ESTIMATION OF THE MEAN AND STANDARD DEVIATION OF THE NORMAL DISTRIBUTION BASED ON MULTIPLY TYPE-II CENSORED SAMPLES*

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Technical Report # 91-37C

Department of Statistics
Purdue University

July, 1991

*Research supported in part by NSF Grants DMS–8923071 and DMS–8717799 at Purdue University.

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Keywords and Phrases

Order statistics, multiply Type–II censored samples, maximum likelihood estimators, approximate maximum likelihood estimators, life–times, bias, mean square error, Monte Carlo simulations, best linear unbiased estimators.

Abstract

In this paper, we consider the problem of estimating the mean and standard deviation of a normal population based on multiply Type–II censored samples. We first describe the best linear unbiased estimators and the maximum likelihood estimators of these parameters. Then by noting that the best linear unbiased estimators need the construction of some tables for its coefficients and the maximum likelihood estimators do not exist explicitly and that they need to be determined by numerical methods, we derive approximate maximum likelihood estimators by appropriately approximating the likelihood equations. These estimators, in addition to being explicit in nature, are shown to be almost as efficient as the best linear unbiased estimators and the maximum likelihood estimators. We derive the asymptotic variances and covariance of these estimators. Finally, we present an example to illustrate the methods of estimation discussed in this paper.

1. Introduction

For the normal distribution, the estimation of the mean μ and standard deviation σ based on doubly Type–II censored samples has been considered by several authors for the past forty years or so. By applying the theory of least–squares estimation based on an ordered sample proposed by Lloyd (1952), Sarhan and Greenberg (1956, 1958, 1962) tabulated the best linear unbiased estimators of μ and σ. Gupta (1952) derived best linear unbiased estimators of μ and σ based on singly censored samples for small sample sizes and proposed an alternative linear estimator for large sample sizes. Dixon (1957, 1960) proposed simplified linear estimators of μ and σ based on complete and censored samples which are nearly as efficient as the best linear unbiased estimators. Saw (1959) also derived simplified linear unbiased estimators based on singly censored samples for sample sizes up to twenty. Downton (1966) proposed linear unbiased estimators with polynomial coefficients. Abe (1971a, b) also gave some simplified linear estimators of μ and σ based on doubly censored samples.

Cohen (1950) discussed the maximum likelihood estimation of μ and σ based on singly and doubly censored samples. He (1955, 1959, 1961) then extended these results; but, his discussion is primarily concerned with Type–I censoring (censoring at a pre–fixed time) instead of Type–II censoring (censoring fixed number of items). Gupta (1952) presented likelihood equations for μ and σ based on singly Type–II censored samples and the asymptotic variances and covariance of the maximum likelihood estimators. Some asymptotic properties of these estimators were studied by Halperin (1952) and Breakwell (1953). Plackett (1958) showed that the maximum likelihood estimators of μ and σ are asymptotically linear and that the best linear unbiased estimators are asymptotically normal and efficient. He also proposed a linearized maximum likelihood estimator for σ and compared it with the best linear unbiased estimator based on censored samples for small sample sizes. The bias and mean square error of the maximum likelihood estimators of μ and σ based on singly and doubly Type–II censored samples were studied extensively through Monte Carlo simulations by Isida and Tagami (1959) and Harter and Moore (1966); see also Harter (1970). By modifying the likelihood equations for μ and σ based on doubly Type–II censored samples, Tiku (1967, 1980) derived the modified maximum likelihood estimators of μ and σ. Recently, Balakrishnan (1989) derived approximate maximum likelihood estimators of μ and σ based on doubly Type–II censored samples by using a linear approximation in the likelihood equations which lends itself to possible extensions. Most of these developments are presented in the recent book on this topic by Balakrishnan and Cohen (1990). In this paper, we consider the problem of estimating the mean μ and standard deviation σ of a normal population based on multiply Type–II censored samples.

Consider the normal distribution with probability density function

$$g(y;\, \mu,\, \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(y-\mu)^2/2\sigma^2}, \qquad -\infty < y < \infty, \tag{1.1}$$

and cumulative distribution function $G(y;\, \mu,\, \sigma)$. Let us assume that the following multiply Type–II censored sample from a sample of size n

$$Y_{r_1+1:n} \le \cdots \le Y_{r_1+s_1:n} \le Y_{r_2+1:n} \le \cdots \le Y_{r_2+s_2:n} \le \cdots \le Y_{r_k+1:n} \le \cdots \le Y_{r_k+s_k:n} \tag{1.2}$$

is available from the normal population in (1.1). That is, among the n items placed on a life–test, the smallest r_1, the largest $n-r_k-s_k$, and in addition some middle life–times are assumed to be not observed. In Section 2, we present the best linear unbiased estimators of μ and σ based on the above multiply Type–II censored sample in (1.2). In Section 3, we discuss the maximum likelihood estimation of μ and σ based on the above multiply Type–II censored sample. By noting that the maximum likelihood estimators do not exist in an explicit algebraic form and that they need to be determined by numerically solving the two likelihood equations, we approximate the likelihood equations by making use of some linear approximations and derive in Section 4 the approximate maximum likelihood estimators of μ and σ based on the multiply Type–II censored sample in (1.2). These estimators are simple explicit estimators which turn out to be almost as efficient as the best linear unbiased estimators and the maximum likelihood estimators. In Section 5, we present the asymptotic variances and covariance of the approximate maximum likelihood estimators of μ and σ which work out in terms of the first two single moments and the product moments of standard normal order statistics. In Section 6, we present an example from a life–testing experiment using which we illustrate the methods of estimation of parameters μ and σ discussed in this paper.

2. Best Linear Unbiased Estimation

Let $X_{i:n} = (Y_{i:n} - \mu)/\sigma$, $i = 1, 2, \ldots, n$. Let us denote $E(X_{i:n})$ by $\alpha^*_{i:n}$, $E(X^2_{i:n})$ by $\alpha^{*(2)}_{i:n}$, $Var(X_{i:n})$ by $\beta^*_{i,i:n}$, $E(X_{i:n}\,X_{j:n})$ by $\alpha^*_{i,j:n}$ and $Cov(X_{i:n}, X_{j:n})$ by $\beta^*_{i,j:n}$. Then, we immediately have $E(Y_{i:n}) = \mu + \sigma\alpha^*_{i:n}$, $Var(Y_{i:n}) = \sigma^2\, \beta^*_{i,i:n}$, and $Cov(Y_{i:n}, Y_{j:n}) = \sigma^2\, \beta^*_{i,j:n}$. Let us further denote

$$\underset{\sim}{Y} = \left[Y_{r_1+1:n} \cdots Y_{r_1+s_1:n}\ Y_{r_2+1:n} \cdots Y_{r_2+s_2:n} \cdots Y_{r_k+1:n} \cdots Y_{r_k+s_k:n}\right]^T,$$

$$\underset{\sim}{\alpha} = \left[\alpha^*_{r_1+1:n} \cdots \alpha^*_{r_1+s_1:n}\ \alpha^*_{r_2+1:n} \cdots \alpha^*_{r_2+s_2:n} \cdots \alpha^*_{r_k+1:n} \cdots \alpha^*_{r_k+s_k:n}\right]^T,$$

$$\underset{\sim}{1} = (1\ \ 1\ \ldots\ldots\ 1)^T_{\sum_1^k s_i \times 1},$$

$$\underset{\sim}{\beta} = \left[\left[\beta^*_{i,j:n}\right]\right] \text{ for } i, j \in I \text{ where } I = \{r_1+1,\ldots,r_1+s_1,r_2+1,\ldots,r_2+s_2,\ldots,r_k+1,\ldots,r_k+s_k\},$$

and

$$\underset{\sim}{\Omega} = \underset{\sim}{\beta}^{-1}.$$

Then, the Best Linear Unbiased Estimators of μ and σ based on the multiply Type–II censored sample in (1.2) may be derived by minimizing the generalized variance (see David, 1981; Balakrishnan and Cohen, 1990) given by

$$\left[\underset{\sim}{Y} - \mu\,\underset{\sim}{1} - \sigma\,\underset{\sim}{\alpha}\right]^T \underset{\sim}{\Omega} \left[\underset{\sim}{Y} - \mu\,\underset{\sim}{1} - \sigma\,\underset{\sim}{\alpha}\right]. \tag{2.1}$$

The best linear unbiased estimators of μ and σ obtained by minimizing the generalized variance in (2.1) are given by

$$\mu^* = \left\{ \frac{\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{\alpha}\, \underset{\sim}{1}^T \underset{\sim}{\Omega} - \underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{1}\, \underset{\sim}{\alpha}^T \underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{\alpha})\,(\underset{\sim}{1}^T \underset{\sim}{\Omega}\, \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{1})^2} \right\} \underset{\sim}{Y}$$

$$= - \underset{\sim}{\alpha}^T \underset{\sim}{\Delta}\, \underset{\sim}{Y}$$

$$= \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} a_j Y_{j:n} \tag{2.2}$$

and

$$\sigma^* = \left\{ \frac{\underset{\sim}{1}^T \underset{\sim}{\Omega}\, \underset{\sim}{1}\, \underset{\sim}{\alpha}^T \underset{\sim}{\Omega} - \underset{\sim}{1}^T \underset{\sim}{\Omega}\, \underset{\sim}{\alpha}\, \underset{\sim}{1}^T \underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{\alpha})\,(\underset{\sim}{1}^T \underset{\sim}{\Omega}\, \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{1})^2} \right\} \underset{\sim}{Y}$$

$$= \underset{\sim}{1}^T \underset{\sim}{\Delta}\, \underset{\sim}{Y}$$

$$= \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} b_j Y_{j:n}, \tag{2.3}$$

where $\underset{\sim}{\Delta}$ is a skew–symmetric matrix of order $\sum_{i=1}^{k} s_i$ given by

$$\underset{\sim}{\Delta} = \frac{\underset{\sim}{\Omega}(\underset{\sim}{1}\, \underset{\sim}{\alpha}^T - \underset{\sim}{\alpha}\, \underset{\sim}{1}^T)\, \underset{\sim}{\Omega}}{(\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{\alpha})\,(\underset{\sim}{1}^T \underset{\sim}{\Omega}\, \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \underset{\sim}{\Omega}\, \underset{\sim}{1})^2}. \tag{2.4}$$

The variances and covariance of the estimators μ^* and σ^* (see David, 1981; Balakrishnan and Cohen, 1990) are given by

$$\mathrm{Var}(\mu^*) = \sigma^2 \left\{ \frac{\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{\alpha}}{(\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{\alpha}) \, (\underset{\sim}{1}^T \, \Omega \, \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{1})^2} \right\}, \tag{2.5}$$

$$\mathrm{Var}(\sigma^*) = \sigma^2 \left\{ \frac{\underset{\sim}{1}^T \, \Omega \, \underset{\sim}{1}}{(\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{\alpha}) \, (\underset{\sim}{1}^T \, \Omega \, \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{1})^2} \right\}, \tag{2.6}$$

and

$$\mathrm{Cov}(\mu^*, \sigma^*) = -\,\sigma^2 \left\{ \frac{\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{1}}{(\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{\alpha}) \, (\underset{\sim}{1}^T \, \Omega \, \underset{\sim}{1}) - (\underset{\sim}{\alpha}^T \, \Omega \, \underset{\sim}{1})^2} \right\}. \tag{2.7}$$

By using the values of means, variances and covariances of standard normal order statistics tabulated by Tietjen, Kahaner and Beckman (1977) for sample sizes up to fifty, we may determine the coefficients a_j and b_j in Eqs. (2.2) and (2.3) and also the variances and covariance of the best linear unbiased estimators from Eqs. (2.5), (2.6) and (2.7), respectively. For sample sizes larger than fifty, we may determine these quantities approximately by using approximate expressions of means, variances and covariances of standard normal order statistics derived by David and Johnson's (1954) method; see, for example, David (1981) and Arnold and Balakrishnan (1989).

3. Maximum Likelihood Estimation

With $X_{i:n} = (Y_{i:n} - \mu)/\sigma$, we have the likelihood function based on the multiply Type–II censored sample in (1.2) to be

$$L = \frac{n!}{\prod_{i=1}^{k+1}\left[r_i - r_{i-1} - s_{i-1}\right]!\,\sigma^{\sum_1^k s_i}} \left\{F\left[X_{r_1+1:n}\right]\right\}^{r_1}$$

$$\times \prod_{i=2}^{k}\left\{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]\right\}^{r_i-r_{i-1}-s_{i-1}}$$

$$\times \left\{1 - F\left[X_{r_k+s_k:n}\right]\right\}^{n-r_k-s_k} \prod_{i=1}^{k}\prod_{j=r_i+1}^{r_i+s_i} f\left[X_{j:n}\right], \qquad (3.1)$$

where $f(x)$ denotes the standard normal density function, $F(x)$ denotes the standard normal cumulative distribution function, $r_0 = s_0 = 0$, and $r_{k+1} = n$. From (3.1), we have the log–likelihood function to be

$$\ell n\, L = \text{Const} - A\, \ell n\, \sigma + r_1\, \ell n\left\{F\left[X_{r_1+1:n}\right]\right\}$$

$$+ \sum_{i=2}^{k}\left[r_i - r_{i-1} - s_{i-1}\right] \ell n\left\{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]\right\}$$

$$+ (n-r_k-s_k)\, \ell n\left\{1 - F\left[X_{r_k+s_k:n}\right]\right\} - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} X_{j:n}^2, \qquad (3.2)$$

where $A = \sum_{i=1}^{k} s_i$ is the size of the available multiply Type–II censored sample. From Eq. (3.2), we obtain the likelihood equations for μ and σ to be

$$\frac{\partial \ell nL}{\partial \mu} = -\frac{1}{\sigma}\left[r_1 \frac{f\left[X_{r_1+1:n}\right]}{F\left[X_{r_1+1:n}\right]} + \sum_{i=2}^{k}\left[r_i - r_{i-1} - s_{i-1}\right] \frac{f\left[X_{r_i+1:n}\right] - f\left[X_{r_{i-1}+s_{i-1}:n}\right]}{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]}\right.$$

$$\left. - (n - r_k - s_k)\frac{f\left[X_{r_k+s_k:n}\right]}{1 - F\left[X_{r_k+s_k:n}\right]} - \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} X_{j:n}\right]$$

$$= 0, \tag{3.3}$$

and

$$\frac{\partial \ell nL}{\partial \sigma} = -\frac{1}{\sigma}\left[A + r_1 X_{r_1+1:n}\frac{f\left[X_{r_1+1:n}\right]}{F\left[X_{r_1+1:n}\right]}\right.$$

$$+ \sum_{i=2}^{k}\left[r_i - r_{i-1} - s_{i-1}\right] \frac{X_{r_i+1:n}\, f\left[X_{r_i+1:n}\right] - X_{r_{i-1}+s_{i-1}:n}\, f\left[X_{r_{i-1}+s_{i-1}:n}\right]}{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]}$$

$$\left. - (n - r_k - s_k) X_{r_k+s_k:n}\frac{f\left[X_{r_k+s_k:n}\right]}{1 - F\left[X_{r_k+s_k:n}\right]} - \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i} X_{j:n}^2\right]$$

$$= 0. \tag{3.4}$$

Eqs. (3.3) and (3.4) do not admit explicit solutions. But, the maximum likelihood estimates of μ and σ may be determined from Eqs. (3.3) and (3.4) by solving them using numerical methods.

4. Approximate Maximum Likelihood Estimation

Let $p_i = i/(n+1)$, $q_i = 1 - p_i$, and $\xi_i = F^{-1}(p_i)$; further, let

$$h_1\left[X_{r_1+1:n}\right] = \frac{f\left[X_{r_1+1:n}\right]}{F\left[X_{r_1+1:n}\right]} \tag{4.1}$$

and

$$h_2\left[X_{r_k+s_k:n}\right] = \frac{f\left[X_{r_k+s_k:n}\right]}{1 - F\left[X_{r_k+s_k:n}\right]} . \tag{4.2}$$

By expanding the functions $h_1\left[X_{r_1+1:n}\right]$ and $h_2\left[X_{r_k+s_k:n}\right]$ in (4.1) and (4.2) around the points ξ_{r_1+1} and $\xi_{r_k+s_k}$ in Taylor series (see David (1981) or Arnold and Balakrishnan (1989) for reasoning), respectively, we may then approximate them by

$$h_1\left[X_{r_1+1:n}\right] = \frac{f\left[X_{r_1+1:n}\right]}{F\left[X_{r_1+1:n}\right]} \simeq \alpha_1 - \beta_1 X_{r_1+1:n} \tag{4.3}$$

and

$$h_2\left[X_{r_k+s_k:n}\right] = \frac{f\left[X_{r_k+s_k:n}\right]}{1 - F\left[X_{r_k+s_k:n}\right]} \simeq \alpha_2 + \beta_2 X_{r_k+s_k:n}, \tag{4.4}$$

where

$$\alpha_1 = f(\xi_{r_1+1}) \left\{1 + \xi^2_{r_1+1} + \xi_{r_1+1} f(\xi_{r_1+1}) / p_{r_1+1}\right\} / p_{r_1+1}, \quad (4.5)$$

$$\beta_1 = f(\xi_{r_1+1}) \left\{f(\xi_{r_1+1}) + p_{r_1+1}\, \xi_{r_1+1}\right\} / p^2_{r_1+1}, \quad (4.6)$$

$$\alpha_2 = f\left[\xi_{r_k+s_k}\right] \left\{1 + \xi^2_{r_k+s_k} - \xi_{r_k+s_k} f\left[\xi_{r_k+s_k}\right]/q_{r_k+s_k}\right\}/q_{r_k+s_k}, \quad (4.7)$$

and

$$\beta_2 = f\left[\xi_{r_k+s_k}\right] \left\{f\left[\xi_{r_k+s_k}\right] - q_{r_k+s_k}\, \xi_{r_k+s_k}\right\} / q^2_{r_k+s_k}. \quad (4.8)$$

From Eqs. (4.6) and (4.8), it can be shown that both β_1 and β_2 are positive. For example, we see easily from (4.6) that $\beta_1 > 0$ whenever $p_{r_1+1} \geq 1/2$. Also when $p_{r_1+1} < 1/2$, we have $\xi_{r_1+1} < 0$ and

$$|\xi_{r_1+1}\, p_{r_1+1}| = |\xi_{r_1+1}\, F(\xi_{r_1+1})| = -\xi_{r_1+1} \int_{-\infty}^{\xi_{r_1+1}} f(x)\, dx$$

$$\leq \int_{-\infty}^{\xi_{r_1+1}} -x\, f(x) dx = f\left[\xi_{r_1+1}\right]$$

by realizing that $f'(x) = -x\, f(x)$, and consequently $\beta > 0$. Now let,

$$k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] = \frac{f\left[X_{r_i+1:n}\right]}{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]} \tag{4.9}$$

and

$$k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] = \frac{f\left[X_{r_{i-1}+s_{i-1}:n}\right]}{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]}. \tag{4.10}$$

By expanding the functions $k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$ and $k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right]$ in (4.9) and (4.10) around the point $\left[\xi_{r_{i-1}+s_{i-1}}, \xi_{r_i+1}\right]$ in bivariate Taylor series, respectively, we may then approximate them by

$$k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \simeq \gamma_{0i} + \gamma_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \gamma_{2i}\, X_{r_i+1:n} \tag{4.11}$$

and

$$k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \simeq \delta_{0i} + \delta_{1i}\, X_{r_{i-1}+s_{i-1}:n} - \delta_{2i}\, X_{r_i+1:n}, \tag{4.12}$$

where

$$\gamma_{1i} = f\left[\xi_{r_{i-1}+s_{i-1}}\right] f\left[\xi_{r_i+1}\right] / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2, \tag{4.13}$$

$$\gamma_{2i} = f\left[\xi_{r_i+1}\right] \left\{f\left[\xi_{r_i+1}\right] + \xi_{r_i+1}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]\right\} / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2, \tag{4.14}$$

$$\gamma_{0i} = \gamma_{2i}\, \xi_{r_i+1} - \gamma_{1i}\, \xi_{r_{i-1}+s_{i-1}} + f\left[\xi_{r_i+1}\right] / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right], \tag{4.15}$$

$$\delta_{1i} = f\left[\xi_{r_{i-1}+s_{i-1}}\right] \left\{f\left[\xi_{r_{i-1}+s_{i-1}}\right] - \xi_{r_{i-1}+s_{i-1}}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]\right\} / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2, \tag{4.16}$$

$$\delta_{2i} = \gamma_{1i} = f\left[\xi_{r_{i-1}+s_{i-1}}\right] f\left[\xi_{r_i+1}\right] / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2, \tag{4.17}$$

and

$$\delta_{0i} = \delta_{2i}\, \xi_{r_i+1} - \delta_{1i}\, \xi_{r_{i-1}+s_{i-1}} + f\left[\xi_{r_{i-1}+s_{i-1}}\right] / \left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]. \tag{4.18}$$

It is readily seen from Eqs. (4.13) and (4.17) that $\gamma_{1i} = \delta_{2i}$ is positive. From Eqs. (4.14) and (4.16), it can be shown that both γ_{2i} and δ_{1i} are positive. For example, we see easily from (4.14) that $\gamma_{2i} > 0$ whenever $p_{r_i+1} \geq 1/2$. Also when $p_{r_i+1} < 1/2$, we have $\xi_{r_i+1} < 0$ and

$$
\begin{aligned}
\left|\xi_{r_i+1}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]\right| &= \left|\xi_{r_i+1}\left\{F\left[\xi_{r_i+1}\right] - F\left[\xi_{r_{i-1}+s_{i-1}}\right]\right\}\right| \\
&= -\,\xi_{r_i+1} \int_{\xi_{r_{i-1}+s_{i-1}}}^{\xi_{r_i+1}} f(x)\,dx \\
&\le \int_{\xi_{r_{i-1}+s_{i-1}}}^{\xi_{r_i+1}} -x\,f(x)dx = f\left[\xi_{r_i+1}\right] - f\left[\xi_{r_{i-1}+s_{i-1}}\right]
\end{aligned}
\tag{4.19}
$$

by using the fact that $f'(x) = -\,x\,f(x)$, and consequently $\gamma_{2i} > 0$. By making use of the approximations in (4.11) and (4.12), we obtain

$$
\begin{aligned}
k\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] &= k_1\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] - k_2\left[X_{r_{i-1}+s_{i-1}:n}, X_{r_i+1:n}\right] \\
&= \frac{f\left[X_{r_i+1:n}\right] - f\left[X_{r_{i-1}+s_{i-1}:n}\right]}{F\left[X_{r_i+1:n}\right] - F\left[X_{r_{i-1}+s_{i-1}:n}\right]} \\
&\simeq \eta_{0i} - \eta_{1i}\,X_{r_{i-1}+s_{i-1}:n} - \eta_{2i}\,X_{r_i+1:n},
\end{aligned}
\tag{4.20}
$$

where

$$\eta_{2i} = \gamma_{2i} - \delta_{2i}$$

$$= \frac{f\left[\xi_{r_i+1}\right]}{\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2}\left\{\xi_{r_i+1}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] + f\left[\xi_{r_i+1}\right] - f\left[\xi_{r_{i-1}+s_{i-1}}\right]\right\}, \tag{4.21}$$

$$\eta_{1i} = \delta_{1i} - \gamma_{1i}$$

$$= \frac{f\left[\xi_{r_{i-1}+s_{i-1}}\right]}{\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]^2}\left\{f\left[\xi_{r_{i-1}+s_{i-1}}\right] - f\left[\xi_{r_i+1}\right]\right.$$

$$\left. - \xi_{r_{i-1}+s_{i-1}}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]\right\}, \tag{4.22}$$

and

$$\eta_{0i} = \gamma_{0i} - \delta_{0i}$$

$$= \eta_{2i}\,\xi_{r_i+1} + \eta_{1i}\,\xi_{r_{i-1}+s_{i-1}} + \left\{f\left[\xi_{r_i+1}\right] - f\left[\xi_{r_{i-1}+s_{i-1}}\right]\right\}/\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]. \tag{4.23}$$

From Eqs. (4.21) and (4.22), it can be shown that both η_{2i} and η_{1i} are positive. For example, we show below that $\eta_{2i} > 0$. When $p_{r_i+1} \geq 1/2$, we have $\xi_{r_i+1} > 0$ and

$$\xi_{r_i+1}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right] = \xi_{r_i+1}\left[F(\xi_{r_i+1}) - F(\xi_{r_{i-1}+s_{i-1}})\right]$$

$$= \xi_{r_i+1} \int_{\xi_{r_{i-1}+s_{i-1}}}^{\xi_{r_i+1}} f(x)\, dx$$

$$\geq \int_{\xi_{r_{i-1}+s_{i-1}}}^{\xi_{r_i+1}} x\, f(x)\, dx = f\left[\xi_{r_{i-1}+s_{i-1}}\right] - f\left[\xi_{r_i+1}\right]$$

and consequently $\eta_{2i} > 0$. Similarly, when $p_{r_i+1} < \frac{1}{2}$ we have $\xi_{r_i+1} < 0$ and

$$\left|\xi_{r_i+1}\left[p_{r_i+1} - p_{r_{i-1}+s_{i-1}}\right]\right| = -\xi_{r_i+1}\left[F(\xi_{r_i+1}) - F(\xi_{r_{i-1}+s_{i-1}})\right]$$

$$= -\xi_{r_i+1} \int_{\xi_{r_{i-1}+s_{i-1}}}^{\xi_{r_i+1}} f(x)\, dx$$

$$\leq \int_{\xi_{r_{i-1}+s_{i-1}}}^{\xi_{r_i+1}} -x\, f(x)\, dx = f\left[\xi_{r_i+1}\right] - f\left[\xi_{r_{i-1}+s_{i-1}}\right]$$

and consequently $\eta_{2i} > 0$. Proceeding similarly, it can also be shown that $\eta_{1i} > 0$.

Now, upon using the approximations in Eqs. (4.3), (4.4) and (4.20) into the likelihood equation for μ in (3.3), we obtain the approximate likelihood equation for μ to be

$$\sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} X_{j:n} + (n-r_k-s_k)\left[\alpha_2 + \beta_2 X_{r_k+s_k:n}\right]$$

$$- \sum_{i=2}^{k} t_i\left[\eta_{0i} - \eta_{1i} X_{r_{i-1}+s_{i-1}:n} - \eta_{2i} X_{r_i+1:n}\right]$$

$$- r_1\left[\alpha_1 - \beta_1 X_{r_1+1:n}\right] = 0, \tag{4.24}$$

which when solved for μ yields the approximate maximum likelihood estimator of μ to be

$$\hat{\mu} = B - \sigma C, \tag{4.25}$$

where

$$t_i = r_i - r_{i-1} - s_{i-1}, \qquad i = 2, 3, ..., k,$$

$$A = \sum_{i=1}^{k} s_i,$$

$$m = r_1\beta_1 + \sum_{i=2}^{k} t_i \eta_{1i} + \sum_{i=2}^{k} t_i \eta_{2i} + (n-r_k-s_k)\beta_2 + A,$$

$$B = \frac{1}{m}\Big\{r_1 \beta_1 Y_{r_1+1:n} + \sum_{i=2}^{k} t_i \eta_{1i} Y_{r_{i-1}+s_{i-1}:n} + \sum_{i=2}^{k} t_i \eta_{2i} Y_{r_i+1:n}$$

$$+ (n-r_k-s_k) \beta_2 Y_{r_k+s_k:n} + \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} Y_{j:n}\Big\},$$

and

$$C = \frac{1}{m}\Big\{r_1 \alpha_1 + \sum_{i=2}^{k} t_i \eta_{0i} - (n-r_k-s_k)\alpha_2\Big\}. \tag{4.26}$$

Next, upon using the approximations in Eqs. (4.3), (4.4), (4.11) and (4.12) into the likelihood equation for σ in (3.4), we obtain the approximate likelihood equation for σ to be

$$A + r_1 X_{r_1+1:n} \Big[\alpha_1 - \beta_1 X_{r_1+1:n}\Big]$$

$$+ \sum_{i=2}^{k} t_i X_{r_i+1:n} \Big[\gamma_{0i} + \gamma_{1i} X_{r_{i-1}+s_{i-1}:n} - \gamma_{2i} X_{r_i+1:n}\Big]$$

$$- \sum_{i=2}^{k} t_i X_{r_{i-1}+s_{i-1}:n} \Big[\delta_{0i} + \delta_{1i} X_{r_{i-1}+s_{i-1}:n} - \delta_{2i} X_{r_i+1:n}\Big]$$

$$- (n-r_k-s_k) X_{r_k+s_k:n} \Big[\alpha_2 + \beta_2 X_{r_k+s_k:n}\Big] - \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} X_{j:n}^2$$

$$= 0, \tag{4.27}$$

which when solved for σ (simultaneously, by using the solution for μ in (4.25)) yields the approximate maximum likelihood estimator of σ to be

$$\hat{\sigma} = \left\{- D + (D^2+4AE)^{1/2}\right\}/2A, \tag{4.28}$$

where

$$A = \sum_{i=1}^{k} s_i \text{ as before,}$$

$$D = r_1 \alpha_1 Y_{r_1+1:n} + \sum_{i=2}^{k} t_i \gamma_{0i} Y_{r_i+1:n} - \sum_{i=2}^{k} t_i \delta_{0i} Y_{r_{i-1}+s_{i-1}:n}$$

$$- (n-r_k-s_k) \alpha_2 Y_{r_k+s_k:n} - mBC,$$

and

$$E = r_1 \beta_1 Y^2_{r_1+1:n} + \sum_{i=2}^{k} t_i \eta_{1i} Y^2_{r_{i-1}+s_{i-1}:n} + \sum_{i=2}^{k} t_i \eta_{2i} Y^2_{r_i+1:n}$$

$$+ (n-r_k-s_k) \beta_2 Y^2_{r_k+s_k:n} + \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} Y^2_{j:n}$$

$$+ \sum_{i=2}^{k} t_i \gamma_{1i} \left[Y_{r_i+1:n} - Y_{r_{i-1}+s_{i-1}:n}\right]^2 - mB^2$$

$$= r_1\, \beta_1 \left[Y_{r_1+1:n} - B \right]^2 + \sum_{i=2}^{k} t_i\, \eta_{1i} \left[Y_{r_{i-1}+s_{i-1}:n} - B \right]^2$$

$$+ \sum_{i=2}^{k} t_i\, \eta_{2i} \left[Y_{r_i+1:n} - B \right]^2 + (n-r_k-s_k)\, \beta_2 \left[Y_{r_k+s_k:n} - B \right]^2$$

$$+ \sum_{i=1}^{k} \sum_{j=r_i+1}^{r_i+s_i} \left[Y_{j:n} - B \right]^2 + \sum_{i=2}^{k} t_i\, \gamma_{1i} \left[Y_{r_i+1:n} - Y_{r_{i-1}+s_{i-1}:n} \right]^2 . \tag{4.29}$$

It is important to mention here that upon solving Eq. (4.27) we obtain a quadratic equation in σ which has two roots; however, one of them becomes negative and hence inadmissible since $\beta_1, \beta_2, \eta_{1i}, \eta_{2i}$ and γ_{1i} are all positive and consequently $E > 0$.

<u>Remark 1</u>: For the special case when $s_1 = s_2 = \ldots = s_{k-1} = 1$, $r_1 = r$, $r_2 = r+1, \ldots,$ $r_k = r+k-1$ and $s_k = n-r-s-k+1$, then the available sample simply becomes a Type–II censored sample $Y_{r+1:n}, Y_{r+2:n}, \ldots, Y_{n-s:n}$, where the smallest r and the largest s observations have been censored. In this case, the estimators $\hat{\mu}$ and $\hat{\sigma}$ in Eqs. (4.25) and (4.28), respectively, simply reduce to the approximate maximum likelihood estimators of μ and σ derived by Balakrishnan (1989).

<u>Remark 2</u>: For the special case when the available multiply Type–II censored sample in (1.2) is symmetric (that is, if $Y_{i:n}$ is available then so also is $Y_{n-i+1:n}$), it can be shown from Eq. (4.26) that $C = 0$. As a result, the estimator $\hat{\mu}$ in (4.25) simply becomes

$$\hat{\mu} = B$$

which is just a linear function of the available order statistics with equal weights for the symmetric order statistics. Due to the symmetry of the standard normal distribution and hence the relation $E\left[X_{i:n}\right] = -E\left[X_{n-i+1:n}\right]$ (see David (1981) or Arnold and Balakrishnan (1989), for example), it is easy to show that the above estimator $\hat{\mu}$ is unbiased for μ.

5. Approximate Variances and Covariance of the Estimators

By using the linear approximations in (4.3), (4.4), (4.11), (4.12) and (4.20), we also obtain from the likelihood equations for μ and σ in Eqs. (3.3) and (3.4) that

$$E\left[-\frac{\partial^2 \ln L}{\partial\mu^2}\right] \simeq \frac{m}{\sigma^2}, \tag{5.1}$$

$$E\left[-\frac{\partial^2 \ln L}{\partial\mu\,\partial\sigma}\right] \simeq \frac{m}{\sigma^2} V_1, \tag{5.2}$$

and

$$E\left[-\frac{\partial^2 \ln L}{\partial\sigma^2}\right] \simeq \frac{m}{\sigma^2} V_2, \tag{5.3}$$

where, as before,

$$m = \sum_{i=1}^{k} s_i + r_1\beta_1 + \sum_{i=2}^{k} t_i\,\eta_{1i} + \sum_{i=2}^{k} t_i\,\eta_{2i} + (n-r_k-s_k)\,\beta_2, \tag{5.4}$$

and

$$V_1 = \frac{2}{m}\Big\{r_1\,\beta_1\,\alpha^*_{r_1+1:n} + \sum_{i=2}^{k} t_i\,\eta_{1i}\,\alpha^*_{r_{i-1}+s_{i-1}:n} + \sum_{i=2}^{k} t_i\,\eta_{2i}\,\alpha^*_{r_i+1:n}$$

$$+ (n-r_k-s_k)\,\beta_2\,\alpha^*_{r_k+s_k:n} + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i}\alpha^*_{j:n}\Big\} - C, \qquad (5.5)$$

and

$$V_2 = \frac{3}{m}\Big\{r_1\,\beta_1\,\alpha^{*(2)}_{r_1+1:n} + \sum_{i=2}^{k} t_i\,\gamma_{2i}\,\alpha^{*(2)}_{r_i+1:n} + \sum_{i=2}^{k} t_i\,\delta_{1i}\,\alpha^{*(2)}_{r_{i-1}+s_{i-1}:n}$$

$$+ (n-r_k-s_k)\,\beta_2\,\alpha^{*(2)}_{r_k+s_k:n} + \sum_{i=1}^{k}\sum_{j=r_i+1}^{r_i+s_i}\alpha^{*(2)}_{j:n}$$

$$- 2\sum_{i=2}^{k} t_i\,\gamma_{1i}\,\alpha^*_{r_{i-1}+s_{i-1},r_i+1:n}\Big\}$$

$$- \frac{2}{m}\Big\{r_1\alpha_1\,\alpha^*_{r_1+1:n} + \sum_{i=2}^{k} t_i\,\gamma_{0i}\,\alpha^*_{r_i+1:n}$$

$$- \sum_{i=2}^{k} t_i\,\delta_{0i}\,\alpha^*_{r_{i-1}+s_{i-1}:n} - (n-r_k-s_k)\,\alpha_2\,\alpha^*_{r_k+s_k:n}\Big\}$$

$$- \frac{A}{m}. \qquad (5.6)$$

In the above formulae, $\alpha^*_{i:n}$, $\alpha^{*(2)}_{i:n}$ and $\alpha^*_{i,j:n}$ denote $E(X_{i:n})$, $E(X^2_{i:n})$ and $E(X_{i:n}\,X_{j:n})$, respectively, where $X_{i:n}$ is the i^{th} order statistic in a sample of size n from the standard normal distribution. From these expressions, we may compute

$$\mathrm{Var}(\hat{\mu}) \simeq \frac{\sigma^2}{m}\left\{\frac{V_2}{V_2-V_1^2}\right\}, \tag{5.7}$$

$$\mathrm{Var}(\hat{\sigma}) \simeq \frac{\sigma^2}{m}\left\{\frac{1}{V_2-V_1^2}\right\}, \tag{5.8}$$

and

$$\mathrm{Cov}(\hat{\mu}, \hat{\sigma}) \simeq -\frac{\sigma^2}{m}\left\{\frac{V_1}{V_2-V_1^2}\right\}; \tag{5.9}$$

see, for example, Kendall and Stuart (1973) or Rao (1975).

Approximate variances and covariance of the estimators $\hat{\mu}$ and $\hat{\sigma}$ may be computed from Eqs. (5.7) – (5.9) either by directly using the tables of means, variances and covariances of standard normal order statistics prepared by Tietjen, Kahaner and Beckman (1977) for sample sizes up to fifty or by using approximations of means, variances and covariances of standard normal order statistics presented by David (1981) and Arnold and Balakrishnan (1989).

The asymptotic distribution of the estimators $\hat{\mu}$ and $\hat{\sigma}$ is presented in the following theorem.

<u>Theorem 1</u>: Asymptotically, $\hat{\mu}$ and $\hat{\sigma}$ jointly have a bivariate normal distribution with mean vector $\begin{bmatrix} \mu \\ \sigma \end{bmatrix}$ and variance–covariance matrix

$$\frac{\sigma^2}{m\left[V_2-V_1^2\right]}\begin{bmatrix} V_2 & -V_1 \\ -V_1 & 1 \end{bmatrix},$$

where m, V_1 and V_2 are as given in Eqs. (5.4), (5.5) and (5.6), respectively.

For a proof, one may refer to Kendall and Stuart (1973) or Rao (1975).

Remark 3: For the special case when the available multiply Type–II censored sample in (1.2) is symmetric (that is, if $Y_{i:n}$ is available then so also is $Y_{n-i+1:n}$), by using the facts that $\alpha^*_{i:n} = -\alpha^*_{n-i+1:n}$ and $C = 0$, it can be very easily shown from Eq. (5.5) that $V_1 = 0$. As a result, we have the estimators $\hat{\mu}$ and $\hat{\sigma}$ to be uncorrelated in this case. Furthermore, we obtain from Eqs. (5.7) and (5.8) that

$$\text{Var}(\hat{\mu}) \simeq \frac{\sigma^2}{m} \quad \text{and} \quad \text{Var}(\hat{\sigma}) \simeq \frac{\sigma^2}{mV_2}.$$

6. Illustrative Example

Let us consider the following data on lifetimes (in hours) of 20 electronic units that were placed on a test:

– ,	– ,	128.887,	132.585,	133.196,	140.734,	141.816,
146.864,	148.350,	– ,	– ,	154.671,	159.188,	163.117,
166.252,	166.770,	172.017,	174.744,	– ,	–	

The first two units failed before the measurement started, the central two observations are censored as the failure times of those two units were not recorded due to experimental problems, and the experiment was stopped as soon as the eighteenth unit failed resulting in the censoring of the last two observations.

By assuming that the above given multiply Type–II censored sample has come from a normal $N(\mu, \sigma^2)$ population, we shall estimate the unknown parameters μ and σ and also construct approximate confidence intervals for them.

For the approximate maximum likelihood estimation developed in this paper, we have:

$n = 20,$

$r_1 = 2,\ s_1 = 7,\ r_2 = 11,\ s_2 = 7,$

$t_2 = 2,$

$A = s_1 + s_2 = 14,$

i	p_i	ξ_i	$f(\xi_i)$
3	0.1429	– 1.0676	0.2256
9	0.4286	– 0.1800	0.3925
12	0.5714	0.1800	0.3925
18	0.8571	1.0676	0.2256

$$\alpha_1 = \frac{0.2256}{0.1429}\left\{1 + (-1.0676)^2 - 1.0676\left[\frac{0.2256}{0.1429}\right]\right\} = 0.7172,$$

$$\beta_1 = \frac{0.2256}{(0.1429)^2}\left\{0.2256 - 1.0676(0.1429)\right\} = 0.8069,$$

$$\alpha_2 = \frac{0.2256}{0.1429}\left\{1 + (1.0676)^2 - 1.0676\left[\frac{0.2256}{0.1429}\right]\right\} = 0.7172,$$

$$\beta_2 = \frac{0.2256}{(0.1429)^2}\left\{0.2256 - 1.0676(0.1429)\right\} = 0.8069,$$

$$\gamma_{12} = (0.3925)^2/(0.5714 - 0.4286)^2 = 7.5548,$$

$$\gamma_{22} = \frac{0.3925}{(0.5714 - 0.4286)^2}\left\{0.3925 + 0.1800(0.5714 - 0.4286)\right\} = 8.0495,$$

$$\gamma_{02} = 8.0495\ (0.1800) + 7.5548\ (0.1800) + \frac{0.3925}{0.5714 - 0.4286} = 5.5574,$$

$$\delta_{12} = \frac{0.3925}{(0.5714 - 0.4286)^2}\left\{0.3925 + 0.1800\ (0.5714-0.4286)\right\} = 8.0495,$$

$$\delta_{22} = \gamma_{12} = 7.5548,$$

$$\delta_{02} = 7.5548\ (0.1800) + 8.0495\ (0.1800) + \frac{0.3925}{0.5714 - 0.4286} = 5.5574,$$

$$\eta_{22} = \gamma_{22} - \delta_{22} = 8.0495 - 7.5548 = 0.4947,$$

$$\eta_{12} = \delta_{12} - \gamma_{12} = 8.0495 - 7.5548 = 0.4947,$$

$$\eta_{02} = \gamma_{02} - \delta_{02} = 5.5574 - 5.5574 = 0,$$

$$m = 2(0.8069) + 2(0.4947 + 0.4947) + 2(0.8069) + 14 = 19.2064,$$

$$B = 2918.9997/19.2064 = 151.9806,$$

$$C = \{1.4344 - 1.4344\}\ /\ 19.2064 = 0,$$

$$D = 4.4794,$$

$$E = 5377.4058,$$

and hence

$$\hat{\sigma} = \left\{- D + (D^2 + 4AE)^{1/2}\right\} / 2A = 19.4392$$

and

$$\hat{\mu} = B - \hat{\sigma}C = B = 151.9806.$$

Also, from Eqs. (5.5) and (5.6) we have

$$V_1 = 0 \quad \text{and} \quad V_2 = 1.5945$$

so that we have the approximate standard errors of the estimates $\hat{\mu}$ and $\hat{\sigma}$ to be

$$SE(\hat{\mu}) = \hat{\sigma}/\sqrt{m} = 19.4392/(19.2064)^{1/2} = 4.4356$$

and

$$SE(\hat{\sigma}) = \hat{\sigma}/(mV_2)^{1/2} = 19.4392/(19.2064 \times 1.5945)^{1/2} = 3.5127$$

By using the asymptotic normality of the estimators $\hat{\mu}$ and $\hat{\sigma}$ (see Theorem 1), we now obtain 95% confidence intervals for μ and σ to be

$$[151.9806 - 1.96\ (4.4356),\ 151.9806 + 1.96\ (4.4356)] = [143.2868,\ 160.6744]$$

and

$$[19.4392 - 1.96\ (3.5127),\ 19.4392 + 1.96\ (3.5127)] = [12.5543,\ 26.3241],$$

respectively.

By using the results presented in Section 2 and making use of the tables of means, variances and covariances of normal order statistics prepared by Tietjen, Kahaner and Beckman (1977), we find the best linear unbiased estimates of μ and σ to be

$$\begin{aligned}\mu^* &= 0.1374\ (128.887) + 0.0517\ (132.585) + 0.0518\ (133.196) + 0.0519\ (140.734)\\ &\quad + 0.0519\ (141.816) + 0.0520\ (146.864) + 0.1033\ (148.350) + 0.1033\ (154.671)\\ &\quad + 0.0520\ (159.188) + 0.0519\ (163.117) + 0.0519\ (166.252) + 0.0518\ (166.770)\\ &\quad + 0.0517\ (172.017) + 0.1374\ (174.744)\\ &= 151.9804\end{aligned}$$

and

$$\begin{aligned}\sigma^* &= -\ 0.3025\ (128.887) - 0.0694\ (132.585) - 0.0563\ (133.196) - 0.0446\ (140.734)\\ &\quad - 0.0339\ (141.816) - 0.0239\ (146.864) - 0.0157\ (148.350) + 0.0157\ (154.671)\\ &\quad + 0.0239\ (159.188) + 0.0339\ (163.117) + 0.0446\ (166.252) + 0.0563\ (166.770)\\ &\quad + 0.0694\ (172.017) + 0.3025\ (174.744)\\ &= 20.7525\end{aligned}$$

and the standard errors of the estimates μ^* and σ^* to be

$$SE(\mu^*) = \sigma^*(0.0520)^{1/2} = 20.7525\ (0.0520)^{1/2} = 4.7323$$

and

$$SE(\sigma^*) = \sigma^*(0.0380)^{1/2} = 20.7525\ (0.0380)^{1/2} = 4.0454.$$

Making use of the asymptotic normality of the best linear unbiased estimators (since they are linear functions of order statistics), we obtain approximate 95% confidence intervals for μ and σ to be

$$[151.9804 - 1.96(4.7323),\ 151.9804 + 1.96(4.7323)] = [142.7051,\ 161.2557]$$

and

$$[20.7525 - 1.96(4.0454),\ 20.7525 + 1.96(4.0454)] = [12.8235,\ 28.6815],$$

respectively.

Upon comparing the results based on the two methods, we observe that the best linear unbiased estimates of μ and σ are numerically close to the approximate maximum likelihood estimates of μ and σ. But, the best linear unbiased estimates have slightly larger standard errors than the corresponding approximate maximum likelihood estimates and, consequently, the confidence intervals based on the best linear unbiased estimates turn out to be slightly wider than the corresponding confidence intervals based on the approximate maximum likelihood estimates.

Acknowledgements

The first author would like to thank the Natural Sciences and Engineering Research Council of Canada while the second author would like to thank the National Science Foundation for funding this research. The authors would also like to thank Mrs. Edna Pathmanathan for the excellent typing of the manuscript.

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