HAUSDORFF AND BOX DIMENSIONS OF CERTAIN SELF-AFFINE FRACTALS

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Abstract

The Hausdorff-Besicovich and Bouligand-Minkowski (box) dimensions δ_H and δ_B are computed for a class of self-affine sets. Necessary and sufficient conditions are given for $\delta_H = \delta_B$; it is found that typically $\delta_H \neq \delta_B$. The methods are largely probabilistic, with certain exponential families of probability measures playing a prominent role.

1. Introduction

Let $S_1, S_2, \ldots, S_r$ be contractions of $\mathbb{R}^2$, i.e., each $S_i: \mathbb{R}^2 \to \mathbb{R}^2$ has Lipschitz constant < 1, the Lipschitz constant being defined as Lip $(S_i) = \sup\{|S_i x - S_i y|/|x-y|: x \neq y\}$. A result of Hutchinson [Hu] states that there exists a unique nonempty compact subset Λ of $\mathbb{R}^2$ such that

$$\Lambda = \bigcup_{i=1}^{r} S_i(\Lambda);$$

we will refer to Λ as the limit set of the semigroup generated by $S_1, S_2, \ldots, S_r$. Many interesting "fractal" sets arise in this manner, and questions concerning the Hausdorff-Besicovich and Bouligand-Minkowski (box) dimensions δ_H and δ_B of Λ are of considerable interest (see [Ma] for definitions). We shall discuss a number of such questions in the case where the contractions S_i are affine mappings of a certain special type. (NOTE: For similarities $S_i, \delta_H = \delta_B$ and δ_H is the "similarity dimension": see [Hu].)

Our interest stems from papers of Bedford [Be], McMullen [Mc], and Falconer [Fa]. McMullen studied the special case in which each S_i has the form

$$S_i(x) = \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} x + \begin{pmatrix} k_i/n \\ \ell_i/m \end{pmatrix},$$

where $1 < m < n, 0 \leq k_i < n$, and $0 \leq \ell_i < m$ (k_i and ℓ_i are integers); thus each S_i maps the unit square onto an $n^{-1} \times m^{-1}$ rectangle R_i contained in the unit square. He found formulas for δ_H and δ_B and discovered that $\delta_H = \delta_B$ only in exceptional circumstances, namely, when the number of R_i in each "row" of the unit square with at least one R_i is the same (i.e., $\forall\ \ell \in \{0, 1, \ldots, m\}$ such that $\ell = \ell_i$ for some i, the number of i with $\ell = \ell_i$ is the same). Bedford found similar formulas (by entirely different methods) when the maps S_i have the form

$$S_i(x) = \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\pm 1} x + \begin{pmatrix} k_i/n \\ \ell_i/m \end{pmatrix}$$

with certain other restrictions, so that Λ is the graph of a continuous function on [0, 1] taking values in [0, 1]. Again, $\delta_H = \delta_B$ only in exceptional circumstances. Thus, the results of Bedford and McMullen, in addition to providing explicit calculations of the HB and BM dimensions, suggest that in such constructions it is atypical for the two dimensions to be equal.

Falconer considers a far more general setup, in which

$$S_i(x) = T_i x + a_i$$

where T_i is an arbitrary invertible matrix of norm $< \frac{1}{3}$. He does not, however, obtain a formula for δ_H in each such case; instead, he shows that there is a constant δ depending on $T_1, T_2, \ldots, T_r$ (but not $a_1, a_2, \ldots, a_r$) such that for almost every choice of $a_1, a_2, \ldots, a_r$ one has $\delta = \delta_H = \delta_B$. This runs counter to the spirit of the Bedford/McMullen results, in that it suggests $\delta_H = \delta_B$ is typical rather than exceptional for self-affine fractals.

We shall consider a class of self-affine sets more general than those of McMullen but less general than those of Falconer. Specifically, we will study the limit set Λ of the semigroup generated by the mappings A_{ij} given by

$$A_{ij}(x) = \begin{pmatrix} a_{ij} & 0 \\ 0 & b_i \end{pmatrix} x + \begin{pmatrix} c_{ij} \\ d_i \end{pmatrix}, \qquad (i,j) \in \mathcal{J}.$$

Here $\mathcal{J} = \{(i,j) \colon 1 \leq i \leq m \text{ and } 1 \leq j \leq n_i\}$ is a finite index set. We assume $0 < a_{ij} < b_i < 1$, for each pair (i,j), $\Sigma_{i=1}^{m} b_i \leq 1$, and $\Sigma_{j=1}^{n_i} a_{ij} \leq 1$ for each i. Also, $0 \leq d_1 < d_2 < \ldots < d_m < 1$ with $d_{i+1} - d_i \geq b_i$ and $1 - d_m \geq b_m$ and for each $i, 0 \leq c_{i1} < c_{i2} < \ldots < c_{in_i} < 1$ with $c_{i(j+1)} - c_{ij} \geq a_{ij}$ and $1 - c_{in_i} \geq a_{in_i}$. These hypotheses guarantee that the open rectangles

$$R_{ij} = A_{ij}((0,1) \times (0,1))$$

are pairwise disjoint subsets of $(0,1) \times (0,1)$ with edges parallel to the x- and y-axes, are arranged in "rows" of height b_i, and have height > width (see Figure 1).

We shall assume throughout the paper that $|\mathcal{G}| > 1$ to avoid the trivial case in which Λ consists of just a single point.

Figure 1

Our results about the limit set Λ are as follows:

(1) We determine the value of the Bouligand-Minkowski (box) dimension δ_B from the parameters a_{ij}, b_i. Specifically, if $p \in \mathbb{R}$ is the unique real such that $\Sigma_{i=1}^{m} b_i^p = 1$ then $\delta_B = \delta$ is the unique real such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} b_i^p a_{ij}^{\delta - p} = 1.$$

(2) We determine the Hausdorff-Besicovich dimension δ_H from the parameters a_{ij}, b_i. Specifically, we prove that

$$\delta_H = \max\left\{ \frac{\sum_i \sum_j p_{ij} \log p_{ij}}{\sum_i \sum_j p_{ij} \log a_{ij}} + \sum_i q_i \log q_i \left(\frac{1}{\sum_i q_i \log b_i} - \frac{1}{\sum_i \sum_j p_{ij} \log a_{ij}} \right) \right\},$$

where the maximum is over all probability distributions $\{p_{ij}\}$ on the set $\mathcal{J} = \{(i,j): i = 1,2,\ldots,m; j = 1,2,\ldots,n_i\}$ and $q_i = \Sigma_j p_{ij}$. The expression in braces is the Hausdorff dimension of the "iid" probability measure μ on Λ determined by $\{p_{ij}\}$ (sec. 3).

(3) We characterize those sets of parameters a_{ij}, b_i for which $\delta_H = \delta_B$. In fact, we prove that the following three conditions are equivalent:

(a) $\delta_H = \delta_B$;

(b) $0 < H_{\delta_H}(\Lambda) < \infty$; and

(c) $\sum_j a_{ij}^{\delta_H - r} = 1, \quad \forall\, i = 1,\ldots,m.$

Here r is the unique real number such that $\Sigma_i b_i^r = 1$. This shows that $\delta_H = \delta_B$ is highly atypical. Moreover, it answers a problem posed in [Mc] (even in the special case considered there): when is $0 < H_{\delta_H}(\Lambda) < \infty$? Observe that the collection of sets Λ considered by Bedford and McMullen is countable; our results provide a smoothly parameterized family of self-affine sets Λ with $\delta_H \neq \delta_B$.

Finally, observe that Λ is a repelling invariant set for a certain expansive (noninvertible) map $T: \mathbb{R}^2 \to \mathbb{R}^2$. T is any mapping which maps each R_{ij} onto the square $(0,1)\times(0,1)$ in such a way that $T|R_{ij} = A_{ij}^{-1}$. If the *closed* rectangles $\overline{R}_{ij}$ are nonoverlapping then T may be made C^∞ (in this case Λ is a Cantor set). Our formula for the HB dimension δ_H shows that δ_H is the maximum HB dimension of a T-invariant probability measure supported by Λ (see, e.g., [Yo] for the definition). This leads one to wonder whether it is generally true that the HB dimension of (say) a repelling, invariant, hyperbolic set in a smooth dynamical system is the maximum HB dimension of an invariant probability measure.

Definitions and Notation

If the closed rectangles $\overline{R}_{ij}$ are nonoverlapping then Λ is a Cantor set homeomorphic in a natural way to the sequence space $\Omega = \mathcal{J}^{\mathbb{N}}$. If, however, some of the $\overline{R}_{ij}$ have nonempty intersection then Λ and Ω are no longer homeomorphic, but as will be seen there is a natural continuous projection $\pi: \Omega \to \Lambda$ which is at most 4 to 1. We will repeatedly pass back and forth between Ω and Λ; the reader should beware that there are occasional subtleties in this when π is not 1 to 1.

We shall use the following notational conventions. Elements of Ω will always be represented by a small ω, possibly with primes ω' or superscripts, and

$$\begin{aligned}
\omega &= (\omega_1, \omega_2, \ldots), \quad \omega_n = (i_n, j_n) \in \mathcal{J}; \\
\omega' &= (\omega'_1, \omega'_2, \ldots), \quad \omega'_n = (i'_n, j'_n) \in \mathcal{J}; \\
\omega^{(u)} &= (\omega^{(u)}_1, \omega^{(u)}_2, \ldots), \quad \omega^{(u)}_n = (i^{(u)}_n, j^{(u)}_n) \in \mathcal{J}.
\end{aligned}$$

Let

$$\mathcal{Z} = \bigcup_{k=1}^{\infty} \mathcal{J}^k$$

be the set of *finite* sequences; elements of $\mathcal{Z}$ will always be represented by a small z, possibly with primes or a superscript. The same convention regarding the entries of z will be followed: e.g.,

$$z^{(u)} = (z^{(u)}_1, z^{(u)}_2, \ldots, z^{(u)}_k), z^{(u)}_n = (i^{(u)}_n, j^{(u)}_n) \in \mathcal{J}.$$

Given $\omega \in \Omega$ and $k = 1, 2, \ldots$, define

$$\begin{aligned}
A(\omega; k) &= A_{i_1 j_1} \circ A_{i_2 j_2} \circ \ldots \circ A_{i_k j_k}, \\
\Lambda(\omega; k) &= A(\omega; k)(\Lambda), \\
R(\omega; k) &= A(\omega; k)((0,1) \times (0,1)), \\
\overline{R}(\omega; k) &= \text{ closure } R(\omega; k),
\end{aligned}$$

Similarly, if $z \in \mathcal{Z}$ is of length $l = \ell(z)$, define

$$\begin{aligned}
A(z) &= A_{i_1 j_1} \circ A_{i_2 j_2} \circ \ldots \circ A_{i_l j_l}, \\
\Lambda(z) &= A(z)(\Lambda), \\
R(z) &= A(z)((0,1) \times (0,1)), \\
\overline{R}(z) &= \text{ closure } (R(z)), \\
\Omega(z) &= \{\omega \in \Omega : \omega_n = z_n \quad \forall\, 1 \le n \le l\}.
\end{aligned}$$

Note that if $\omega \in \Omega(z)$ then $A(\omega; \ell(z)) = A(z), \Lambda(\omega; \ell(z)) = \Lambda(z)$, etc. Also, each $R(z)$, $R(\omega; k)$ is an open rectangle, and

$$\begin{aligned}
&R(\omega; k) \cap \Lambda \subset \Lambda(\omega; k) \subset \overline{R}(\omega; k) \cap \Lambda, \\
&\Lambda(\omega; k+1) \subset \Lambda(\omega; k), \\
&R(\omega; k+1) \subset R(\omega; k), \\
&\text{height } (R(\omega; k)) = \prod_{\nu=1}^{k} b_{i_\nu}, \\
&\text{width } (R(\omega; k)) = \prod_{\nu=1}^{k} a_{i_\nu j_\nu}.
\end{aligned}$$

Since $b_i < 1$ and $a_{ij} < 1$, the diameter of $R(\omega; k)$ shrinks to zero as $k \to \infty$, for any $\omega \in \Omega$. Consequently, $\forall\, \omega \in \Omega$

$$\bigcap_{k=1}^{\infty} \overline{R}(\omega; k) = \{\pi(\omega)\}$$

consists of a single point $\pi(\omega) \in \Lambda$. This defines a map $\pi: \Omega \to \Lambda$. It is easily seen that π is continuous, surjective, and at most 4 to 1 (see [Hu]).

Essential to the arguments of this paper will be certain sets which we will call *approximate squares.* Analogous sets were defined in [Mc]. They will be used repeatedly in calculations involving Hausdorff and box dimensions. Given $\omega \in \Omega$ and $k \in \mathbb{N}$, define

$$(1.1) \qquad L_k(\omega) = \max\left\{n \geq 0: \quad \prod_{\nu=1}^{k} b_{i_\nu} \leq \prod_{\nu=1}^{n} a_{i_\nu i_\nu}\right\},$$

where $\Pi_{\nu=1}^{0} a_{i_\nu j_\nu} = 1$ (convention); observe that as $k \to \infty, k - L_k \to \infty$ because $a_{ij} < b_i$ for all i, j. Define the *approximate square*

$$(1.2) \qquad B_k(\omega) = \{\omega' \in \Omega: i'_\nu = i_\nu, \nu = 1, \ldots, k \text{ and } j'_\nu = j_\nu, \nu = 1, \ldots, L_k(\omega)\}.$$

Observe that each approximate square $B_k(\omega)$ is a finite union of cylinder sets $\Omega(z)$, and that approximate squares are "nested": i.e., for any two, say $B_k(\omega)$ and $B_{k'}(\omega')$, either $B_k(\omega) \cap B_{k'}(\omega') = \varnothing$ or $B_k(\omega) \subset B_{k'}(\omega')$ or $B_{k'}(\omega') \subset B_k(\omega)$. In addition,

$$\tilde{B}_k(\omega) \cap \Lambda \subset \pi(B_k(\omega)) \subset \Lambda \cap \text{ closure } (\tilde{B}_k(\omega))$$

where $\tilde{B}_k(\omega)$ is an open rectangle in $\mathbb{R}^2$ with sides parallel to the x- and y-axes, height $\Pi_{\nu=1}^{k} b_{i_\nu}$, and width $\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}$. (The rectangle $\tilde{B}_k(\omega)$ is the intersection of the rectangle $R(\omega; L_k(\omega))$ with the horizontal strip of height $\Pi_{\nu=1}^{k} b_{i_\nu}$ containing $R(\omega; k)$. See Figure 2.) By (1.1),

$$(1.3) \qquad 1 \leq \frac{\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}}{\Pi_{\nu=1}^{k} b_{i_\nu}} \leq \max a_{ij}^{-1},$$

so the width/height ratios of the rectangles $\tilde{B}_k(\omega)$ are bounded away from 0 and ∞ – hence the term "approximate square". Furthermore, observe that since $\pi(B_k(\omega))$ contains $\Lambda(\omega; k)$, its diameter is at least the height of $\Lambda(\omega; k)$), which is height $(\Lambda)\Pi_{\nu=1}^{k} b_{i_\nu}$. It follows that there are constants $0 < C_1 \leq C_2 < \infty$ such that $\forall\, \omega \in \Omega, \ \forall\, k \geq 1$,

$$(1.4) \qquad C_1 \leq \frac{\text{diam } (\pi(B_k(\omega)))}{\Pi_{\nu=1}^{k} b_{i_\nu}} \leq C_2.$$

Figure 2

Because of their nesting property, approximate squares and their π-images are easier to work with than balls in Λ. Observe that for any $x \in \Lambda$ and any $r > 0$ there is an approximate square $B_k(\omega)$ such that $x \in \pi(B_k(\omega))$ and such that both the height and width of the rectangle $\tilde{B}_k(\omega)$ are between r and $r(\max a_{ij}^{-1})(\max b_i^{-1})$. (Just take $\omega \in \pi^{-1}(\{x\})$ and $k = \max\{k^*\colon \Pi_{\nu=1}^{k^*} b_{i_\nu} \geq r\}$, and refer to (1.3).) Consequently, for any $x \in \Lambda$ and $r > 0$ the ball $B(r, x) \cap \Lambda$ is contained in the union of (at most) four $\pi(B_k(\omega))$, each with height and width between r and $r(\max a_{ij}^{-1}(\max b_i^{-1})$. Therefore, for any covering of Λ by balls $B(r, x)$ there is a covering by sets $\pi(B_k(\omega))$ which is essentially just as efficient.

2. The Bouligand-Minkowski (Box) Dimension

In this section we calculate the Bouligand-Minkowski dimension δ_B of Λ. This is defined as follows:

$$\delta_B = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon^{-1}},$$

where $N(\varepsilon)$ is the minimum number of squares of side ε needed to cover Λ (here and throughout this section the term *square* means square with sides parallel to the coordinate axes). We will begin by showing how to reduce the calculation of δ_B to a counting problem concerning approximate squares.

For $\varepsilon > 0$ define

$$\mathcal{F}_\varepsilon = \cup_{k=0}^{\infty} \{B_k(\omega)\colon \omega \in \Omega \text{ and } \Pi_{\nu=1}^{k} b_{i_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k+1} b_{i_\nu}\}.$$

2.1 Lemma

$$\delta_B = \limsup_{\varepsilon \to 0} \frac{\log |\mathcal{F}_\varepsilon|}{\log \varepsilon^{-1}}.$$

NOTE: Here $|\cdot|$ denotes cardinality.

PROOF: Clearly $\mathcal{F}_\varepsilon$ is a covering of Ω by approximate squares for each $\varepsilon > 0$, because for every $\omega \in \Omega$ there is a $k \geq 0$ so that $\Pi_{\nu=1}^{k} b_{i_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k+1} b_{i_\nu}$ (recall that $\forall\, i, 0 < b_i < 1$). Moreover, for each $B_k(\omega) \in \mathcal{F}_\varepsilon$ the corresponding rectangle $\tilde{B}_k(\omega)$ in $\mathbb{R}^2$ has sides $\Pi_{\nu=1}^{k} b_{i_\nu}$ and $\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}$ between ε and $\varepsilon(\max b_i^{-1})(\max a_{ij}^{-1})$. Hence $\mathcal{F}_\varepsilon$ determines a covering of Λ by squares of side $\varepsilon(\max b_i^{-1})(\max a_{ij}^{-1})$ (for each $B_k(\omega) \in \mathcal{F}_\varepsilon$, blow up $\tilde{B}_k(\omega)$ to a square). Therefore, $|\mathcal{F}_\varepsilon| \geq N(\varepsilon(\max b_i^{-1})(\max a_{ij}^{-1}))$, which shows that

$$\delta_B \leq \limsup_{\varepsilon \to 0} \frac{\log |\mathcal{F}_\varepsilon|}{\log \varepsilon^{-1}}.$$

To prove the reverse inequality it suffices to show that there is a constant $C < \infty$, not depending on ε, such that

$$|\mathcal{F}_\varepsilon| \leq CN(\varepsilon) \qquad \forall\, \varepsilon > 0.$$

For this we will show that for each $\varepsilon > 0$ there is a subset $\tilde{\mathcal{F}}_\varepsilon$ of $\mathcal{F}_\varepsilon$ such that $\tilde{\mathcal{F}}_\varepsilon$ is a covering of Ω by *pairwise disjoint* approximate squares, and such that

$$C^* = \sup |\mathcal{F}_\varepsilon| / |\tilde{\mathcal{F}}_\varepsilon| < \infty.$$

It will then follow that any square in $\mathbb{R}^2$ of side ε can intersect at most four $\pi(B_k(\omega))$ with $B_k(\omega) \in \mathcal{F}_\varepsilon$ (since the sets $\pi(B_k(\omega))$ are (essentially) the intersections of Λ with nonoverlapping rectangles with sides $\geq \varepsilon$), and hence that $|\tilde{\mathcal{F}}_\varepsilon| \leq 4N(\varepsilon)$ and $|\mathcal{F}_\varepsilon| \leq 4(C^*)N(\varepsilon)$.

Let $\tilde{\mathcal{F}}_\varepsilon$ consist of those $B_k(\omega) \in \mathcal{F}_\varepsilon$ such that $B_k(\omega)$ is not properly contained in any other $B_{k'}(\omega') \in \mathcal{F}_\varepsilon$. The elements of $\tilde{\mathcal{F}}_\varepsilon$ are pairwise disjoint, because if two approximate squares overlap then one is properly contained in the other. Moreover, since $\mathcal{F}_\varepsilon$ is a covering of Ω, so is $\tilde{\mathcal{F}}_\varepsilon$: every $\overline{\omega} \in \Omega$ is an element of some $B_k(\omega) \in \mathcal{F}_\varepsilon$, so it must be an element of a maximal $B_k(\omega)$.

Each element of $\mathcal{F}_\varepsilon$ is properly contained in exactly one element of $\tilde{\mathcal{F}}_\varepsilon$. Consider $B_k(\omega) \in \tilde{\mathcal{F}}_\varepsilon$; if $B_{k'}(\omega') \subset B_k(\omega)$ for some $B_{k'}(\omega') \in \mathcal{F}_\varepsilon$ then $k \leq k', L_k(\omega) \leq L_{k'}(\omega'), i_n = i'_n$ for each $n = 1, 2, \ldots, k$, and $j_n = j'_n$ for each $n = 1, 2, \ldots, L_k(\omega)$. But since both $B_k(\omega)$ and $B_{k'}(\omega')$ are elements of $\mathcal{F}_\varepsilon$,

$$\begin{aligned}
&\Pi_{\nu=1}^{k} b_{i_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k+1} b_{i_\nu}, \\
&\Pi_{\nu=1}^{k'} b_{i'_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k'+1} b_{i'_\nu}, \\
&\Pi_{\nu=1}^{\ell} a_{i_\nu j_\nu} \geq \Pi_{\nu=1}^{k} b_{i_\nu} > \Pi_{\nu=1}^{\ell+1} a_{i_\nu j_\nu}, \\
&\Pi_{\nu=1}^{\ell'} a_{i'_\nu j'_\nu} \geq \Pi_{\nu=1}^{k'} b_{i'_\nu} > \Pi_{\nu=1}^{\ell'+1} a_{i'_\nu j'_\nu},
\end{aligned}$$

where $\ell = L_k(\omega)$ and $\ell' = L_{k'}(\omega')$. Consequently,

$$\begin{aligned}
&\Pi_{\nu=k+1}^{k'} b_{i'_\nu} \geq \min_i b_i, \\
&\Pi_{\nu=\ell+1}^{\ell'} a_{i'_\nu j'_\nu} \geq (\min_{ij} a_{ij})(\min_i b_i)
\end{aligned}$$

which implies

$$(k' - k) + (\ell' - \ell) \leq K$$

for a suitable $K < \infty$ (independent of $B_k(\omega), B_{k'}(\omega')$, and $\varepsilon > 0$). Since $B_{k'}(\omega')$ was obtained from $B_k(\omega)$ by tacking on no more than K entries to $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_\ell)$, it follows that there are no more than $|\mathcal{J}|^K$ elements of $\mathcal{F}_\varepsilon$ contained in any $B_k(\omega) \in \tilde{\mathcal{F}}_\varepsilon$. This proves that $|\mathcal{F}_\varepsilon| \leq |\mathcal{J}|^K |\tilde{\mathcal{F}}_\varepsilon|$. □

The set $\mathcal{F}_\varepsilon$ is a collection of approximate squares, each of which is uniquely specified by a pair of finite sequences $(i_1, i_2, \ldots, i_{k+1}; j_1, j_2, \ldots, j_{\ell+1})$ subject to certain restrictions. It will be easier to work with a modification $\mathcal{F}_\varepsilon^*$ of $\mathcal{F}_\varepsilon$, defined by

$$\mathcal{F}_\varepsilon^* = \{(i_1, i_2, \ldots, i_{k+1}; j_1, j_2, \ldots, j_{\ell+1}) \colon (i_\nu, j_\nu) \in \mathcal{J} \;\; \forall\, 1 \leq \nu \leq \ell + 1; \\ \Pi_{\nu=1}^k b_{i_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k+1} b_{i_\nu}; \text{ and } \Pi_{\nu=1}^{\ell} a_{i_\nu j_\nu} \geq \varepsilon > \Pi_{\nu=1}^{\ell+1} a_{i_\nu j_\nu}\}.$$

2.2 Lemma

$$\delta_B = \limsup_{\varepsilon \to 0} \frac{\log |\mathcal{F}_\varepsilon^*|}{\log \varepsilon^{-1}}.$$

PROOF: Define

$$\mathcal{F}_\varepsilon' = \{(i_1, i_2, \ldots, i_{k+1}; j_1, j_2, \ldots, j_{\ell+1}) \colon (i_\nu, j_\nu) \in \mathcal{J} \;\; \forall\, 1 \leq \nu \leq \ell + 1; \\ \Pi_{\nu=1}^k b_{i_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k+1} b_{i_\nu}; \text{ and} \\ \Pi_{\nu=1}^{\ell} a_{i_\nu j_\nu} \geq \varepsilon > \Pi_{\nu=1}^{\ell+1} a_{i_\nu j_\nu}\}.$$

Clearly, $\mathcal{F}_\varepsilon$ is in one-to-one correspondence with $\mathcal{F}_\varepsilon'$. Moreover, for each $f \in \mathcal{F}_\varepsilon^*$ there is a unique $f' \in \mathcal{F}_\varepsilon'$ such that $f' \leq f$, i.e., $k' = k, \ell' \leq \ell, i_\nu = i_\nu' \;\; \forall\, \nu = 1, 2, \ldots, k$, and $j_\nu' = j_\nu \;\; \forall\, \nu = 1, 2, \ldots, \ell'$. On the other hand, for each $f' \in \mathcal{F}_\varepsilon'$ there is at least one $f \in \mathcal{F}_\varepsilon^*$ such that $f' \leq f$. Hence, $|\mathcal{F}_\varepsilon| \leq |\mathcal{F}_\varepsilon^*|$.

Suppose $f' \leq f$ for some $f' \in \mathcal{F}_\varepsilon'$ and $f \in \mathcal{F}_\varepsilon^*$. Then

$$\Pi_{\nu=\ell'+1}^{\ell} a_{i_\nu j_\nu} \geq \varepsilon / \Pi_{\nu=1}^k b_{i_\nu} \geq \min_i b_i,$$

so

$$\ell - \ell' \leq K = \max\{n \colon (\max a_{ij})^n \geq \min b_i\}.$$

Consequently, there are at most $|\mathcal{J}|^K$ elements $f' \in \mathcal{F}_\varepsilon'$ satisfying $f' \leq f$ for any $f \in \mathcal{F}_\varepsilon^*$. This proves that $|\mathcal{F}_\varepsilon^*| \leq |\mathcal{J}|^K |\mathcal{F}_\varepsilon|$ for all $\varepsilon > 0$. The result now follows from Lemma 2.1. □

For $\varepsilon > 0$ and $t \in \mathbb{R}$ define

$$\mathcal{G}_\varepsilon = \{z \in \mathcal{Z} \colon \Pi_{\nu=1}^{\ell(z)-1} a_{i_\nu j_\nu} \geq \varepsilon > \Pi_{\nu=1}^{\ell(z)} a_{i_\nu j_\nu}\};$$

$$\mathcal{H}_\varepsilon = \{(i_1, i_2, \ldots, i_{k+1}) \colon \Pi_{\nu=1}^k b_{i_\nu} \geq \varepsilon > \Pi_{\nu=1}^{k+1} b_{i_\nu}\};$$

$$H(t) = |\mathcal{H}_{e^{-t}}|;$$

$$F(t) = \sum_{z \in \mathcal{G}_{e^{-t}}} H(t + \Sigma_{\nu=1}^{\ell(z)} \log b_{i_\nu}).$$

Recall that $\mathcal{Z}$ is the set of (nonempty) finite sequences with entries in $\mathcal{G}$; for $z \in \mathcal{Z}, \ell(z)$ is the length of z and $z_\nu = (i_\nu, j_\nu)$ for $\nu = 1, 2, \ldots, \ell(z)$. Observe that

$$F(t) = |\mathcal{F}^*_{e^{-t}}|, \tag{2.1}$$

because elements of $\mathcal{F}^*_\varepsilon$ are obtained from elements of $\mathcal{G}_\varepsilon$ by tacking on $i_{\ell+1}, i_{\ell+2}, \ldots, i_{k+1}$ so that the appropriate inequalities are satisfied. Note that $H(t) = 0 \;\; \forall\, t < 0$.

2.3 Lemma

Let $r \geq 0$ be the unique real number such that $\Sigma_{i=1}^m b_i^r = 1$. Then there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 e^{rt} \leq H(t) \leq C_2 e^{rt} \quad \forall\, t \geq 0.$$

PROOF: Every element $(i_1, i_2, \ldots, i_{k+1})$ of $\mathcal{H}_\varepsilon, \varepsilon > 0$, has at least one entry i_1. Decomposing $\mathcal{H}_\varepsilon$ into m disjoint subsets, one for each possible initial letter i_1, yields the functional equation

$$H(t) = \sum_{i=1}^m H(t + \log b_i) + R(t), \quad t \geq 0,$$

where

$$R(t) = |\{i \in \{i \in \{1, 2, \ldots, m\}: t + \log b_i < 0\}|.$$

Observe that $R(t)$ is bounded, piecewise continuous with finitely many discontinuities, and has compact support in $[0, \infty)$. Also, R is strictly positive on $[0, \max_i, \log b_i^{-1})$.

The functional equation for H may be transformed into a renewal equation ([Fe]$_2$, Ch. 11). Define $\tilde{H}(t) = e^{-rt} H(t)$ and $\tilde{R}(t) = e^{-rt} R(t)$; then

$$\begin{aligned} \tilde{H}(t) &= \sum_{i=1}^m b_i^r \tilde{H}(t + \log b_i) + \tilde{R}(t) \\ &= \int_{s \in [0,t]} \tilde{H}(t - s)\mu(ds) + \tilde{R}(t), \quad t \geq 0, \end{aligned}$$

where μ is the probability distribution on $(0, \infty)$ which puts mass b_i^r at $-\log b_i \;\; \forall\, i$. There are now two cases to consider: the *arithmetic case*, where $\log b_1^{-1}, \log b_2^{-1}, \ldots, \log b_m^{-1}$ are contained in a discrete additive subgroup of $\mathbb{R}$, and the *nonarithmetic case*. In the nonarithmetic case, the Renewal Theorem ([Fe]$_2$, Ch. 11) implies that

$$\lim_{t \to \infty} \tilde{H}(t) = \int_0^\infty \tilde{R}(s) ds \Big/ \int s\mu(ds) > 0$$

since $\tilde{R}$ is, clearly, directly Riemann integrable (recall that R has compact support and only finitely many discontinuities). In the arithmetic case, the Renewal Theorem ([Fe]$_1$, sec. XIII.10) implies that

$$\lim_{n \to \infty} \tilde{H}(n\gamma) = \sum_{n=0}^\infty \tilde{R}(n\gamma) \Big/ \int s\mu(ds) > 0$$

where $\gamma\mathbb{Z}$ is the additive subgroup of $\mathbb{R}$ generated by $\{\log b_i^{-1} : i = 1, 2, \ldots, m\}$ (and $\gamma > 0$). Since $H(t)$ is nondecreasing in t and is strictly positive on $[0, \infty)$, the result follows. □

2.4 Theorem

Let $r \geq 0$ be the unique real number such that $\Sigma_{i=1}^m b_i^r = 1$, and let $\delta > 0$ be the unique real number such that $\Sigma_{i=1}^m \Sigma_{j=1}^{n_i} b_i^r a_{ij}^{\delta - r} = 1$. Then

$$\delta_B = \delta.$$

PROOF: By (2.1) and Lemmas 2.1–2.2 it suffices to prove that

$$\delta = \lim_{t \to \infty} \frac{\log F(t)}{t}.$$

Define

$$G(t) = 0 \text{ if } t < 0,$$
$$G(t) = \sum_{z \in \mathcal{G}_{e^{-t}}} \exp\{rt\} \Pi_{\nu=1}^{\ell(z)} b_{i_\nu}^r \text{ if } t \geq 0;$$

then by Lemma 2.3 and the definition of $F(t)$, there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 G(t) \leq F(t) \leq C_2 G(t) \quad \forall\, 0 \leq t < \infty.$$

$G(t)$ is a sum over a collection of finite sequences z, each with a first entry $z_1 = (i_1, j_1)$. Breaking this sum into $|\mathcal{J}|$ disjoint sums, one for each possible value of (i_1, j_1), gives the functional equation

$$G(t) = \sum_{(i,j) \in \mathcal{J}} b_i^r a_{ij}^{-r} G(t + \log a_{ij}) + R(t), \ t \geq 0,$$

where $R(t)$ is piecewise continuous with only finitely many discontinuities, $R(t) > 0$ for all $t \in [0, \max_{ij} \log a_{ij}^{-1})$, and $R(t) = 0$ for all $t \geq \max \log a_{ij}^{-1}$. Define $\tilde{G}(t) = e^{-\delta t} G(t)$ and $\tilde{R}(t) = e^{-\delta t} R(t)$; then the functional equation transcribes to

$$\tilde{G}(t) = \int_{[0,t]} \tilde{G}(t - s) \mu(ds) + \tilde{R}(t), \ t \geq 0$$

where μ is the probability measure on $(0, \infty)$ which puts mass $b_i^r a_{ij}^{\delta - r}$ at $-\log a_{ij}$ $\forall\, (i,j) \in \mathcal{J}$. By the same arguments as in the proof of Lemma 2.3 it follows that there exist constants $0 < C_1' \leq C_2' < \infty$ such that $C_1' \leq \tilde{G}(t) \leq C_2'$ $\forall\, t > 0$, and hence that $C_1' e^{\delta t} \leq G(t) \leq C_2' e^{\delta t}$. By the preceding paragraph, therefore,

$$C_1 C_1' e^{\delta t} \leq F(t) \leq C_2 C_2' e^{\delta t},$$

from which it follows that $t^{-1} \log F(t) \to \delta$ as $t \to \infty$.

NOTE: The fact that $\delta > 0$ follows from our standing assumption that $|\mathcal{J}| \geq 2$. Since $0 < b_i < 1$ and $m \geq 1$ we must have $r \geq 0$ in order that $\Sigma_{i=1}^m b_i^r = 1$. Since $0 < a_{ij} < b_i$ we must have $b_i^r a_{ij}^{-r} \geq 1$, so we can only have $\Sigma\Sigma b_i^r a_{ij}^{\delta-r} = 1$ with $\delta > 0$. □

3. The Volume Lemma

In this section we obtain a lower bound for the Hausdorff-Besicovich dimension δ_H of Λ, using the auxiliary notion of the Hausdorff dimension of a measure supported by Λ. Recall ([Ma]) that for a Borel subset K of $\mathbb{R}^n$ the *δ-dimensional Hausdorff measure* of K is defined by

$$H_\delta(K) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U} \in \mathcal{C}(\varepsilon)} \sum_{U \in \mathcal{U}} (\text{diam } U)^\delta$$

where $\mathcal{C}(\varepsilon)$ is the set of all open covers $\mathcal{U}$ of K whose elements U have diameters $< \varepsilon$. The *Hausdorff-Besicovich dimension* of K is defined to be $\inf\{\delta > 0 \colon H_\delta(K) < \infty\}$.

Let μ be a finite Borel measure on $\mathbb{R}^k$. The *Hausdorff dimension* of μ is defined by

$$HD(\mu) = \inf\{HD(Y) \colon Y \text{ Borel and } \mu(\mathbb{R}^k \backslash Y) = 0\}.$$

The following lemma is useful in determining $HD(\mu)$. Its proof may be found in [Yo].

3.1 Lemma (The Volume Lemma)

If μ is a Borel probability measure on a Euclideon space and if there exists $\delta > 0$ such that

$$\lim_{r \to 0} \frac{\log \mu(B(r,x))}{\log r} = \delta, \quad \text{for } \mu \text{ - } a.e.x,$$

then $HD(\mu) = \delta$. Here $B(r,x)$ denotes the ball of radius r centered at x.

Recall the mapping $\pi \colon \Omega \to \Lambda$ defined in sec. 1. Since π is continuous, it is measurable with respect to the Borel σ-fields on Λ and Ω respectively, and hence, given a Borel probability measure μ on Ω, the set function defined by

$$\mu\pi^{-1}(E) = \mu(\pi^{-1}(E)), \quad E \in \mathcal{B}(\Lambda), \tag{3.1}$$

defines a Borel probability measure on Λ.

Let Σ be the simplex defined by

$$\Sigma = \{\boldsymbol{p} = (p_{ij})_{(i,j)\in\mathcal{J}}) \colon p_{ij} \in [0,1] \text{ and } \sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} = 1\}.$$

If $\boldsymbol{p} \in \Sigma$, then $\boldsymbol{p}$ defines a probability distribution on the set of indices $\mathcal{J}$. For each $\boldsymbol{p} \in \Sigma$, define a probability measure $\mu_{\boldsymbol{p}}$ on the Borel subsets of Ω by requiring that for any $z \in \mathcal{Z}$, the $\mu_{\boldsymbol{p}}$-measure of the cylinder set $\Omega(z)$ be given by

$$\mu_{\boldsymbol{p}}(\Omega(z)) = \prod_{\nu=1}^{\ell(z)} p_{i_\nu j_\nu}, \tag{3.2}$$

where $\ell(z)$ is the length of z. Note that (3.2) uniquely determines a Borel probability measure on Ω, by Kolmogorov's existence theorem (see [Bi]); $\mu_{\boldsymbol{p}}$ is the distribution of a sequence of i.i.d., $\mathcal{J}$-valued random vectors $X_1, X_2, \ldots$, each of which has distribution $\boldsymbol{p}$. For each $\boldsymbol{p} \in \Sigma$ the measure $\mu_{\boldsymbol{p}}$ induces (by (3.1)) a Borel probability measure on Λ, which we will denote by $\tilde{\mu}_{\boldsymbol{p}}$ (thus, $\tilde{\mu}_{\boldsymbol{p}} = \mu \circ \pi^{-1}$). Define, for any $\boldsymbol{p} \in \Sigma$,

$$D(\boldsymbol{p}) = HD(\tilde{\mu}_{\boldsymbol{p}});$$

observe that $\sup\{D(\boldsymbol{p}) : \boldsymbol{p} \in \Sigma\}$ is a lower bound for the Hausdorff dimension δ_H of Λ. We will ultimately prove (sec. 5) that this supremum in fact equals δ_H.

3.2 Proposition

For every $\boldsymbol{p} \in \Sigma$,

$$(3.3) \qquad D(\boldsymbol{p}) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} \log p_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} \log a_{ij}} + \left(1 - \frac{\sum_{i=1}^{m} q_i \log b_i}{\sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} \log a_{ij}}\right) \frac{\sum_{i=1}^{m} q_i \log q_i}{\sum_{i=1}^{m} q_i \log b_i}$$

(with the convention $0 \cdot \log 0 = 0$*), where*

$$q_i = \sum_{j=1}^{n_i} p_{ij}.$$

PROOF: Fix $\boldsymbol{p} \in \Sigma$ and let d denote the right hand side of (3.3). Write $\mu = \mu_{\boldsymbol{p}}$ and $\tilde{\mu} = \tilde{\mu}_{\boldsymbol{p}} = \mu \circ \pi^{-1}$. By the Volume Lemma it suffices to prove that for $\tilde{\mu}$ - a.e. $x \in \Lambda$,

$$\lim_{r \to 0} \frac{\log \tilde{\mu}(B(r,x))}{\log r} = d.$$

We will argue that balls $B(r,x)$ may be replaced by π-images of approximate squares in the preceding formula. Recall (from sec. 1) that for any $x \in \Lambda$ and $r > 0$ the ball $B(r,x) \cap \Lambda$ is contained in the union of (at most) four $\pi(B_k(\omega))$ each with height and width between r and $r(\max a_{ij}^{-1})(\max b_i^{-1})$. Furthermore, if $x \in \Lambda$ and $r > 0$ then for any $\omega \in \pi^{-1}\{x\}$ (recall $|\pi^{-1}\{x\}| \leq 4$) there is an approximate square $B_k(\omega)$ such that $\pi(B_k(\omega)) \subset B(r,x)$ *and* $Cr \leq \operatorname{diam}(\pi(B_k(\omega))) \leq r$ for a suitable constant $C > 0$ (independent of x, r, ω), by (1.4). Thus $B(r,x) \cap \Lambda$ is bracketed by approximate squares of diameter on the order of r. By (1.4), it therefore suffices to prove that

$$(3.4) \qquad \lim_{k \to \infty} \frac{\log \mu(B_k(\omega))}{\Sigma_{\nu=1}^{k} \log b_{i_\nu}} = d$$

for μ – a.e. (ω).

But (3.4) is an elementary consequence of Kolmogorov's strong law of large numbers (SLLN). For any $k \geq 1$ and $\omega \in \Omega$,

$$\mu(B_k(\omega)) = \left(\prod_{\nu=1}^{L_k(\omega)} p_{i_\nu j_\nu}\right)\left(\prod_{\nu=L_k(\omega)+1}^{k} q_{i_\nu}\right). \tag{3.5}$$

By (1.3),

$$\frac{1}{k}\left(\sum_{\nu=1}^{k} \log b_{i_\nu} - \sum_{\nu=1}^{L_k(\omega)} \log a_{i_\nu j_\nu}\right) \longrightarrow 0$$

and by SLLN,

$$\frac{1}{k}\sum_{\nu=1}^{k} \log b_{i_\nu} \longrightarrow \sum_{i=1}^{m} q_i \log b_i;$$

$$\frac{1}{L_k(\omega)}\sum_{\nu=1}^{L_k(\omega)} \log a_{i_\nu j_\nu} \longrightarrow \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij} \log a_{ij};$$

and

$$\frac{1}{L_k(\omega)}\sum_{\nu=1}^{L_k(\omega)} \log p_{i_\nu j_\nu} \longrightarrow \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij} \log p_{ij}.$$

for μ - $a.e.(\omega)$. It follows from the first three of these that

$$\frac{L_k(\omega)}{k} \longrightarrow \frac{\sum_{i=1}^{m} q_i \log b_i}{\sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij} \log a_{ij}}, \quad \mu \text{ - } a.e.(\omega),$$

and now by another (double) application of SLLN,

$$\frac{1}{k - L_k(\omega)}\sum_{\nu=L_k(\omega)+1}^{k} \log q_{i_\nu} \longrightarrow \sum_{i=1}^{m} q_i \log q_i\ .$$

Taking logs in (3.5) and applying the last three listed convergences one obtains (3.4). □

3.4 Proposition

There exists a $\mathbf{p}^* = (p^*_{ij})_{(i,j)\in\mathcal{J}} \in \Sigma$, *such that*

$$D(\mathbf{p}^*) = \max_{\mathbf{p}\in\Sigma} D(\mathbf{p}). \tag{3.6}$$

Furthermore any $\mathbf{p}^*$ *satisfying* (3.6) *must be an interior point of* Σ *and have the form*

$$p^*_{ij} = C a^{\theta}_{ij} b^{\lambda}_i \left(\sum_{l=1}^{n_i} a^{\theta}_{il}\right)^{\rho-1}, \qquad (i,j) \in \mathcal{J}, \tag{3.7}$$

for some constants C, θ, λ and ρ. The constants θ, λ, ρ satisfy

$$(3.8)\qquad \begin{aligned} \theta &= \frac{\Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^* \log p_{ij}^* - \Sigma_{i=1}^{m} q_i^* \log q_i^*}{\Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^* \log a_{ij}} \\ \lambda &= \frac{\Sigma_{i=1}^{m} q_i^* \log q_i^*}{\Sigma_{i=1}^{m} q_i^* \log b_i} \\ \rho &= \frac{\Sigma_{i=1}^{m} q_i^* \log b_i}{\Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^* \log a_{ij}} \end{aligned}$$

where $q_i^* = \Sigma_{j=1}^{n_i} p_{ij}^*, \quad i = 1, \ldots, m$.

REMARK 1: Comparing the equations (3.8) for θ and λ with the formula (3.3) for $D(\boldsymbol{p})$ shows that

$$\theta + \lambda = \max_{\boldsymbol{p} \in \Sigma} D(\boldsymbol{p}).$$

REMARK 2: In certain special cases the maximum value of $D(\boldsymbol{p}), \boldsymbol{p} \in \Sigma$, can be characterized more explicitly. For exmaple, if there exists $\eta \in (0,1)$ such that $a_{ij}^{\eta} = b_i$ for every $(i,j) \in \mathcal{J}$ then $\max_\Sigma D(\boldsymbol{p})$ is the unique $\delta > 0$ such that

$$\sum_{i=1}^{m} b_i^{\delta} n_i^{\eta} = 1.$$

This is proved in Lemma 5.1 below. Observe that the cases considered in [Mc] are subsumed by this.

PROOF: First observe that by (3.3), the function $D: \Sigma \to [0,2]$, given by $\boldsymbol{p} \to D(\boldsymbol{p})$, is continuous. Since Σ is compact, the function D has a maximum. Let $\boldsymbol{p}^*$ be a point in Σ at which D attains its maximum. We claim that

$$(3.9)\qquad \boldsymbol{p}^* \in \operatorname{int} \Sigma.$$

Once (3.9) has been proved a routine use of the Lagrange multipliers method shows that $\boldsymbol{p}^*$ must satisfy (3.7) for some constants C, θ, λ and ρ and that θ, λ, ρ must satisfy (3.8).

To complete the proof it remains to show (3.9). So assume $\boldsymbol{p}^* \notin \operatorname{int} \Sigma$ and suppose that $p_{i_0 j_0}^* = 0$. Choose $p_{i_1 j_1}^* > 0$. Such a $p_{i_1 j_1}^*$ always exists since $\boldsymbol{p}^* \in \Sigma$. For $0 < \varepsilon < p_{i_1 j_1}^*$ define $\boldsymbol{p}^{(\varepsilon)}$ by $p_{i_0 j_0}^{(\varepsilon)} = \varepsilon, \quad p_{i_1 j_1}^{(\varepsilon)} = p_{i_1 j_1}^* - \varepsilon$ and $p_{ij}^{(\varepsilon)} = p_{ij}^*$ for all other, i, j. Then $\boldsymbol{p}^{(\varepsilon)}$ is certainly in Σ. We will show that for some $\varepsilon > 0, \quad D(\boldsymbol{p}^{(\varepsilon)}) > D(\boldsymbol{p}^*)$.

Let

$$\begin{aligned} P^* &= \Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^* \log p_{ij}^*, \\ A^* &= \Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^* \log a_{ij}, \\ P(\varepsilon) &= \Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^{(\varepsilon)} \log p_{ij}^{(\varepsilon)} \end{aligned}$$

and

$$A(\varepsilon) = \Sigma_{i=1}^{m}\Sigma_{j=1}^{n_i} p_{ij}^{(\varepsilon)} \log a_{ij}.$$

Then, as $\varepsilon \downarrow 0$,

$$\frac{P(\varepsilon)}{A(\varepsilon)} \longrightarrow \frac{P^*}{A^*}.$$

On the other hand

$$\frac{d}{d\varepsilon}\left(\frac{P(\varepsilon)}{A(\varepsilon)}\right) \longrightarrow \infty, \quad \text{as } \varepsilon \downarrow 0.$$

Hence the function $\varepsilon \to P(\varepsilon)/A(\varepsilon)$, is increasing in some interval $(0, \varepsilon_1)$ and so

$$\frac{P(\varepsilon)}{A(\varepsilon)} > \frac{P^*}{A^*}, \qquad 0 < \varepsilon < \varepsilon_1.$$

Next let

$$\begin{aligned} Q^* &= \Sigma_{i=1}^m q_i^* \log q_i^*, \\ B^* &= \Sigma_{i=1}^m q_i^* \log b_i, \\ Q(\varepsilon) &= \Sigma_{i=1}^m q_i^{(\varepsilon)} \log q_i^{(\varepsilon)} \end{aligned}$$

and

$$B(\varepsilon) = \Sigma_{i=1}^m q_i^{(\varepsilon)} \log b_i,$$

where $q_i^{(\varepsilon)} = \Sigma_{j=1}^{n_i} p_{ij}^{(\varepsilon)}, i = 1, \ldots, m$. Then as above there exists $\varepsilon_2 > 0$ such that

$$Q(\varepsilon)\left[\frac{1}{B(\varepsilon)} - \frac{1}{A(\varepsilon)}\right] > Q^*\left[\frac{1}{B^*} - \frac{1}{A^*}\right], \qquad 0 < \varepsilon < \varepsilon_2.$$

However then we have that

$$\begin{aligned} D(\boldsymbol{p}^{(\varepsilon)}) &= \frac{P(\varepsilon)}{A(\varepsilon)} + Q(\varepsilon)\left[\frac{1}{B(\varepsilon)} - \frac{1}{A(\varepsilon)}\right] \\ &> \frac{P^*}{A^*} + Q^*\left[\frac{1}{B^*} - \frac{1}{A^*}\right] = D(\boldsymbol{p}), \qquad 0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}, \end{aligned}$$

by (3.3) and this contradicts the fact that $D(\boldsymbol{p}^*) = \max_{\boldsymbol{p}\in\Sigma} D(\boldsymbol{p})$. □

Define $\boldsymbol{p}(\theta, \lambda, \rho) \in \Sigma$, by defining $p_{ij}(\theta, \lambda, \rho)$ to be the right-hand side of (3.7). Then by the previous proposition, $\max_{\boldsymbol{p}\in\Sigma} D(\boldsymbol{p})$ may be found by maximizing the function $d(\theta, \lambda, \rho) = D(\boldsymbol{p}(\theta, \lambda, \rho))$, for (θ, λ, ρ) in some compact subset of $\mathbb{R}^3$. (The compact set over which $d(\theta, \lambda, \rho)$ is to be maximized is determined by (3.8).) Proposition 3.3 provides a formula for $d(\theta, \lambda, \rho)$ and so $\max_{\boldsymbol{p}\in\Sigma} D(\boldsymbol{p})$ is computable. As will be seen in section 5, this number is in fact the Hausdorff dimension of Λ. Observe that, by definition 3.1, we already have shown that the Hausdorff dimension of Λ satisfies

$$\delta_H \geq \max_{\boldsymbol{p}\in\Sigma} D(\boldsymbol{p}) = \max_{\theta,\lambda,\rho} d(\theta, \lambda, \rho).$$

4. Comparison of δ_H with δ_B

In this section we show that $0 < H_{\delta_H}(\Lambda) < \infty$ iff $\delta_H = \delta_B$, and give an easily checked necessary and sufficient condition for $\delta_H = \delta_B$ in terms of the parameters a_{ij} and b_i. Throughout this section we will write δ rather than δ_H for the HB dimension of Λ. For $s \geq 0$ we will write H_s for s-dimensional Hausdorff measure.

4.1 Lemma

For any two distinct $z, z' \in \mathcal{J}^k, k \in \mathbb{N}$,

$$H_\delta(\Lambda(z) \cap \Lambda(z')) = 0. \tag{4.1}$$

NOTE: It follows that if $z, z' \in \mathcal{Z}$ are distinct finite sequences such that $z_n \neq z'_n$ for some $n \leq \min(\ell(z), \ell(z'))$, then

$$H_\delta(\Lambda(z) \cap \Lambda(z')) = 0. \tag{4.2}$$

For suppose $\ell(z) < \ell(z')$. Define $z'' = (z'_1, z'_2, \ldots, z'_{\ell(z)})$; then $\Lambda(z') \subset \Lambda(z'')$. But Lemma 4.1 implies that $H_\delta(\Lambda(z) \cap \Lambda(z'')) = 0$, since $z \neq z''$.

PROOF: Let $S = [0,1] \times [0,1]$ and let ∂S be the boundary of S. Let F_1, F_2, F_3, F_4 be the sides of ∂S. We shall assume that there is no F_i such that each $\overline{R}_{i'j'}$ intersects F_i. (If this were so then $\Lambda \subset F_i$ and every nonempty intersection $\Lambda(z) \cap \Lambda(z')$, where $z, z' \in \mathcal{Z}^k$, would consist of a single point, in which case (4.1) would reduce to a triviality.)

Now assume that $\Lambda(z) \cap \Lambda(z') \neq \varnothing$ for distinct $z, z' \in \mathcal{J}^k$. Since $z \neq z'$, the open rectangles $R(z)$ and $R(z')$ do not intersect; since $\Lambda(z)$ and $\Lambda(z')$ are contained in the closed rectangles $\overline{R}(z)$ and $\overline{R}(z')$, respectively, $\overline{R}(z)$ and $\overline{R}(z')$ must meet in a single edge, either vertical or horizontal. In either case $\Lambda(z) \cap \Lambda(z')$ is contained in this edge. Now the intersection of $\Lambda(z)$ with this edge is precisely $A(z)(\Lambda \cap F_i)$ for one of the four sides F_i of ∂S, and (recall) $A(z)$ is an affine mapping. Consequently, to prove (4.1) it suffices to prove that

$$H_\delta(\Lambda \cap F_i) = 0 \qquad \forall\, i = 1,2,3,4.$$

We shall prove this for $F_1 = \{(x,0) : 0 \leq x \leq 1\}$; the other three cases are similar.

Let Λ^* be the limit set of the semigroup of similarity mappings of $\mathbb{R}$ generated by $T_1, T_2, \ldots, T_{n_1}$, defined by

$$T_j(x) = a_{1j}x + c_{1j}, \quad x \in \mathbb{R}$$

(recall the definitions of A_{ij}, sec. 1). Then $\Lambda^* \times \{0\} = \Lambda \cap F_1$, so $H_\delta(\Lambda \cap F_1) = H_\delta(\Lambda^*)$. Now Λ^* is a *self-similar* set, so its Hausdorff-Besicovich dimension is the unique $s \geq 0$ such that $\Sigma_{j=1}^{n_1} a_{1j}^s = 1$ (see [Hu]).

Consider the probability vector $\boldsymbol{p} \in \Sigma$ defined by

$$\begin{aligned} p_{ij} &= a_{1j}^s \quad \forall\, j = 1,2,\ldots,n_1, \\ p_{ij} &= 0 \qquad \forall\, i > 1; \end{aligned}$$

note that $q_1 = 1$ and $q_i = 0$ for $i > 1$. Proposition 3.3 implies that $D(\boldsymbol{p}) = s$. But $\boldsymbol{p}$ lies on the boundary of Σ, because there is at least one pair $(i,j) \in \mathcal{J}$ such that $i > 1$ (otherwise, every $\overline{R}_{ij}$ would intersect F_1). Proposition 3.4 states that D attains its maximum only at interior point(s) of Σ, and this maximum is $\leq \delta$. Hence

$$s < \delta$$

which shows

$$0 = H_\delta(\Lambda^*) = H_\delta(\Lambda \cap F_1). \qquad \square$$

4.2 Lemma

Let $\mathbf{p}$ be any probability vector in the interior of Σ, i.e., such that $p_{ij} > 0$ for every $(i,j) \in \mathcal{J}$. Then for any two distinct $z, z' \in \mathcal{J}^k$, any $k \geq 1$,

$$\tilde{\mu}_{\mathbf{p}}(\Lambda(z) \cap \Lambda(z')) = 0. \tag{4.3}$$

NOTE: Again it follows that if $z, z' \in \mathcal{Z}$ are distinct finite sequences such that $z_n \neq z'_n$ for some $n \leq \min(\ell(z), \ell(z'))$, then

$$\tilde{\mu}_{\mathbf{p}}(\Lambda(z) \cap \Lambda(z')) = 0.$$

This also implies that $\forall\, z \in \mathcal{Z}$,

$$\tilde{\mu}_{\mathbf{p}}(\Lambda(z)) = \mu_{\mathbf{p}}(\Omega(z)). \tag{4.4}$$

PROOF: Without loss of generality we may assume that $z_n = z'_n \ \forall\, n \leq k-1$ and $z_k \neq z'_k$, since the sets $\Lambda(z)$ are nested. There are two possibilities (assuming $\Lambda(z) \cap \Lambda(z')) \neq \varnothing$): either $\overline{R}(z)$ and $\overline{R}(z')$ share a single horizontal edge, or they share a single vertical edge. We shall only consider the first of these (the second is quite similar); in this case $i_n \neq i'_n$, and we may assume $i'_n = i_n + 1$.

Consider $\tilde{\omega} \in \pi^{-1}(\Lambda(z) \cap \Lambda(z'))$. It must either be that $\tilde{i}_k = i_k$, in which case $\tilde{i}_n = m \ \ \forall\, n > k$, or $\tilde{i}_k = i'_k$, in which case $\tilde{i}_n = 1 \ \ \forall\, n > k$. Thus $\pi^{-1}(\Lambda(z) \cap \Lambda(z'))$ consists entirely of sequences $\tilde{\omega}$ for which the coordinates $\tilde{\omega}_n, n > k$, are all constrained to lie in one of a finite collection of proper subsets of $\mathcal{J}$. But $p_{ij} > 0 \ \ \forall\, (i,j) \in \mathcal{J}$; hence the $\mu_{\mathbf{p}}$-measure of any such constrained subset of Ω must be zero (e.g., by Kolmogorov's SLLN). $\square$

4.3 Proposition

Let r be the unique real number such that $\Sigma\Sigma b_i^r a_{ij}^{\delta-r} = 1$, and let $\mathbf{p} \in \Sigma$ be the probability vector defined by

$$p_{ij} = b_i^r a_{ij}^{\delta-r} \ \ \forall\, (i,j) \in \mathcal{J}. \tag{4.5}$$

If

$$0 < H_\delta(\Lambda) < \infty \tag{4.6}$$

then

$$\Sigma_{j=1}^{n_i} a_{ij}^{\delta-r} = 1 \quad \forall\, i = 1, 2, \ldots, m, \tag{4.7}$$

$$\Sigma_{i=1}^{m} b_i^r = 1, \tag{4.8}$$

and the restriction $H_\delta|\Lambda$ of δ-dimensional Hausdorff measure to δ is equivalent (mutually absolutely continuous) to $\tilde{\mu}_{\boldsymbol{p}}$.

NOTE: In view of Th. 2.4 it follows from (4.7)–(4.8) that if $0 < H_\delta(\Lambda) < \infty$ then $\delta = \delta_B$.

PROOF: The main step will be to show that (4.6) implies that $H_\delta|\Lambda$ is equivalent to $\tilde{\mu}_{\boldsymbol{p}}$. It will then follow that $D(\boldsymbol{p}) = \delta$ and hence that $\boldsymbol{p} = \boldsymbol{p}^*$, where $\boldsymbol{p}^*$ is as in (3.7); reconciliation of (4.5) with (3.7)–(3.8) will then require that (4.7) and (4.8) hold. So assume (4.6).

Consider first the special case considered in [Mc], specifically, $a_{ij} = n^{-1}$ and $b_i = m^{-1}$ for all $(i,j) \in \mathcal{J}$. In this case $\boldsymbol{p}$ defined by (4.5) is just the uniform distribution on $\mathcal{J}$. Moreover, for all $z, z' \in \mathcal{J}^k$ (and $k \geq 1$) the sets $\Lambda(z) = A(z)(\Lambda)$ and $\Lambda(z') = A(z')(\Lambda)$ are congruent, so they have the same H_δ-measure. By Lemma 4.1, if $z \neq z'$ then $H_\delta(\Lambda(z) \cap \Lambda(z')) = 0$; consequently, for each $z \in \mathcal{J}^k$,

$$H_\delta(\Lambda(z)) = \frac{H_\delta(\Lambda)}{|\mathcal{J}|^k}.$$

But by Lemma 4.2, (cf. (4.4))

$$\tilde{\mu}_{\boldsymbol{p}}(\Lambda(z)) = \mu_{\boldsymbol{p}}(\Omega(z)) = \frac{1}{|\mathcal{J}|^k},$$

since $\boldsymbol{p}$ is the uniform distribution on $\mathcal{J}$. Now the sets $\Lambda(z), z \in \mathcal{Z}$ generate the σ-algebra of Borel subsets of Λ, so this proves that $\tilde{\mu}_{\boldsymbol{p}}$ is in fact equal to the normalized δ-dimensional Hausdorff measure on Λ.

Unfortunately, this argument does not extend to the general case because in general different $\Lambda(z)$'s will not be congruent. So instead we will use different $\Lambda(z)$'s which are approximately *similar*. For each $t > 0$ and $\omega \in \Omega$ define

$$n(\omega, t) = \min\left\{ n \in \mathbb{N} \colon \prod_{\nu=1}^{n} \left(\frac{b_{i_\nu}}{a_{i_\nu j_\nu}} \right) \geq t \right\};$$

set

$$\mathcal{N}_t = \{(\omega_1, \omega_2, \ldots, \omega_{n(\omega;t)}) \colon \omega \in \Omega\}.$$

Observe that $\mathcal{N}_t$ is a *finite* set of *finite* sequences z. For any two $z, z' \in \mathcal{N}_t$ the sets $\Lambda(z)$ and $\Lambda(z')$ are approximately similar (although they may differ greatly in size) because the y-direction/x-direction contraction ratios for $A(z)$ and $A(z')$ are approximately the same (both about t). More precisely, the affine map $A(z')A(z)^{-1}$ taking $\Lambda(z)$ onto $\Lambda(z')$ is approximately a similarity; its x-direction and y-direction expansion factors are

$$\frac{\prod_{\nu=1}^{\ell(z)} a_{i_\nu j_\nu}}{\prod_{\nu=1}^{\ell(z')} a_{i'_\nu j'_\nu}} \quad \text{and} \quad \frac{\prod_{\nu=1}^{\ell(z)} b_{i_\nu}}{\prod_{\nu=1}^{\ell(z')} b_{i'_\nu}},$$

respectively, where $\ell(z)$ and $\ell(z')$ are the lengths of z and z'. Since $\Pi_{\nu=1}^{\ell(z)}(b_{i_\nu}/a_{i_\nu j_\nu})$ and $\Pi_{\nu=1}^{\ell(z')}(b_{i'_\nu}/a_{i'_\nu j'_\nu})$ are both within a bounded factor of t, it follows that there are constants $0 < C_1 < C_2 < \infty$, independent of t, such that $\forall\ z, z' \in \mathcal{N}_t$

$$C_1 \leq \frac{H_\delta(\Lambda(z))/H_\delta(\Lambda(z'))}{\Pi_{\nu=1}^{\ell(z)} b_{i_\nu}^r a_{i_\nu j_\nu}^{\delta-r} / \Pi_{\nu=1}^{\ell(z')} b_{i'_\nu}^r a_{i'_\nu j'_\nu}^{\delta-r}} \leq C_2.$$

By Lemmas 4.1–4.2 (see the notes following these lemmas) the overlaps $\Lambda(z) \cap \Lambda(z')$ have H_δ- and $\tilde{\mu}_{\boldsymbol{p}}$-measure zero. Also, for each $t > 0, \Lambda = \cup_{z \in \mathcal{N}_t} \Lambda(z)$, and by (4.4), $\forall z'' \in \mathcal{Z}$,

$$\tilde{\mu}_{\boldsymbol{p}}(\Lambda(z'')) = \Pi_{\nu=1}^{\ell(z'')} b_{i''_\nu}^r a_{i''_\nu j''_\nu}^{\delta-r}.$$

Hence, $\forall\ t > 0$ and $\forall\ z \in \mathcal{N}_t$,

$$C_1 \leq \frac{H_\delta(\Lambda(z))/H_\delta(\Lambda)}{\tilde{\mu}_{\boldsymbol{p}}(\Lambda(z))} \leq C_2.$$

But the sets $\Lambda(z), z \in \cup_{t>0} \mathcal{N}_t$, generate the σ-algebra of Borel subsets of Λ. Therefore, the preceding inequalities imply that $H_\delta|\Lambda$ and $\tilde{\mu}_{\boldsymbol{p}}$ are mutually absolutely continuous, with Radon-Nikodym derivative bounded away from 0 and ∞.

The preceding result implies that $D(\boldsymbol{p})(= HD(\tilde{\mu}_{\boldsymbol{p}})) = \delta$. For if $S \subset \Lambda$ is any Borel set such that $\tilde{\mu}_{\boldsymbol{p}}(S) = 1$ then $0 < H_\delta(S) < \infty$. But Prop. 3.3 also specifies $D(\boldsymbol{p})$: comparing (3.3) with the equation $D(\boldsymbol{p}) = \delta$ shows that

$$\sum_{i=1}^{m} b_i^r \gamma_i \log \gamma_i = 0 \tag{4.9}$$

where

$$\gamma_i = \sum_{j=1}^{n_i} a_{ij}^{\delta-r}.$$

Moreover, since $\boldsymbol{p}$ maximizes $D(\cdot)$, Prop. 3.4 implies that $\boldsymbol{p} = \boldsymbol{p}^*$ where $\boldsymbol{p}^*$ satisfies (3.7)–(3.8). Using (4.9) in the formula (3.8) for θ shows that $\theta = \delta - r$ and using (4.9) in the formula (3.8) for λ shows that $\lambda = r$. Consequently, by (3.7),

$$b_i^r a_{ij}^{\delta-r} = C b_i^r a_{ij}^{\delta-r} \gamma_i^{\rho-1}\ \ \forall\ (i,j) \in \mathcal{J},$$

so $C\gamma_i^{\rho-1} = 1\ \ \forall\ i$. Finally, the formula (3.8) for ρ shows that $\rho < 1$, since $0 < a_{ij} < b_i < 1\ \ \forall\ (i,j) \in \mathcal{J}$, so $\gamma_i = \gamma$ is independent of i. Now, by (4.9), $\gamma = 1$. This proves (4.7), and (4.8) follows because r was chosen so that $\Sigma\Sigma b_i^r a_{ij}^{\delta-r} = 1$. □

4.4 Proposition

If there exists a constant $\theta \in \mathbb{R}$ such that

$$\sum_{j=1}^{n_i} a_{ij}^\theta = 1\ \ \forall\ i = 1, 2, \ldots, m$$

then $\delta = \delta_B$ *and* $0 < H_\delta(\Lambda) < \infty$.

PROOF: Let r be the unique real number such that $\Sigma_{i=1}^m b_i^r = 1$. Then $\Sigma_{i=1}^m \Sigma_{j=1}^{n_i} a_{ij}^\theta b_i^r = 1$, so by Th. 2.3 $\theta + r = \delta_B$. Define $p \in \Sigma$ by

$$p_{ij} = a_{ij}^\theta b_i^r, \quad (i,j) \in \mathcal{J}.$$

Recall the approximate square $B_k(\omega)$ defined in (1.2), and recall that in computing Hausdorff measures of Λ coverings by sets $\pi(B_k(\omega))$ may be used in place of coverings by open balls. By (1.3)–(1.4), there exist constants $0 < C_1' \le C_2' < \infty$ such that $\forall\, k = 1, 2, \ldots$ and $\forall\, \omega \in \Omega$

$$C_1' \le \frac{\text{diam } (\pi(B_k(\omega)))^{\theta+r}}{\left(\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}\right)^\theta \left(\Pi_{\nu=1}^k b_{i_\nu}\right)^r} \le C_2',$$

which implies

$$C_1' \le \frac{\text{diam } (\pi(B_k(\omega)))^{\theta+r}}{\tilde{\mu}_p(\pi(B_k(\omega)))} \le C_2'$$

(since $\tilde{\mu}_p(\pi(B_k(\omega))) = \mu_p(B_k(\omega)) = \Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}^\theta \Pi_{\nu=1}^k b_i^r$, by (1.2) and (4.4)).

Now consider any covering $\{F_1, F_2, \ldots\}$ of Λ by sets F_i of the form $\pi(B_k(\omega))$. By the preceding inequality, together with the fact that $\tilde{\mu}_p$ is a probability measure on Λ, $\Sigma\text{diam } (F_n)^{\theta+r} \ge C_1' > 0$. Therefore, $H_{\theta+r}(\Lambda) > 0$.

Next let $\mathcal{F}_k = \{B_k(\omega) : \omega \in \Omega\}$ be the collection of all k^{th} generation approximate squares; observe that each $\mathcal{F}_k$ is a (finite) covering of Ω. Recall that distinct approximate squares are either disjoint or one is properly contained in the other. Let $\mathcal{G}_k = \{B \in \mathcal{F}_k : B$ is not properly contained in any other $B' \in \mathcal{F}_k\}$; then $\mathcal{G}_k$ is a covering of Ω. For any $\pi(B) \in \pi\mathcal{G}_k$,

$$\text{diam } (\pi(B)) \le C_2 b^k$$

where $b = \max(b_1, b_2, \ldots, b_m) < 1$, by (1.4). By taking k large, we can make $C_2 b^k$ arbitrarily small. But $\forall\, k \ge 1$

$$\sum_{B \in \mathcal{G}_k} \text{diam } (\pi(B))^{\theta+r} \le C_2' \sum_{B \in \mathcal{G}_k} \tilde{\mu}_p(\pi(B)) \le C_2'.$$

Thus

$$H_{\theta+r}(\Lambda) \le C_2' < \infty. \qquad \square$$

4.5 Proposition

Let r *be the unique real number such that* $\Sigma_{i=1}^m b_i^r = 1$. *If* $\delta = \delta_B$ *then*

(4.10) $$\sum_{j=1}^{n_i} a_{ij}^{\delta - r} = 1 \quad \forall\, i = 1, 2, \ldots, m.$$

PROOF: Since $\delta = \delta_B$, Th. 2.3 implies that $\Sigma_{i=1}^{m} \Sigma_{j=1}^{n_i} b_i^r a_{ij}^{\delta - r} = 1$. Define $\gamma_i = \Sigma_{j=1}^{n_i} a_{ij}^{\delta-r}$; then $\Sigma b_i^r \gamma_i = 1 = \Sigma b_i^r$. We will show that if $\gamma_i \neq 1$ for some i, then for some $\varepsilon > 0$

$$H_{\delta-\varepsilon}(\Lambda) < \infty,$$

contradicting the fact that $\delta = \delta_H$ is the Hausdorff-Besicovich dimension of Λ.

So assume that $\gamma_i \neq 1$ for some i. For some $\theta \in [0,1]$, define

$$\varphi(\theta) = \sum_{i=1}^{m} b_i \gamma_i^{1-\theta}.$$

Observe that $\varphi(0) = \varphi(1) = 1$ and that $\varphi''(\theta) = \Sigma_i b_i \gamma_i^{1-\theta} (\log \gamma_i)^2 > 0$, since $\log \gamma_i \neq 0$ for at least one i. Consequently,

$$0 < \varphi(\theta) < 1 \quad \forall\, \theta \in (0,1).$$

Next, define for each $\theta \in [0,1]$ a probability vector $\boldsymbol{p}^\theta \in \Sigma$ by

$$p_{ij}^\theta = b_i^r a_{ij}^{\delta-r} \gamma_i^{-\theta} / \varphi(\theta), \quad (i,j) \in \mathcal{J}.$$

Write μ_θ and $\tilde{\mu}_\theta$ in place of $\mu_p\theta$ and $\tilde{\mu}_p\theta$. The measures μ_θ form a one-parameter exponential family of probability measures on Ω; the assumption that $\gamma_i \neq 1$ for some i assures that μ_θ and $\mu_{\theta'}$ are distinct (in fact mutually singular) when $\theta \neq \theta'$. For any approximate square $B_k(\omega)$ (see (1.2)),

$$\mu_\theta(B_k(\omega)) = \varphi(\theta)^{-k} \left(\prod_{\nu=1}^{k} b_{i_\nu} \right)^r \left(\prod_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu} \right)^{\delta - r} \frac{(\Pi_{\nu=1}^{k} \gamma_{i_\nu})^{1-\theta}}{\Pi_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}}.$$

But by (1.3)–(1.4), there exist constants $0 < C_1' \leq C_2' < \infty$ such that for every $k = 1, 2, \ldots$ and every $\omega \in \Omega$,

$$C_1' \leq \frac{\operatorname{diam}\,(\pi(B_k(\omega)))^\delta}{\left(\Pi_{\nu=1}^{k} b_{i_\nu}\right)^r \left(\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}\right)^{\delta - r}} \leq C_2';$$

hence $\forall\, \theta \in [0,1]$,

$$\operatorname{diam}\,(\pi(B_k(\omega)))^\delta \leq C_2' \varphi(\theta)^k \mu_\theta(B_k(\omega)) \frac{\Pi_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}}{\left(\Pi_{\nu=1}^{k} \gamma_{i_\nu}\right)^{1-\theta}}. \tag{4.11}$$

We will use this inequality along with the fact that $\varphi(\theta) < 1 \ \ \forall\, \theta \in (0,1)$ to produce efficient coverings of Λ by π-images of approximate squares.

Recall that the ratios $a_{ij}/b_i \in (0,1)$ for all $(i,j) \in \mathcal{G}$. Hence, by (3.3), the ratios $L_k(\omega)/k$ are bounded away from 0 and 1, at least for large k; i.e., there exist constants $0 < \theta_2 < \theta_1 < 1$ such that $\forall\, \omega \in \Omega$ and

$$1 - \theta_1 \leq \liminf_{k \to \infty} \frac{L_k(\omega)}{k} \leq \limsup_{k \to \infty} \frac{L_k(\omega)}{k} \leq 1 - \theta_2.$$

Now for any $\omega \in \Omega$, it must be the case that

$$\liminf_{k\to\infty} \left\{ \frac{\Pi_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}}{\left(\Pi_{\nu=1}^{k} \gamma_{i_\nu}\right)^{1-\theta_1}} \right\}^{1/k} \leq 1 \tag{4.12}$$

or

$$\liminf_{k\to\infty} \left\{ \frac{\Pi_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}}{\left(\Pi_{\nu=1}^{k} \gamma_{i_\nu}\right)^{1-\theta_2}} \right\}^{1/k} \leq 1 . \tag{4.13}$$

This may be seen by taking a sequence $k_1 < k_2 < \dots$ of integers such that, with $\ell_n = L_{k_n}(\omega)$,

$$\begin{aligned} &\ell_n/k_n \to \theta \in [1-\theta_1, 1-\theta_2], \\ &\ell_n^{-1}\Sigma_{\nu=1}^{\ell_n} \log \gamma_{i_\nu} \to \lambda \leq \limsup_{k\to\infty} k^{-1}\Sigma_{\nu=1}^{k} \log \gamma_{i_\nu}, \\ &k_n^{-1}\Sigma_{\nu=1}^{k_n} \log \gamma_{i_\nu} \to \limsup_{k\to\infty} k^{-1}\Sigma_{\nu=1}^{k} \log \gamma_{i_\nu}; \end{aligned}$$

one or the other of (4.12)–(4.13) will hold, depending on whether lim sup $k^{-1}\Sigma_{\nu=1}^{k} \log \gamma_{i_\nu}$ is negative or nonnegative.

Since $0 < \theta_2 < \theta_1 < 1$ it must be that $\varphi(\theta_1) < 1$ and $\varphi(\theta_2) < 1$. Set $b = \min_{1\leq i\leq m} b_i$; then $0 < b < 1$ and there exists $\varepsilon > 0$ such that $b^{3\varepsilon} > \varphi(\theta_i)$ for $i = 1$ and 2. For each $k = 1, 2, \dots$ let $\mathcal{G}_k$ be the set consisting of those $\pi(B_k(\omega))$, $\omega \in \Omega$, for which

$$\frac{\Pi_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}}{\left(\Pi_{\nu=1}^{k} \gamma_{i_\nu}\right)^{1-\theta_1}} \leq b^{-\varepsilon k}$$

or

$$\frac{\Pi_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}}{\left(\Pi_{\nu=1}^{k} \gamma_{i_\nu}\right)^{1-\theta_2}} \leq b^{-\varepsilon k}$$

and for which there is no $\omega' \in \Omega\backslash\{\omega\}$ such that $B_k(\omega)$ is properly contained in $B_k(\omega')$ (recall that $\forall\, \omega, \omega' \in \Omega, B_k(\omega) \cap B_k(\omega') = \varnothing$ or one of $B_k(\omega), B_k(\omega')$ is contained in the other). By (4.12)–(4.13), $\cup_{k=N}^{\infty}\mathcal{G}_k$ is a covering of Λ for each $N = 1, 2, \dots$.

Let $\pi(B_k(\omega)) \in \mathcal{G}_k$; by (1.4), diam $(\pi(B_k(\omega)))^{-\varepsilon} \leq (C_1 b^k)^{-\varepsilon}$. Consequently, by (4.11) (recall that $\varphi(\theta_i) < b^{3\varepsilon}$)

$$\text{diam } (\pi(B_k(\omega)))^{\delta-\varepsilon} \leq \tilde{C} b^{\varepsilon k} \mu_{\theta_i}(B_k(\omega))$$

for $i = 1$ or 2 and a suitable constant $\tilde{C} < \infty$. Now for any two distinct $\pi(B_k(\omega)), \pi(B_k(\omega'))$ $\in \mathcal{G}_k, B_k(\omega) \cap B_k(\omega') = \varnothing$; since μ_{θ_i} is a probability measure, it follows that for each $k = 1, 2, \dots$

$$\begin{aligned} &\sum_{\mathcal{G}_k} \text{diam } (\pi(B_k(\omega)))^{\delta-\varepsilon} \leq 2\tilde{C} b^{\varepsilon k} \\ \Longrightarrow\ &\sum_{k=N}^{\infty} \sum_{\mathcal{G}_k} \text{diam } (\pi(B_k(\omega)))^{\delta-\varepsilon} \leq 2\tilde{C}/(1-b^{\varepsilon}) < \infty \\ \Longrightarrow\ &H_{\delta-\varepsilon}(\Lambda) \leq 2\tilde{C}/(1-b^{\varepsilon}) < \infty. \end{aligned}$$

□

In summary, we have proved

4.6 Theorem

The following are equivalent:

$$0 < H_\delta(\Lambda) < \infty;$$
$$\delta = \delta_B;$$
$$\Sigma_{j=1}^{n_i} a_{ij}^{\delta - r} = 1 \ \ \forall\, i = 1, 2, \dots, m.$$

Here r is the unique real number such that $\Sigma_{i=1}^{m} b_i^r = 1$.

5. The Hausdorff-Besicovich Dimension

In this section we complete the computation of the Hausdorff-Besicovich dimension δ_H of Λ. Recall from section 3 that

$$\delta_H \geq \max_{\boldsymbol{p}\in\Sigma} D(\boldsymbol{p}). \tag{5.1}$$

Here we will show that the reverse inequality is also true. For ease of notation we will set $\delta = \max_{\boldsymbol{p}\in\Sigma} D(\boldsymbol{p})$ throughout this section.

We start by considering a 3-parameter family of probability measures on $\mathcal{J}$. For $\theta \in \mathbb{R}, \lambda \in \mathbb{R}$ and $\rho \in (0,1)$ define a probability vector $\boldsymbol{p}(\theta, \lambda, \rho) \in \Sigma$, by defining

$$p_{ij}(\theta, \lambda, \rho) = C(\theta, \lambda, \rho) a_{ij}^{\theta} b_i^{\lambda - \theta} (\gamma_i(\theta))^{\rho - 1}, \ \ (i, j) \in \mathcal{J}, \tag{5.2}$$

where

$$\gamma_i(\theta) = \sum_{j=1}^{n_i} a_{ij}^{\theta}, \ \ i = 1, \dots, m \tag{5.3}$$

and

$$C(\theta, \lambda, \rho) = \left[\sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij}^{\theta} b_i^{\lambda - \theta} (\gamma_i(\theta))^{\rho - 1} \right]^{-1}. \tag{5.4}$$

Write $\mu_{\theta,\lambda,\rho}$ and $\tilde{\mu}_{\theta,\lambda,\rho}$ in place of $\mu_{\boldsymbol{p}(\theta,\lambda,\rho)}$ and $\tilde{\mu}_{\boldsymbol{p}(\theta,\lambda,\rho)}$, respectively. We will construct an efficient cover of Λ with the aid of the measures $\mu_{\theta,\lambda,\rho}$.

5.1 Lemma

There exists a real-valued continuous function $\theta(\rho), \rho \in (0,1)$, such that for every $\rho \in (0,1)$,

$$C(\theta(\rho), \delta, \rho) = 1 \tag{5.5}$$

PROOF: There are two distinct cases, each requiring a separate argument.

CASE 1: *Assume that there is no* $\eta \in (0,1)$ *such that* $a_{ij}^{\eta} = b_i \ \ \forall\, (i,j) \in \mathcal{J}$.

First, we show that for each $\rho \in (0,1)$ there exist $\theta \in \mathbb{R}$ and $\lambda \leq \delta$ such that $C(\theta,\lambda,\rho) = 1$. For this it suffices to show that for each $\rho \in (0,1)$ there exist $\theta, \lambda \in \mathbb{R}$ such that $C(\theta,\lambda,\rho) = 1$ and $D(\boldsymbol{p}(\theta,\lambda,\rho)) = \lambda$, because then λ must be $\leq \delta$.

Fix $\rho \in (0,1)$ and $\theta \in \mathbb{R}$. The function $C(\theta,\lambda,\rho)$ is strictly increasing and continuous in λ, with limit $+\infty$ as $\lambda \to +\infty$ and limit 0 as $\lambda \to -\infty$ (since $0 < b_i < 1$). Consequently, by the intermediate value theorem, there is a *unique* $\lambda = \lambda(\theta,\rho) \in \mathbb{R}$ such that $C(\theta,\lambda,\rho) = 1$. Moreover, $\lambda(\theta,\rho)$ is jointly continuous in θ,ρ since $C(\theta,\lambda,\rho)$ is jointly continuous in θ,λ,ρ and strictly increasing in λ.

By Prop. 3.3,

$$D(\boldsymbol{p}(\theta,\lambda,\rho)) = \lambda + \frac{\log C(\theta,\lambda,\rho)}{b(\theta,\lambda,\rho)} + \frac{\gamma(\theta,\lambda,\rho)}{b(\theta,\lambda,\rho)}\left\{\rho - \frac{b(\theta,\lambda,\rho)}{a(\theta,\lambda,\rho)}\right\} \tag{5.6}$$

where

$$a(\theta,\lambda,\rho) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}(\theta,\lambda,\rho)\log a_{ij},$$

$$b(\theta,\lambda,\rho) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}(\theta,\lambda,\rho)\log b_i,$$

$$\gamma(\theta,\lambda,\rho) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}(\theta,\lambda,\rho)\log \gamma_i(\theta).$$

Note that each of these is jointly continuous in θ,λ,ρ; hence, $\gamma(\theta,\lambda(\theta,\rho),\rho)$ is jointly continuous in θ,ρ. Now $\gamma(\theta,\lambda,\rho)$ is a weighted average of the functions $\gamma_i(\theta)$ defined by (5.3), and clearly $\gamma_i(\theta) \to 0$ as $\theta \to +\infty$ and $\gamma_i(\theta) \to \infty$ as $\theta \to -\infty$, since $0 < a_{ij} < 1$. Consequently, for each fixed $\rho \in (0,1)$,

$$\lim_{\theta\to\infty} \gamma(\theta,\lambda(\theta,\rho),\rho) = -\infty, \quad \lim_{\theta\to-\infty} \gamma(\theta,\lambda(\theta,\rho),\rho) = \infty.$$

By the intermediate value theorem, for each $\rho \in (0,1)$ there exists $\theta \in \mathbb{R}$ such that $\gamma(\theta,\lambda(\theta,\rho),\rho) = 0$, and therefore $C(\theta,\lambda(\theta,\rho),\rho) = 1$ and $D(\theta,\lambda(\theta,\rho),\rho) = \lambda$, by (5.6). This proves that for each $\rho \in (0,1)$ there exist $\lambda \leq \delta$ and $\theta \in \mathbb{R}$ such that $C(\theta,\lambda,\rho) = 1$.

Second, we show that there is a *continuous* function $\theta(\rho)$ satisfying (5.5). It is here that we use the assumption that there is no $\eta \in (0,1)$ such that $a_{ij}^{\eta} = b_i \ \ \forall\, (i,j) \in \mathcal{J}$. Define

$$F_{\lambda,\rho}(\theta) = C(\theta,\lambda,\rho)^{-1} = \sum_{i=1}^{m} b_i^{\lambda-\theta}\gamma_i(\theta)^{\rho};$$

then

$$F''_{\lambda,\rho}(\theta) = \sum_{i=1}^{m}\left\{\left[-\log b_i + \rho\frac{\gamma_i'(\theta)}{\gamma_i(\theta)}\right]^2 + \rho\left[\frac{\gamma_i''(\theta)}{\gamma_i(\theta)} - \frac{\gamma_i'(\theta)^2}{\gamma_i(\theta)^2}\right]\right\} b_i^{\lambda-\theta}\gamma_i(\theta)^{\rho}$$

and

$$\gamma_i'(\theta) = \sum_{j=1}^{n_i} a_{ij}^{\theta}\log a_{ij},$$

$$\gamma_i''(\theta) = \sum_{j=1}^{n_i} a_{ij}^{\theta}(\log a_{ij})^2.$$

By the Cauchy-Schwartz inequality, $\gamma_i''\gamma_i - \gamma_i'^2 > 0$ unless $\log a_{ij} = \log a_i \;\; \forall\, j$, and in this case $\gamma_i'/\gamma_i \equiv \log a_i$, so $F''_{\lambda,\rho}(\theta) > 0$ unless $\rho \log a_{ij} = \log b_i \;\; \forall\, (i,j) \in \mathcal{J}$. But we have assumed that this is not the case; consequently, for each pair $(\lambda, \rho) \in \mathbf{R} \times (0,1)$ the function $F_{\lambda,\rho}(\theta)$ is strictly convex in θ.

Now $F_{\lambda,\rho}(\theta)$ is strictly decreasing in λ. In the first part of the proof we showed that for each ρ there exists $\lambda \leq \delta$ such that $F_{\lambda,\rho}$ attains the value 1; it therefore follows that for each ρ the function $F_{\delta,\rho}$ attains a value ≤ 1. But $F_{\delta,\rho}(\theta)$ is strictly convex for $\theta \in \mathbf{R}$, so $F_{\delta,\rho}(\theta) \to \infty$ either as $\theta \to \infty$ or as $\theta \to -\infty$, and hence $F_{\delta,\rho}(\theta) = 1$ has either one or two solutions θ. Define $\theta(\rho)$ to be the larger of these two solutions; then $F'_{\delta,\rho}(\theta(\rho)) \geq 0$. If $F'_{\delta,\rho}(\theta(\rho)) > 0$ then $\theta(\cdot)$ is continuous at ρ, by the implicit function theorem. If $F'_{\delta,\rho}(\theta(\rho)) = 0$ then for each $\varepsilon > 0$,

$$
\begin{aligned}
F_{\delta,\rho}(\theta(\rho) - \varepsilon) &> 1, \\
F_{\delta,\rho}(\theta(\rho) + \varepsilon) &> 1, \\
F'_{\delta,\rho}(\theta(\rho) - \varepsilon) &< 0, \\
F'_{\delta,\rho}(\theta(\rho) + \varepsilon) &> 0,
\end{aligned}
$$

since $F''_{\delta,\rho} > 0$. Since $F_{\delta,\rho}(\theta)$ and $F'_{\delta,\rho}(\theta)$ are continuous in ρ, it follows that for $\tilde{\rho}$ sufficiently near ρ

$$
\begin{aligned}
F_{\delta,\tilde{\rho}}(\theta(\rho) - \varepsilon) &> 1, \\
F_{\delta,\tilde{\rho}}(\theta(\rho) + \varepsilon) &> 1, \\
F'_{\delta,\tilde{\rho}}(\theta(\rho) - \varepsilon) &< 0, \\
F'_{\delta,\tilde{\rho}}(\theta(\rho) + \varepsilon) &> 0,
\end{aligned}
$$

and therefore any solution θ of $F_{\delta,\tilde{\rho}}(\theta) = 1$ must lie between $\theta(\rho) - \varepsilon$ and $\theta(\rho) + \varepsilon$, by convexity. This proves that $\theta(\cdot)$ is continuous at ρ. □

CASE 2: *Assume that there exists $\eta \in (0,1)$ such that $a_{ij}^{\eta} = b_i$ for all $(i,j) \in \mathcal{J}$.*

The argument used in Case 1 fails here because $F_{\lambda,\rho}(\theta)$ is no longer strictly convex in θ when $\rho = \eta$. In fact, the family $\{\boldsymbol{p}(\theta, \lambda, \rho)\}$ of probability measures on $\mathcal{J}$ is over-parametrized in this case: in particular, $\boldsymbol{p}(\theta, \lambda, \eta) = \boldsymbol{p}(0, \lambda, \eta)$ for all $\theta, \lambda \in \mathbf{R}$, because

$$
a_{ij}^{\theta} b_i^{\lambda-\theta} \gamma_i(\theta)^{\eta-1} = b_i^{\theta/\eta} b_i^{\lambda-\theta} (n_i b_i^{\theta/\eta})^{\eta-1} = b_i^{\lambda} n_i^{\eta-1} = b_i^{\lambda} \gamma_i(0)^{\eta-1},
$$

and so $F_{\lambda,\eta}(\theta) = C(\theta, \lambda, \eta)^{-1}$ is *constant* in θ.

Consider a probability vector $\boldsymbol{p}^*$ which maximizes $D(\boldsymbol{p})$. By Prop. 3.4, $\boldsymbol{p}^* = \boldsymbol{p}(\theta, \lambda + \theta, \rho)$ for some triple (θ, λ, ρ) satisfying (3.8). But the equations (3.8) imply that $\rho = \eta$ (since $a_{ij}^{\eta} = b_i$) and $\lambda + \theta = \delta$ (this is always the case – compare (3.8) with (3.3)). Consequently, by the previous paragraph, $\boldsymbol{p}^* = \boldsymbol{p}(0, \delta, \eta)$. Now (5.6) implies that

$$
\delta = D(\boldsymbol{p}(0, \delta, \eta)) = \delta + \frac{\log C(0, \delta, \eta)}{b(0, \delta, \eta)}
$$

because $b(\theta,\lambda,\rho)/a(\theta,\lambda,\rho) \equiv \eta$. Therefore, $C(0,\delta,\eta) = 1$, i.e.,

$$\sum_{i=1}^{m} b_i^{\delta} n_i^{\eta} = 1. \tag{5.7}$$

This implies that $F_{\delta,\eta}(\theta) = 1$ for all $\theta \in \mathbf{R}$.

Since $0 < b_i < 1$ for each $i = 1,\ldots,m$, there exists, for each $\rho \neq \eta$, a unique $\theta = \theta(\rho)$ such that

$$F_{\delta,\rho}(\theta) = \sum_{i=1}^{m} b_i^{\delta-\theta(1-\rho/\eta)} n_i^{\rho} = 1;$$

moreover, (5.7) guarantees that $\theta(1-\rho/\eta) \to 0$ as $\rho \to \eta$. The implicit function theorem implies that $\theta = \theta(\rho)$ is continuous and differentiable at each $\rho \neq \eta$, and

$$\frac{d\theta}{d\rho} = (1-\rho/\eta)^{-1} \frac{\Sigma_{i=1}^{m}\{\log n_i + (\theta/\rho)\log b_i\} b_i^{\delta-\theta(1-\rho/\eta)} n_i^{\rho}}{\Sigma_{i=1}^{m}\{\log b_i\} b_i^{\delta-\theta(1-\rho/\eta)} n_i^{\rho}}.$$

Consequently,

$$\lim_{\rho\to\eta} \theta(\rho) = -\eta \frac{\Sigma_{i=1}^{m}(\log n_i) b_i^{\delta} n_i^{\eta}}{\Sigma_{i=1}^{m}(\log b_i) b_i^{\delta} n_i^{\eta}},$$

because if this were not the case it would be impossible for $\theta(1-\rho/\eta) \to 0$ as $\rho \to \eta$, in view of the differential equation. Thus, we may define $\theta(\eta) = \lim_{\rho\to\eta}\theta(\rho)$, making $\theta(\rho)$ continuous at every $\rho \in (0,1)$. Since $F_{\delta,\eta}(\theta) = 1 \;\; \forall\, \theta \in \mathbf{R}$, it follows that (5.5) holds for every $\rho \in (0,1)$. □

5.2 Lemma

For each $\omega \in \Omega$, there exists a triple $(\theta,\lambda,\rho) \in \mathbf{R}\times\mathbf{R}\times(0,1)$ such that $C(\theta,\lambda,\rho) = 1$ and

$$\limsup_{k\to\infty} \left\{(b_{i_1}\ldots b_{i_k})^{-\delta} \mu_{\theta,\lambda,\rho}(B_k(\omega))\right\}^{1/k} \geq 1, \tag{5.8}$$

where $\delta = \max_{\mathbf{p}\in\Sigma} D(\mathbf{p})$ and $B_k(\omega)$ is defined by (1.2).

PROOF: Fix $\omega \in \Omega$. Then for any $(\theta,\lambda,\rho) \in \mathbf{R}\times\mathbf{R}\times(0,1)$ and $k \in \mathbf{N}$,

$$\mu_{\theta,\lambda,\rho}(B_k(\omega)) = (C(\theta,\lambda,\rho))^k \left(\prod_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}\right)^{\theta} \frac{(\Pi_{\nu=1}^{k}\gamma_{i_\nu}(\theta))^{\rho}}{\Pi_{\nu=1}^{L_k(\omega)}\gamma_{i_\nu}(\theta)} \left(\prod_{\nu=1}^{k} b_{i_\nu}\right)^{\lambda-\theta}. \tag{5.9}$$

Set $\lambda = \delta$. By Lemma 5.1, for each $\rho \in (0,1)$ there exists $\theta = \theta(\rho)$ varying continuously with ρ such that $C(\theta,\delta,\rho) = 1$. For any such pair ρ,θ, equation (5.9) may be rewritten as follows:

$$\begin{aligned} &\left\{(b_{i_1}\ldots b_{i_k})^{-\delta}\mu_{\theta,\delta,\rho}(B_k(\omega))\right\}^{1/k} \\ &= \left(\frac{\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}}{\Pi_{\nu=1}^{k} b_{i_\nu}}\right)^{\theta/k} \exp\left\{\rho\frac{1}{k}\sum_{\nu=1}^{k}\log\gamma_{i_\nu}(\theta) - \frac{1}{k}\sum_{\nu=1}^{L_k(\omega)}\log\gamma_{i_\nu}(\theta)\right\}. \end{aligned}$$

Since by (1.3)

$$\lim_{k\to\infty}\left(\frac{\Pi_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu}}{\Pi_{\nu=1}^{k} b_{i_\nu}}\right)^{1/k} = 1,$$

it suffices to show that there exists $\rho \in (0,1)$ such that for $\theta = \theta(\rho)$,

$$\limsup_{k\to\infty} \exp\left\{\rho\frac{1}{k}\sum_{\nu=1}^{k}\log\gamma_{i_\nu}(\theta) - \frac{1}{k}\sum_{\nu=1}^{L_k(\omega)}\log\gamma_{i_\nu}(\theta)\right\} \geq 1. \tag{5.10}$$

Define

$$\rho_0 = \lim\ \inf_{k\to\infty} L_k(\omega)/k,$$
$$\rho_1 = \lim\ \sup_{k\to\infty} L_k(\omega)/k.$$

Observe that $0 < \rho_0 \leq \rho_1 < 1$. Recall (Lemma 5.1) that $\theta = \theta(\rho)$ is a continuous function of ρ, hence so is $\gamma_i(\theta(\rho))$ for each $i = 1, \dots, m$, and therefore also are

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\log\gamma_{i_\nu}(\theta(\rho))$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\log\gamma_{i_\nu}(\theta(\rho)).$$

Consequently, by the intermediate value theorem, at least one of the following must be true:

$$(i)\ \limsup_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\log\gamma_{i_\nu}(\theta(\rho_1)) > 0;$$
$$(ii)\ \liminf_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\log\gamma_{i_\nu}(\theta(\rho_0)) < 0;$$
$$(iii)\ \limsup_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\log\gamma_{i_\nu}(\theta(\rho_*)) = 0 \text{ for some } \rho_0 \leq \rho_* \leq \rho_1.$$

In case (i), (5.10) holds with $\rho = \rho_1$ and $\theta = \theta(\rho_1)$; in case (ii); (5.10) holds with $\rho = \rho_0$ and $\theta = \theta(\rho_0)$; and in case (iii), (5.10) holds with $\rho = \rho_*$ and $\theta = \theta(\rho_*)$. For instance, consider case (i) (the other two cases are similar). Set $\rho = \rho_1$ and $\theta = \theta(\rho_1)$, and choose a sequence $k_1 < k_2 < \dots$ such that

$$\lim_{j\to\infty}\frac{1}{k_j}\sum_{\nu=1}^{k_j}\log\gamma_{i_\nu}(\theta) = \limsup_{k\to\infty}\frac{1}{k}\sum_{\nu=1}^{k}\log\gamma_{i_\nu}(\theta) > 0$$

and such that both $\{k^{-1}L_k(\omega)\}$ and $\{(L_k(\omega))^{-1}\Sigma_{\nu=1}^{L_k(\omega)}\log\gamma_{i_\nu}(\theta)\}$ converge along the subsequence $\{k_j\}$. Set $l_j = L_{k_j}(\omega), j = 1, 2, \dots$; then

$$\lim_{j\to\infty}\left\{\rho\frac{1}{k_j}\sum_{\nu=1}^{k_j}\log\gamma_{i_\nu}(\theta) - \frac{l_j}{k_j}\cdot\frac{1}{l_j}\sum_{\nu=1}^{l_j}\log\gamma_{i_\nu}(\theta)\right\} \geq 0,$$

since $\rho = \lim\ \sup_{k\to\infty} k^{-1}L_k(\omega) \geq \lim_{j\to\infty} k_j^{-1}l_j$. This implies (5.10). □

5.3 Theorem

$\delta_H = \delta \left(= \max_{p \in \Sigma} D(p) \right).$

PROOF: By (5.1) we need only show that $\delta_H \leq \delta$. Fix $\varepsilon > 0$ and $\eta > 0$. We will construct a covering $\mathcal{U} = \mathcal{U}_{\varepsilon,\eta}$ of Λ consisting of π-images of approximate squares $B_k(\omega)$ such that each element of $\mathcal{U}$ has diameter $\leq \eta$ and

$$\sum_{U \in \mathcal{U}} (\text{diam } U)^{\delta+3\varepsilon} < C_\varepsilon$$

for a constant $C_\varepsilon < \infty$ depending on ε but not η. This will imply that $H_{\delta+3\varepsilon}(\Lambda) < \infty \;\; \forall\, \varepsilon > 0$, which in turn will prove that $\delta_H \leq \delta$.

Let $F_1 \subset F_2 \subset \ldots$ be a sequence of compact subsets of $\mathbb{R} \times \mathbb{R} \times (0,1)$ such that $\cup_{n=1}^{\infty} F_n = \mathbb{R} \times \mathbb{R} \times (0,1)$. For each $k = 1, 2, \ldots$ let $\mathcal{G}_k$ be the collection of all $\pi(B_k(\omega)), \omega \in \Omega$, such that $B_k(\omega)$ is not properly contained in any other $B_k(\omega')$ and such that

$$\left(\prod_{\nu=1}^{k} b_{i_\nu} \right)^{-\delta} \mu_{\theta,\lambda,\rho}(B_k(\omega)) \geq (\max_i b_i)^{\varepsilon k} \tag{5.11}$$

for some $(\theta, \lambda, \rho) \in F_k$ satisfying $C(\theta, \lambda, \rho) = 1$. (Recall that for distinct $\omega, \omega' \in \Omega$ either $B_k(\omega)$ and $B_k(\omega')$ are disjoint or one is contained in the other.) By Lemma 5.2, $\mathcal{U}_K = \cup_{k=K}^{\infty} \mathcal{G}_k$ is a covering of Λ for each $K = 1, 2, \ldots$. By (1.4) there is a constant $C < \infty$ such that for any approximate square $B_k(\omega)$,

$$\text{diam } (\pi(B_k(\omega))) \leq C \prod_{\nu=1}^{k} b_{i_\nu}. \tag{5.12}$$

Since $\max_i b_i < 1$, it follows that for sufficiently large K all elements of $\mathcal{U}_K$ have diameter $\leq \eta$. Thus, to complete the proof it suffices to show that for some choice of the sets F_k

$$\sum_{k=1}^{\infty} \sum_{\mathcal{G}_k} \text{diam } (\pi(B_k(\omega)))^{\delta+3\varepsilon} < \infty. \tag{5.13}$$

Choose $F_1 \subset F_2 \subset \ldots$ so that $\cup_{k=1}^{\infty} F_k = \mathbb{R} \times \mathbb{R} \times (0,1)$ and so that each F_k has a finite subset $E_k = \{(\theta_i, \lambda_i, \rho_i) : i = 1, 2, \ldots, k\}$ of cardinality k with the following property: For each $(\theta, \lambda, \rho) \in F_k$ there exists $(\theta_i, \lambda_i, \rho_i) \in E_k$ such that $\forall\, \omega \in \Omega$,

$$\frac{\mu_{\theta,\lambda,\rho}(B_k(\omega))}{\mu_{\theta_i,\lambda_i,\rho_i}(B_k(\omega))} \leq (\max_i b_i)^{-\varepsilon k}.$$

That this is possible follows from the continuity of the functions $C(\theta, \lambda, \rho)$, and $\gamma_i(\theta)$, together with the formula

$$\mu_{\theta,\lambda,\rho}(B_k(\omega)) = C(\theta, \lambda, \rho)^k \left(\prod_{\nu=1}^{k} b_{i_\nu} \right)^{\lambda-\theta} \left(\prod_{\nu=1}^{L_k(\omega)} a_{i_\nu j_\nu} \right)^{\theta} \frac{\left(\prod_{\nu=1}^{k} \gamma_{i_\nu}(\theta) \right)^{\rho}}{\prod_{\nu=1}^{L_k(\omega)} \gamma_{i_\nu}(\theta)},$$

valid $\forall\ \omega \in \Omega$ and $\forall\ (\theta, \lambda, \rho) \in \mathbf{R} \times \mathbf{R} \times (0,1)$. It now follows from (5.11)-(5.12) that for every $\pi(B_k(\omega)) \in \mathcal{G}_k$ there exists $(\theta_i, \lambda_i, \rho_i) \in E_k$ such that

$$\text{diam } (\pi(B_k(\omega)))^{\delta+3\varepsilon} \leq C\mu_{\theta_i,\lambda_i,\rho_i}(B_k(\omega))(\max_i b_i)^{\varepsilon k}.$$

Recall that each $\mu_{\theta,\lambda,\rho}$ is a probability measure on Ω and that distinct sets $\pi(B_k(\omega))$ in $\mathcal{G}_k$ are disjoint. Therefore,

$$\begin{aligned}
&\sum_{\mathcal{G}_k} \text{diam } (\pi(B_k(\omega)))^{\delta+3\varepsilon} \\
&\qquad \leq C(\max_i b_i)^{\varepsilon k} \sum_{i=1}^{k} \sum_{\mathcal{G}_k} \mu_{\theta_i,\lambda_i,\rho_i}(B_k(\omega)) \\
&\qquad \leq Ck(\max_i b_i)^{\varepsilon k};
\end{aligned}$$

since $(\max_i b_i) < 1$, this proves (5.13). □

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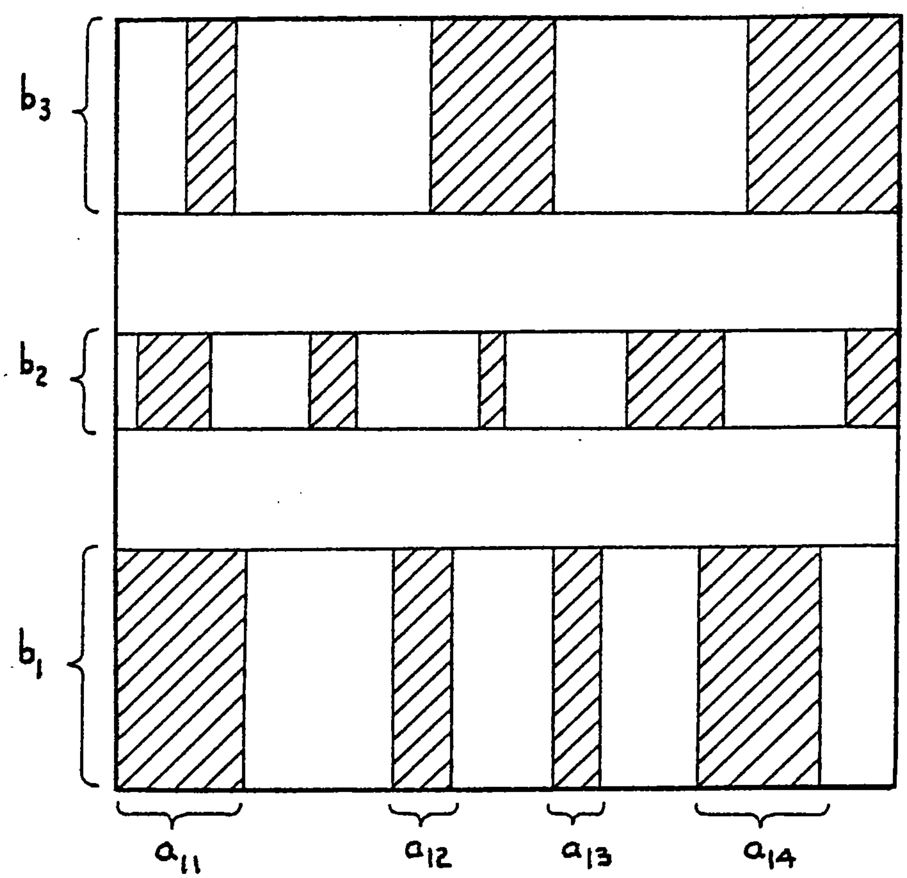

Figure 1

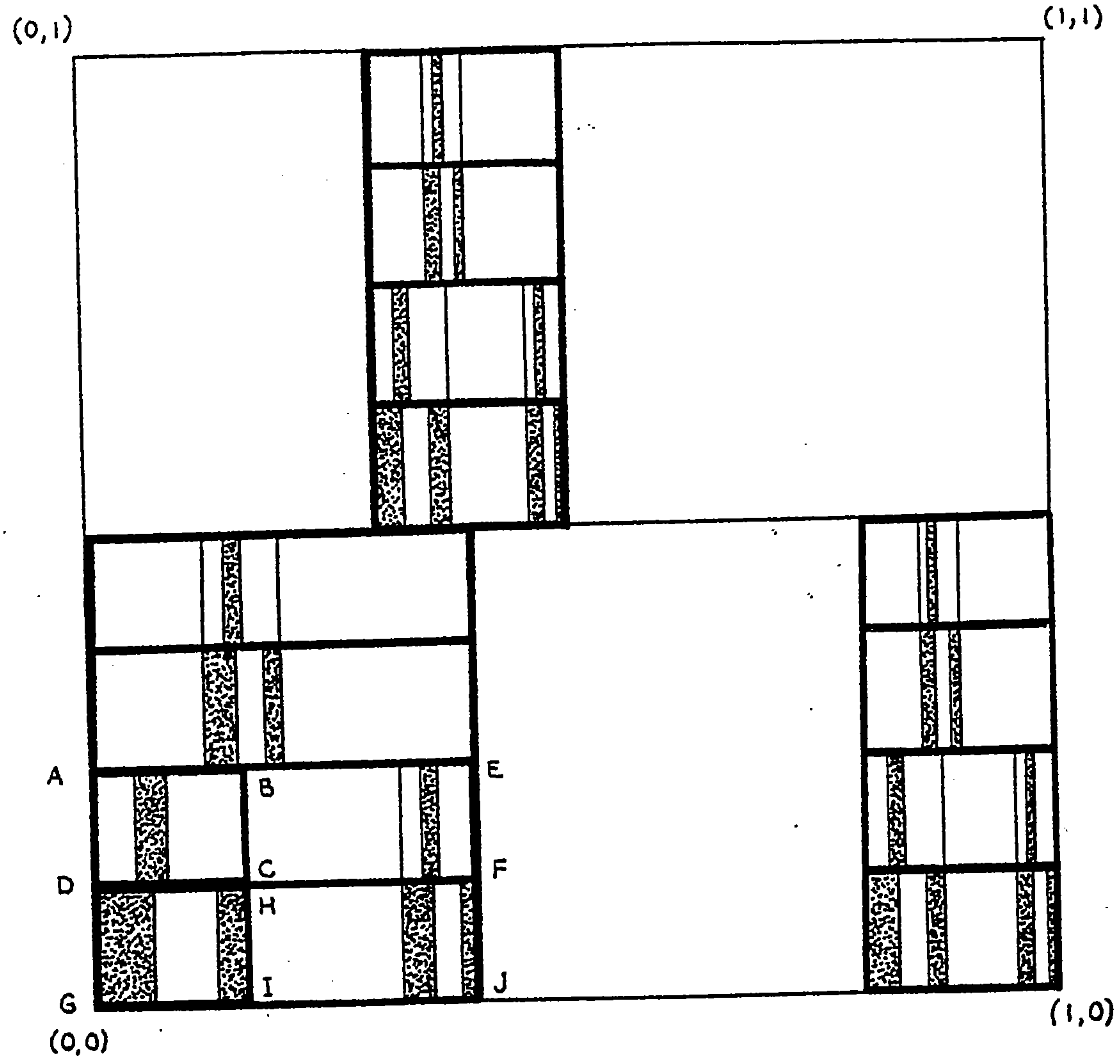

Figure 2: *In this example* $b_1 = b_2 = .5$*;* $a_{11} = .4, a_{12} = a_{21} = .2; c_{11} = 0, c_{12} = .8, c_{21} = .3; d_1 = 0, d_2 = .5$. *The shaded rectangles are the various* $R(z), z \in \mathcal{J}^3$ *(third generation);* Λ *is contained in their union. The rectangles surrounded by thick lines are the various* $\tilde{B}_3(\omega)$*; observe that each of* $ABCD, AEFD, DHIG$, *and* $DFJG$ *is a* $\tilde{B}_3(\omega)$.