\hat{\sigma}_N^{-1}(U_N-\Delta_N(n))}| = O(|t|N^{-5/2}),$$

as $N \to \infty$ uniformly over $|t| \geq \varepsilon_2 N^{1/2}$. From (22), (39), (41) and (42), we get

$$(43) \qquad |Eit\hat{\sigma}_N^{-1}\Delta_N(n)e^{it\hat{\sigma}_N^{-1}(U_N-\Delta_N(n))}| = O(|t|nN^{-3/2}),$$

as $N \to \infty$ uniformly over $|t| \geq \varepsilon_2 N^{1/2}$. Now it follows from (38), (40) and (43) that

$$|\phi_N(t)| = O(N^{-1} + |t|nN^{-3/2} + |t|^p nN^{1-3p/2}),$$

and from the definition of n, we have

$$(44) \qquad \int_{\varepsilon_2 N^{1/2} \leq |t| \leq N^{1/2}\log N} |\phi_N(t)/t|dt = o(N^{-1/2}),$$

as $N \to \infty$. Proposition 3 follows from (35) and (44). □

5 Appendix

Lemma 1 *Suppose that (3), (4) are satisfied and r is a fixed nonnegative integer. Then*

$$Ee^{it\hat{\sigma}_N^{-1}S_{a,b}^{(r)}} = e^{-t^2/2}(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}],$$

as $N \to \infty$ uniformly over $1 \leq a < b \leq N$ and $|t| \leq N^{\varepsilon_1}$, where

$$\begin{aligned} \hat{\kappa}_3 &= \sigma_g^{-3}E\{g^3(X_1) + 3\sum_{j=1}^{m}[g^2(X_1)g(X_{j+1}) + g(X_1)g^2(X_{j+1})] \\ &\quad +6\sum_{j=2}^{m+1}\sum_{k=j+1}^{j+m} g(X_1)g(X_j)g(X_k)\}. \end{aligned}$$

PROOF. Let $\hat{\sigma}_{a,b}^{(r)}$ denote the standard deviation of $S_{a,b}^{(r)}$. We observe that the third cumulant of $(\hat{\sigma}_{a,b}^{(r)})^{-1}S_{a,b}^{(r)}$ is asymptotically $\hat{\kappa}_3 N^{-1/2}$ with error $O(N^{-3/2})$ uniformly over $1 \leq a < b \leq N$. Hence it follows from Heinrich (1982) p.513 that

$$Ee^{it(\hat{\sigma}_{a,b}^{(r)})^{-1}S_{a,b}^{(r)}} = e^{-t^2/2}(1 - \frac{i\hat{\kappa}_3}{6}N^{-1/2}t^3) + o[(|t|^2 + |t|^5)e^{-t^2/4}N^{-1/2}],$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $|t| \le N^{\epsilon_1+\delta}$, where δ is a small positive constant. We remark that Heinrich stated his result only for the case of a sum of 1-dependent random variables. However the extension to m-dependence is straightforward. Since $1-(\hat\sigma^{(r)}_{a,b}/\hat\sigma_N)^2 = O(N^{-1})$ uniformly over $1 \le a < b \le N$, we have

$$\begin{aligned} Ee^{it\hat\sigma_N^{-1}S^{(r)}_{a,b}} &= Ee^{it(\hat\sigma_N^{-1}\hat\sigma^{(r)}_{a,b})(\hat\sigma^{(r)}_{a,b})^{-1}S^{(r)}_{a,b}} \\ &= e^{-t^2/2}(1-\frac{i\hat\kappa_3}{6}N^{-1/2}t^3) + o[(|t|^2+|t|^5)e^{-t^2/4}N^{-1/2}], \end{aligned}$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $|t| \le N^{\epsilon_1}$. This completes the proof of Lemma 1. □

Lemma 2 *Let $5/3 < p \le 2$, $p^{-1}+q^{-1} = 1$ and $1 \le a < b \le N$ with $b-a > 3m$. Then*

$$\begin{aligned} & Eit\hat\sigma_N^{-1}\psi(X_a,X_b)\{\exp[it\hat\sigma_N^{-1}(S^{(0)}_{a,b}-S^{(1)}_{a,b})]-1\} \\ = \; & -it^3\sigma_g^{-3}N^{-5/2}E\sum_{j=(a-m)\vee 1}^{a+m}\sum_{k=b-m}^{(b+m)\wedge N}\psi(X_a,X_b)g(X_j)g(X_k) \\ & +O(|t|^3N^{-7/2}+|t|^{2+3/q}N^{-2-3/(2q)}), \end{aligned}$$

as $N \to \infty$ uniformly in a, b and t.

PROOF. We observe that

$$S^{(0)}_{a,b}-S^{(1)}_{a,b} = (N-6m-1)[\sum_{j=(a-m)\vee 1}^{a+m} g(X_j) + \sum_{k=b-m}^{(b+m)\wedge N} g(X_k)].$$

For $1 \le c \le N$, we define

$$R_c = it\hat\sigma_N^{-1}(N-6m-1)\sum_{j=(c-m)\vee 1}^{(c+m)\wedge N} g(X_j). \tag{45}$$

Then

$$\begin{aligned} & Eit\hat\sigma_N^{-1}\psi(X_a,X_b)\{\exp[it\hat\sigma_N^{-1}(S^{(0)}_{a,b}-S^{(1)}_{a,b})]-1\} \\ = \; & Eit\hat\sigma_N^{-1}\psi(X_a,X_b)[(e^{R_a}-1-R_a)(e^{R_b}-1-R_b) \\ & +R_a(e^{R_b}-1-R_b)+R_b(e^{R_a}-1-R_a)+R_aR_b]. \end{aligned} \tag{46}$$

The last equality uses the observation that

$$E\psi(X_a, X_b) = E[\psi(X_a, X_b)|R_a] = E[\psi(X_a, X_b)|R_b] = 0.$$

Next we observe that

$$\begin{aligned} & Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_aR_b \\ = & -it^3\hat{\sigma}_N^{-3}(N-6m-1)^2 E \sum_{j=(a-m)\vee 1}^{a+m} \sum_{k=b-m}^{(b+m)\wedge N} g(X_j)g(X_k)\psi(X_a, X_b) \\ = & -it^3\sigma_g^{-3}N^{-5/2}E \sum_{j=(a-m)\vee 1}^{a+m} \sum_{k=b-m}^{(b+m)\wedge N} g(X_j)g(X_k)\psi(X_a, X_b) \\ & +O(|t|^3N^{-7/2}), \qquad (47) \end{aligned}$$

as $N \to \infty$ uniformly in a, b and t. Furthermore it follows from (10) that

$$\begin{aligned} & E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)[(e^{R_a} - 1 - R_a)(e^{R_b} - 1 - R_b) \\ & +R_a(e^{R_b} - 1 - R_b) + R_b(e^{R_a} - 1 - R_a)]| \\ \leq & \ 6E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_aR_b^{3/q}| + 2E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)R_bR_a^{3/q}| \\ \leq & \ 6|t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}[(E|R_a|^q)^{1/q}(E|R_b|^3)^{1/q} \\ & +(E|R_b|^q)^{1/q}(E|R_a|^3)^{1/q}] \\ = & \ O(|t|^{2+3/q}N^{-2-3/(2q)}), \qquad (48) \end{aligned}$$

as $N \to \infty$ uniformly in a, b and t. Lemma 2 now follows from (46), (47) and (48). □

Lemma 3 *Let r be a fixed positive integer, $5/3 < p \leq 2$ and $1 \leq a < b \leq N$ with $b - a > 3m$. Then*

$$|Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]| = O(|t|^3N^{-5/2}|tN^{-1/2}|^{r-1}),$$

and

$$\begin{aligned} & |Eit\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)} - S_{a,b}^{(l)})} - 1]e^{it\hat{\sigma}_N^{-1}S_{a,b}^{(r)}}| \\ = & \ O(|t|^3N^{-5/2}|tN^{-1/2}|^{r-1}), \end{aligned}$$

as $N \to \infty$ uniformly in a, b and t.

PROOF. Let R_a and R_b be defined as in (45). We observe that

$$\begin{aligned}
&|Eit\hat{\sigma}_N^{-1}\psi(X_a,X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]| \\
=\ & |Eit\hat{\sigma}_N^{-1}\psi(X_a,X_b)[(e^{R_a}-1-R_a)(e^{R_b}-1-R_b) \\
& +R_a(e^{R_b}-1-R_b) \\
& +R_b(e^{R_a}-1-R_a)+R_aR_b]\prod_{l=2}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]| \\
\le\ & 9E|t\hat{\sigma}_N^{-1}\psi(X_a,X_b)R_aR_b\prod_{l=2}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]|. \qquad (49)
\end{aligned}$$

The last inequality uses (10). By Hölder's inequality, the r.h.s. of (49) is less than or equal to

$$\begin{aligned}
& 9|t|\hat{\sigma}_N^{-1}\{E|\psi(X_a,X_b)\prod{}'[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]|^p\}^{1/p} \\
& \times\{E|R_aR_b\prod{}''[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]|^q\}^{1/q}, \qquad (50)
\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $\prod'$ denotes the product over all even integers l, $2\le l\le r$ and $\prod''$ denotes the product over all odd integers l, $3\le l\le r$. By virtue of m-dependence, the r.h.s. of (50) is bounded by

$$\begin{aligned}
& 9|t|\hat{\sigma}_N^{-1}[E|\psi(X_a,X_b)|^p]^{1/p}(E|R_aR_b|^q)^{1/q} \\
& \times\prod_{l=2}^{r}[E|e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1|^3]^{1/3} \\
\le\ & 9|t|\hat{\sigma}_N^{-1}[E|\psi(X_a,X_b)|^p]^{1/p}(E|R_aR_b|^q)^{1/q} \\
& \times\prod_{l=2}^{r}[E|t\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})|^3]^{1/3}. \qquad (51)
\end{aligned}$$

Since

$$[E|t\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})|^3]^{1/3}=O(|t|N^{-1/2}),$$

as $N\to\infty$ uniformly over $1\le a<b\le N$, $2\le l\le r$ and t, it follows from (51) that

$$|Eit\hat{\sigma}_N^{-1}\psi(X_a,X_b)\prod_{l=1}^{r}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}-S_{a,b}^{(l)})}-1]|=O(|t|^3N^{-5/2}|tN^{-1/2}|^{r-1}),$$

This proves the first statement of Lemma 3. The proof of the second statement is similar and is omitted. □

Lemma 4 *Let ε_1 be as in Section 1, u and n be as in (19) and (21) respectively. Then there exists a constant ε_2 satisfying $0 < \varepsilon_2 < 1 \wedge \varepsilon_3$ such that for sufficiently large N, we have*

$$|Ee^{it\hat{\sigma}_N^{-1}S_{a,b}^{(\nu)}(n)}| \leq N^{-1},$$

whenever $N^{\varepsilon_1} \leq |t| \leq \varepsilon_2 N^{1/2}$, $1 \leq a < b \leq N$ and $1 \leq \nu \leq u$.

PROOF. Let $\hat{\sigma}_{a,b}^{(\nu)}(n)$ denote the standard deviation of $S_{a,b}^{(\nu)}(n)$. We observe from Corollary 3.2 of Heinrich (1982) that

$$|Ee^{it[\hat{\sigma}_{a,b}^{(\nu)}(n)]^{-1}S_{a,b}^{(\nu)}(n)}| \leq e^{-t^2/4},$$

whenever $|t| \leq [\hat{\sigma}_{a,b}^{(\nu)}(n)]^3[192\lceil\omega_{a,b}^{(\nu)}(n)/(m+1)\rceil(m+1)^3N^3E|g(X_1)|^3]^{-1}$, where $\omega_{a,b}^{(\nu)}(n)$ denotes the cardinality of $\Omega_{a,b}^{(\nu)}(n)$. Hence

$$\begin{aligned} |Ee^{it\hat{\sigma}_N^{-1}S_{a,b}^{(\nu)}(n)}| &= |Ee^{it[\hat{\sigma}_{a,b}^{(\nu)}(n)/\hat{\sigma}_N][\hat{\sigma}_{a,b}^{(\nu)}(n)]^{-1}S_{a,b}^{(\nu)}(n)}| \\ &\leq e^{-[\hat{\sigma}_{a,b}^{(\nu)}(n)/\hat{\sigma}_N]^2t^2/4}, \end{aligned} \tag{52}$$

whenever

$$|t| \leq \frac{\hat{\sigma}_N[\hat{\sigma}_{a,b}^{(\nu)}(n)]^2}{192\lceil\omega_{a,b}^{(\nu)}(n)/(m+1)\rceil(m+1)^3N^3E|g(X_1)|^3}.$$

By writing out the explicit expression for $\hat{\sigma}_{a,b}^{(\nu)}(n)$ and letting C_1 be as in (20), we observe that for sufficiently large N,

$$\begin{aligned} \frac{t^2}{4}\left(\frac{\hat{\sigma}_{a,b}^{(\nu)}(n)}{\hat{\sigma}_N}\right)^2 &\geq \frac{t^2}{5}\left[\frac{n-5u(m+1)}{N}\right] \\ &\geq \frac{t^2}{5N}\left[\frac{C_1N\log N}{t^2} + \frac{5(m+1)N\log N}{t^2\log\varepsilon_4}\right] \\ &= \log N, \end{aligned} \tag{53}$$

for all $1 \leq a < b \leq N$ and $1 \leq \nu \leq u$. In a similar way, it can be seen that

$$\begin{aligned} &N^{-1/2}\frac{\hat{\sigma}_N[\hat{\sigma}_{a,b}^{(\nu)}(n)]^2}{192\lceil\omega_{a,b}^{(\nu)}(n)/(m+1)\rceil(m+1)^3N^3E|g(X_1)|^3} \\ &\rightarrow \frac{\sigma_g^3}{192(m+1)^2E|g(X_1)|^3}, \end{aligned} \tag{54}$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $1 \le \nu \le u$. Let ε_2 satisfy

$$0 < \varepsilon_2 < \min\{\frac{\sigma_g^3}{192(m+1)^2 E|g(X_1)|^3}, \varepsilon_3, 1\}.$$

Then it follows from (52), (53) and (54) that for sufficiently large N,

$$|Ee^{it\hat{\sigma}_N^{-1} S_{a,b}^{(\nu)}(n)}| \le N^{-1},$$

whenever $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$, $1 \le a < b \le N$ and $1 \le \nu \le u$. This proves Lemma 4. □

Lemma 5 *Let u and n be defined as in (19) and (21) respectively. Then for $5/3 < p \le 2$ we have*

$$E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{u}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}(n)-S_{a,b}^{(l)}(n))} - 1]| = O(|t|N^{-5/2}),$$

and

$$E\sum_{r=2}^{u}|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{r-1}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}(n)-S_{a,b}^{(l)}(n))} - 1]| = O(|t|N^{-3/2}),$$

as $N \to \infty$ uniformly over $1 \le a < b \le N$ and $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$.

PROOF. As in the proof of Lemma 3, we have

$$\begin{aligned} & E|t\hat{\sigma}_N^{-1}\psi(X_a, X_b)\prod_{l=1}^{u}[e^{it\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}(n)-S_{a,b}^{(l)}(n))} - 1]| \\ \le\ & |t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}\prod_{l=1}^{u}[E|t\hat{\sigma}_N^{-1}(S_{a,b}^{(l-1)}(n) - S_{a,b}^{(l)}(n))|^3]^{1/3} \\ \le\ & |t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}\{5(m+1)\varepsilon_2 N^{3/2}\hat{\sigma}_N^{-1}[E|g(X_1)|^3]^{1/3}\}^u. \qquad (55) \end{aligned}$$

Since $\varepsilon_2 < \varepsilon_3$, it follows from (18) that for sufficiently large N the r.h.s. of (55) is bounded by

$$\begin{aligned} |t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}\varepsilon_4^u &\le |t|\hat{\sigma}_N^{-1}[E|\psi(X_a, X_b)|^p]^{1/p}\varepsilon_4^{-1}N^{-1} \\ &= O(|t|N^{-5/2}), \end{aligned}$$

uniformly over $1 \le a < b \le N$ and $N^{\varepsilon_1} \le |t| \le \varepsilon_2 N^{1/2}$. The last inequality uses the fact that $0 < \varepsilon_2 < 1$. This proves the first statement of Lemma 5. The proof of the second statement is similar and is omitted. □

Lemma 6 *Let $1 \le a < b \le N$. Then with the notation of Proposition 3, we have*

$$|E[e^{it\hat{\sigma}_N^{-1}S_{a,b}^{(0)}(n)}|\mathcal{A}_{a,b,n}]| \le N^{-1},$$

for sufficiently large N uniformly over $1 \le a < b \le N$ and $|t| \ge \varepsilon_2 N^{1/2}$.

PROOF. We observe that

$$\begin{aligned} & E[e^{it\hat{\sigma}_N^{-1}S_{a,b}^{(0)}(n)}|\mathcal{A}_{a,b,n}] \\ (56) \qquad = \; & E[\int e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=1}^n g(X_j)} \prod_l{}^* dF(\xi_{l(m+1)})|\mathcal{A}_{a,b,n}], \end{aligned}$$

where $F(\xi_{l(m+1)})$ denotes the distribution function of the random variable $\xi_{l(m+1)}$ and $\prod_l^*$ denotes the product over all positive odd integers l satisfying $l(m+1) \notin [a-m, a+2m] \cup [b-m, b+2m] \cup [n+1-m, \infty)$. Thus the absolute value of the r.h.s. of (56) is bounded by

$$\begin{aligned} & E[\prod_l{}^* | \int e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=l(m+1)-m}^{l(m+1)} g(X_j)} dF(\xi_{l(m+1)})||\mathcal{A}_{a,b,n}] \\ = \; & \prod_l{}^* E[| \int e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=l(m+1)-m}^{l(m+1)} g(X_j)} dF(\xi_{l(m+1)})||\mathcal{A}_{a,b,n}] \\ = \; & \{E|E[e^{it\hat{\sigma}_N^{-1}(N-6m-1)\sum_{j=1}^{m+1} g(X_j)}|\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_{2m+1}]|\}^{k_0}, \end{aligned}$$

where k_0 equals the number of terms in the product $\prod_l^*$. The second [last] equality uses the independence [stationarity] of the ξ_j's respectively. Since

$$k_0 \ge \lfloor \frac{n-2m}{2(m+1)} \rfloor - 3,$$

it follows from (36) that

$$|E[e^{it\hat{\sigma}_N^{-1}S_{a,b}^{(0)}(n)}|\mathcal{A}_{a,b,n}]| \le (1-\gamma)^{\lfloor (n-2m)/[2(m+1)] \rfloor - 3},$$

whenever $(N-6m-1)\hat{\sigma}_N^{-1}|t| \ge \sigma_g^{-1}\varepsilon_2/2$. Thus we conclude from (37) that

$$|E[e^{it\hat{\sigma}_N^{-1}S_{a,b}^{(0)}(n)}|\mathcal{A}_{a,b,n}]| \le N^{-1},$$

for sufficiently large N uniformly over $1 \le a < b \le N$ and $|t| \ge \varepsilon_2 N^{1/2}$. □

References

[1] BICKEL, P. J. (1974). Edgeworth expansions in nonparametric statistics. *Ann. Statist.* **2** 1-20.

[2] BICKEL, P. J., GÖTZE, F. and VAN ZWET, W. R. (1986). The Edgeworth expansion for U-statistics of degree two. *Ann. Statist.* **14** 1463-1484.

[3] CALLAERT, H. and JANSSEN, P. (1978). The Berry-Esseen theorem for U-statistics. *Ann. Statist.* **6** 417-421.

[4] CALLAERT, H., JANSSEN, P. and VERAVERBEKE, N. (1980). An Edgeworth expansion for U-statistics. *Ann. Statist.* **8** 299-312.

[5] CHAN, Y.-K. and WIERMAN, J. (1977). On the Berry-Esseen theorem for U-statistics. *Ann. Probab.* **5** 136-139.

[6] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

[7] GÖTZE, F. and HIPP, C. (1983). Asymptotic expansions for sums of weakly dependent random vectors. *Z. Wahrsch. Verw. Gebiete.* **64** 211-239.

[8] GRAMS, W. F. and SERFLING, R. J. (1973). Convergence rates for U-statistics and related statistics. *Ann. Statist.* **1** 153-160.

[9] HEINRICH, L. (1982). A method for the derivation of limit theorems for sums of m-dependent random variables. *Z. Wahrsch. Verw. Gebiete.* **60** 501-515.

[10] HEINRICH, L. (1985). Some remarks on asymptotic expansions in the central limit theorem for m-dependent random variables. *Math. Nachr.* **122** 151-155.

[11] HELMERS, R. and VAN ZWET, W. R. (1982). The Berry-Esseen bound for U-statistics. *Statistical Decision Theory and Related Topics, III* (S. S. Gupta and J. O. Berger, eds.) **1** 497-512. Academic Press, New York.

[12] HOEFFDING, W. (1948). A class of statistics with asymptotically normal distributions. *Ann. Math. Statist.* **19** 293-325.

[13] KOKIC, P. N. and WEBER, N. C. (1990). An Edgeworth expansion for U-statistics based on samples from finite populations. *Ann. Probab.* **18** 390-404.

[14] RHEE, W. T. (1988). On asymptotic normality for m-dependent U-statistics. *Internat. J. Math. & Math. Sci.* **11** 187-200.

[15] TIKHOMIROV, A. N. (1980). On the convergence rate in the central limit theorem for weakly dependent random variables. *Theory Probab. Appl.* **25** 790-809.

[16] ZHAO, L. and CHEN, X. (1987). Berry-Esseen bounds for finite-population U-statistics. *Sci. Sinica Ser. A* **30** 113-127.