Finite Range Random Walk on Free Groups and Homogeneous Trees

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ABSTRACT

Local limit theorems and saddlepoint approximations are given for random walks on a free group whose step distributions have finite support. The techniques used to prove these results are necessarily different from those used for random walks in Euclidean spaces, because Fourier analysis is not available; the basic tools are the elementary theory of algebraic functions and the Perron-Frobenius theory of nonnegative matrices. An application to the structure of the boundary process is also given.

1. Introduction

This paper concerns the asymptotic behavior as $n \to \infty$ of the convolution powers p^{*n} of a finitely supported probability measure p on a finitely generated free group $\mathcal{G}$. In probabilistic terms, this amounts to studying the transition probabilities of a homogeneous random walk on $\mathcal{G}$ with bounded step size. The main results are analogues of the local limit theorems and (sharp) large deviations theorems for random walks with bounded step size on the integer lattices $\mathbb{Z}^d$. Because the group $\mathcal{G}$ is nonamenable [KV], the character of these results is somewhat different than that of the corresponding results for Euclidean random walk; and since Fourier transforms are of no use in $\mathcal{G}$ ([FP] notwithstanding) the mathematical technique is considerably different. The methods and results of this paper extend easily to discrete groups whose Cayley graphs are trees, e.g., the free product of an arbitrary number (≥ 3) of copies of $\mathbb{Z}_2$, and to certain inhomogeneous random walks, e.g., the periodic random walks of [Ao], sec. 1, but in the interest of simplicity we shall only consider homogeneous random walks on free groups.

Let $\mathcal{G}$ be the free group with generators $a_1, a_2, \ldots, a_L$. Each $x \in \mathcal{G}$ has a unique representation as a finite reduced word $x = x_1 x_2 \ldots x_n$ from the alphabet $\mathcal{A}$ consisting of the letters $a_1, a_2, \ldots, a_L$ and their inverses. (reduced means that $x_{i+1} \neq x_i^{-1}$); we define $|x|$ to be the length n of the reduced word representing x. The group identity e is represented by the empty word, so $|e| = 0$. A *finite-range random walk* $\{Z_n\}_{n\geq 0}$ is a Markov chain on $\mathcal{G}$ with $Z_0 = e$ (unless otherwise specified) and transition probabilities

$$P\{Z_{n+1} = gx | Z_n = g\} = p_x \quad \forall\, x, g \in \mathcal{G}, n \geq 0,$$

where $p_x, x \in \mathcal{G}$, is a probability distribution on $\mathcal{G}$ with finite support. Thus,for some integer $K \geq 1$

$$(1.1) \qquad p_x > 0 \text{ only if } x \in \mathcal{B}$$

where $\mathcal{B} = \{x \in \mathcal{G} : |x| \leq K\}$.(Throughout the paper the symbols $\mathcal{G}, \mathcal{B}$, and $\mathcal{A}$ will have the meanings assigned here. We assume that the integer K is the *least* integer such that (1.1) holds.) The n-step transition probabilities $P\{Z_n = x\}$ will be denoted by $p^{*n}(x)$. We shall assume that the random walk is *irreducible* and *aperiodic,* i.e., that

$$(1.2) \qquad \sum_{n=1}^{\infty} p^{*n}(x) > 0 \quad \forall\, x \in \mathcal{G}$$

and

$$(1.3) \qquad GCD\{n \geq 1 : p^{*n}(e) > 0\} = 1$$

where GCD indicates the greatest common divisor of the set. In particular, we shall repeatedly make use of the simple consequence of (1.2) and (1.3) stated as Lemma 1.1 below.

Our first main result (Th. 4.1) describes the asymptotic comportment of $p^{*n}(x)$ as $n \to \infty$ for *fixed* $x \in \mathcal{G}$. This corresponds to the local limit theorem for random walk in $\mathbb{Z}^d$. Specifically, we show that there is a constant $1 < R < \infty$ (depending on the distribution $\{p_x\}$) and constants $B_x, x \in \mathcal{G}$, such that

$$p^{*n}(x) \sim \frac{B_x R^{-n}\sqrt{R}}{2\sqrt{\pi}\, n^{3/2}} . \tag{1.4}$$

Special cases of this have been proved before: [GW] for *nearest neighbor* random walk (the special case $K = 1$ of our result) and [Sa], [Pi], [FP] for *isotropic* random walk (the case $p_x = p_{|x|}$). The techniques used here differ considerably from those of [GW] and [Sa], etc. In the isotropic case the theory of spherical functions may be applied, and in the nearest neighbor case a relatively simple and explicit functional equation may be written for the Green's function; neither approach works in the general finite range case. Instead, arguments combining elements of algebraic function theory and the Perron-Frobenius theory of nonnegative matrices are used to determine the character of the smallest positive singularity of the Green's function (sections 2,3). Because algebraic methods are needed, the results are limited to finite range walks (but the author believes that a modification of the method may also apply in the infinite range case–this will be discussed in a subsequent paper).

NOTE: Professor T. Steger of the University of Chicago has informed the author that he also discovered a proof of (1.3) in the special case of *symmetric* random walk ($p_x = p_{x^{-1}}\ \forall\, |x| \leq K$) but has not written his proof down. It is apparently somewhat different from the proof given here.

The asymptotic relation (1.4) is clearly not uniform in $x \in \mathcal{G}$, as $p^{*n}(x) = 0$ if $|x| > nK$. In sec. 5-7 we study the behavior of $p^{*n}(x)$ as $n \to \infty$ for $|x|$ growing linearly with n. Let Λ_+ be the set of infinite reduced words $x_1x_2x_3\ldots$ from the alphabet $\mathcal{A}$ and let Λ be the set of doubly infinite reduced words $\ldots x_{-1}x_0x_1\ldots$. For $x = x_1x_2\ldots x_m \in \mathcal{G}$ let $\xi_x \in \Lambda$ be a periodic sequence of minimal period whose first m entries are $x_1, x_2, \ldots, x_m$. We show (Theorem 6.5) that there are functions $\beta, \ddot{\psi}, C_1, C_2 : \Lambda \times (0, R) \to \mathbb{R}$ such that

$$p^{*n}(x) \sim \frac{\exp\{n\beta(\xi_x, m/n)\}}{\sqrt{S_m\ddot{\psi}(\xi_x, m/n)}}\, C_1(\xi_x, m/n)C_2(\sigma^m\xi_x, m/n) \tag{1.5}$$

as $n \to \infty$, uniformly in the range $\varepsilon \leq m/n \leq 1/I$ for a certain $I > 0$; here σ: $\Sigma(\mathcal{G}) \to \Sigma(\mathcal{G})$ is the forward shift, and $S_m\ddot{\psi}(\overline{x}, t) = \ddot{\psi}(\overline{x}, t) + \ddot{\psi}(\sigma\overline{x}, t) + \ldots + \ddot{\psi}(\sigma^{m-1}\overline{x}, t)$. This corresponds to the classical "saddlepoint approximations" (sharp large deviations theorems) for sums of iid random vectors in $\mathbb{R}^d$. A major difficulty here is the identification of a suitable rate function β: this necessitates a careful study of certain matrix products, resulting in an extension of the Perron-Frobenius theorem to "inhomogeneous" products (sec. 5) and an extension to products of complex perturbations of nonnegative matrices (sec. 6). In the special case $K = 1$ (nearest neighbor random walk) these matrix products reduce to scalar products, trivializing much of the theory. The differences between the nearest neighbor case and the general finite range case become more apparent here. In the nearest neighbor case the rate function $\beta(\xi_x, t)$ is a function only of t and the relative frequencies of the generators in the reduced word $x = x_1 x_2 \dots x_m$ (see [La]), but in general $\beta(\xi_x, t)$ depends on the *order* of the letters as well.

The saddlepoint approximations are of interest for another reason. For large n, nearly all the mass in the probability distribution $p^{*n}(x)$ is concentrated in the region $|x| \geq \varepsilon n$ for a certain $\varepsilon > 0$ (independent of n), where the local limit approximations (1.4) are not accurate. This contrasts with the situation for finite range random walk in euclidean space. In fact, Guivarch [G] has shown that for random walks in $\mathcal{G}$ generally it is the case that as $n \to \infty$ the distance from the identity grows linearly in n, in particular, there is a constant $\beta > 0$ such that $|Z_n|/n \to \beta$. Sawyer and Steger [SS] have further shown that, under some additional hypotheses, $(Z_n - n\beta)/n^{1/2}$ converges in law to a normal distribution. Using our results concerning matrix products from section 6, together with some standard results concerning Ruelle's Perron-Frobenius operators ([Bo], ch. 1) we derive in Theorem 7.2 saddlepoint approximations for the distribution of $|Z_n|$, specifically, we show that there are functions $B(q), C(q)$, and $D(q)$ such that

$$P\{|Z_n| = m\} \sim \frac{\exp\{nB(\frac{m}{n})\}}{\sqrt{2\pi m D(\frac{m}{n})}} C(\frac{m}{n})$$

uniformly for $\frac{m}{n}$ in any compact subset of $(0, I_0)$, for a certain constant $I_0 > 0$. These provide independent proofs of the earlier results of Guivarch and Sawyer and Steger, and show that under the hypotheses (1.1)-(1.3) the limiting normal distribution in the central limit theorem is nondegenerate. In addition they give large deviations theorems and local limit theorems.

It is clear from (1.3), since $R > 1$, (or from Guivarch's theorem) that the random walk Z_n is transient. This is well known – in fact it is known [KV] that an irreducible random walk on any nonamenable discrete group is transient. It is also known [De] that the Martin boundary of any finite range random walk on the free group $\mathcal{G}$ is the space Λ_+ of infinite reduced words (see [DM] for the nearest neighbor case and [CS] for infinite range isotropic random walks). Now the one-step distribution $\{p_x\}$ has support $x \in \mathcal{B}$ so the transition from Z_n to Z_{n+1} affects only the last K letters of the reduced word. It follows from transience that for each $m < \infty$ the first m letters of Z_n stabilize as $n \to \infty$; thus,

(1.6) $$Z_n \xrightarrow{a.s.} Z_\infty = A_1 A_2 A_3 \dots \in \Lambda_+.$$

This explains in part the fact that Λ_+ is the Martin boundary. Our study of the transition probabilities $p^{*n}(x)$ yields as a byproduct some interesting information about the stochastic process $A_1, A_2, \dots$. We prove (sec. 5) that this process is asymptotically stationary, i.e., the joint distribution of $A_n, A_{n+1}, \dots$ converges as $n \to \infty$ to that of a stationary process. The limiting process is a *Gibbs state* in the sense of [Bo], Ch. 1, hence is isomorphic to a Bernoulli shift. In the nearest neighbor case the process $A_1, A_2, \dots$ is a one-step Markov chain, as can be seen by elementary arguments; in general, however, it appears to be non-Markovian.

We conclude this section with a lemma that will be called upon repeatedly in the paper. It will allow us to reduce many arguments to the special case in which p_e and $p_x > 0$ for all $x \in \mathcal{A}$.

LEMMA 1.1: *There exists an integer $n \geq 1$ such that $p^{*n}(e) > 0$ and $p^{*n}(x) > 0$ for every $x \in \mathcal{A}$.*

PROOF: The assumption (1.3) implies, by a well known argument, that $\{n \geq 1 : p^{*n}(e) > 0\}$ includes all sufficiently large integers, i.e., that there exists $n_0 < \infty$ such that for every $n \geq n_0$, $p^{*n}(e) > 0$. By (1.2), there exist integers $n_x \geq 1$ such that $p^{*n}(x) > 0$. Set $n_* = n_0 + max_{x \in \mathcal{A}} n_x$; then the Chapman-Kolmogorov equations imply that for any $n \geq n_*$, $p^{*n}(e) > 0$ and $p^{*n}(x) > 0$ for every $x \in \mathcal{A}$. ☐

2. Green's Function and Associated Generating Functions

The limit theorems stated in section 1 will ultimately devolve from the character of the singularities of the Green's function(s) on the circle of convergence. This section sets forth the key properties of the Green's function and various related generating functions. The arguments are based on the Markov property and the tree structure of the group $\mathcal{G}$. The finite range assumption (1.1) is of fundamental importance: it guarantees that any positive probability path from e to x in $\mathcal{G}$ must pass through each "rosette" $x_1 x_2 \dots x_k \mathcal{B}$ where x has reduced word representation $x = x_1 x_2 \dots x_n$ and $1 \leq k \leq n$.

For $x \in \mathcal{G}$ and $z \in \mathbb{C}$ satisfying $|z| < 1$, define

$$\begin{aligned} G_x(z) &= \sum_{n=0}^{\infty} p^{*n}(x) z^n, \\ G(z) &= G_e(z), \\ T_x &= \inf\{n \geq 0 : Z_n = x\}, \\ F_x(z) &= E z^{T_x} 1\{T_x < \infty\}. \end{aligned}$$

The functions F_x and G_x are clearly analytic in $|z| < 1$; $G(z)$ is called the Green's function for the random walk. Furthermore, F_x, G_x, and G satisfy the following fundamental relations, both simple consequences of the Markov property:

(2.1) $G_x(z) = F_x(z) G(z)$;

(2.2) $$G(z) = 1 + z \left\{ p_e + \sum_{x \neq e} p_x F_{x^{-1}}(z) \right\} G(z) = \left\{ 1 - z p_e - z \sum_{x \neq e} p_x F_{x^{-1}}(z) \right\}^{-1}.$$

Because each of F_x, G_x has a Taylor series with nonnegative coefficients, its radius of convergence coincides with its smallest positive singularity. Using (2.1) – (2.2), we will show that all of the functions F_x, G_x have the same radius of convergence R, that $1 < R < \infty$, and that the singularity at R is algebraic in nature. Most important, we will determine the algebraic character of the singularity (Prop. 3.6).

PROPOSITION 2.1: *Let R be the radius of convergence of $G(z)$. Then*

(2.3) $$1 < R < \infty \text{ and}$$

(2.4) $$G(R) < \infty.$$

PROOF: That $R < \infty$ follows from the fact that the random walker may return to e with positive probability: in particular, $p^{*k}(e) > 0$ for some $k \geq 1$ implies, since $p^{*nk}(e) \geq (p^{*k}(e))^n$, that $\limsup\, (p^{*n}(e))^{1/n} \geq (p^{*k}(e))^{1/k} > 0$, so $R \leq (p^{*k}(e))^{-1/k}$. That $1 < R$ is known for (at least) symmetric random walks – see [K_1], [K_2] for a proof that works in any nonamenable group – and since the result is not used in the subsequent analysis we shall omit the proof. (Observe, however, that $1 \leq R$ is trivial.) That $G(R) < \infty$ is also known – see [G], p. 85. Since this fact is crucial to our analysis, and since the proof in [G] is somewhat sketchy, we shall present the argument here.

Suppose that $G(R) = \infty$. Then the random walk is R-recurrent and there exists a positive R-invariant function $\gamma : \mathcal{G} \to \mathbb{R}_+$, i.e.,

$$\gamma(x) = R \sum_{x'} p_{x'} \gamma(xx') \quad \forall\, x \in \mathcal{G}.$$

Furthermore, the R-invariant function is unique up to multiplication by a scalar. (These are standard results in discrete potential theory – see [Se], sec. 6.2 or [N], sec. 5.1.) Now if γ is R-invariant then so is any left translate of γ, hence by the essential uniqueness of γ it follows that every left translate of γ is a scalar multiple of γ. It follows that if $x \in \mathcal{G}$ has the representation $x = x_1 x_2 \ldots x_m$ as a reduced word, then $\gamma(x) = \gamma(x_1)\gamma(x_2)\ldots\gamma(x_m)\gamma(e)$. Without loss of generality we may take $\gamma(e) = 1$.

Define

$$q_x = R p_x \gamma(x), \quad x \in \mathcal{G}.$$

Then

$$\sum_{x \in \mathcal{G}} q_x = R \sum_{x \in \mathcal{G}} p_x \gamma(xe) = \gamma(e) = 1$$

and, by the multiplicative property of $\gamma(\cdot)$,

$$\begin{aligned}
&q^{*n}(x) = R^n p^{*n}(x)\gamma(x) \Longrightarrow \\
&q^{*n}(e) = R^n p^{*n}(e) \Longrightarrow \\
&\sum_{n=0}^{\infty} q^{*n}(e) = \sum_{n=0}^{\infty} R^n p^{*n}(e) = G(R) = \infty.
\end{aligned}$$

Thus, the random walk with step distribution $\{q_x\}$ is recurrent. But this is impossible, because the free group $\mathcal{G}$ is nonamenable and, as such, admits no recurrent random walks with full support – see [KV], sec. 4. □

COROLLARY 2.2: *For each $x \in \mathcal{G}$ the radius of convergence of $F_x(z)$ is at least R, and $F_x(R) < \infty$. Moreover, the radius of convergence of $\sum_x p_x F_{x^{-1}}(z)$ is exactly R, so there is at least one $x \in \mathcal{B}$ such that $F_{x^{-1}}(z)$ has radius of convergence R. For all $z \in \mathbb{C}$ such that $|z| \leq R$,*

(2.5)
$$|z\{p_e + \sum_x p_x F_{x^{-1}}(z)\}| < 1.$$

PROOF: Fix $x \in G$. By the irreducibility of the random walk there exists $n \geq 1$ and $x_1, x_2, \ldots, x_n \in \mathcal{B}$ such that $x^{-1} = x_1 x_2 \ldots x_n$ and $p_{x_1} p_{x_2} \ldots p_{x_n} > 0$. The Markov property implies that

$$G(z) = \{P(z) + z^n p_{x_1} p_{x_2} \ldots p_{x_n} F_x(z)\} G(z),$$

where $P(z)$ is a polynomial of degree $\leq n$. It follows that $F_x(z)$ has radius of convergence $\geq R$, and $F_x(R) < \infty$.

Since each $F_x(z)$ has nonnegative Taylor coefficients, the maximum of $|z\{p_e + \sum_x p_x F_{x^{-1}}(z)\}|$ for $|z| \leq R$ occurs at $z = R$. Now (2.4) implies that $|G(z)| \leq G(R) < \infty$ for $|z| \leq R$, so

$$1 - zp_e - z \sum_x p_x F_{x^{-1}}(z)$$

has no roots in $|z| \leq R$, by (2.2). But $zp_e + z \sum_x p_x F_{x^{-1}}(z)$ is zero at $z = 0$, so the Intermediate Value Theorem implies that $Rp_e + R \sum_x p_x F_{x^{-1}}(R) < 1$. This proves (2.5). Finally, (2.2) and Proposition 2.1 imply that $z \sum_x p_x F_{x^{-1}}(z)$ has a singularity at $z = R$. □

Recall that $\mathcal{B} = \{x \in \mathcal{G} : |x| \leq K\}$. According to our standing hypotheses about the random walk Z_n, the support of the step distribution $\{p_x\}$ is contained in $\mathcal{B}$ but *not* in $\{x \in \mathcal{G} : |x| < K\}$. For $x \in \mathcal{G}$ and $a, b \in \mathcal{B}$, define

$$\begin{aligned}
\tau_x = \tau(x) &= \inf\{n \geq 0 : Z_n \in x\mathcal{B}\}, \\
H_x^{ab}(z) &= E^a z^{\tau(x)} 1\{Z_{\tau(x)} = xb\}, \\
\phi_x(z) &= E^x z^{T_e} = F_{x^{-1}}(z).
\end{aligned}$$

Here and in the sequel the notation P^x and E^x is used to indicate that the initial point Z_0 of the random walk is x; thus, $P = P^e$. Observe that $H_x(z)$ is a $|\mathcal{B}| \times |\mathcal{B}|$ matrix-valued function of z, analytic in $|z| < 1$. These matrices were introduced in [SS] . We will let $\phi(z)$ denote the $|\mathcal{B}| \times 1$ vector- valued function whose entries are the functions $\phi_x(z), x \in \mathcal{B}$, and $\phi^*(z)$ denote the $(|\mathcal{B}| - 1) \times 1$ vector-valued function whose entries are $\phi_x(z), x \in \mathcal{B} - \{e\}$. Also, u will be the $|\mathcal{B}| \times 1$ vector whose entries are $u_e = 1$ and $u_x = 0 \; \forall x \neq e$.

PROPOSITION 2.3: *Let $x \in \mathcal{G}$ have the reduced word representation $x_1 x_2 \ldots x_m$ and let $b \in \mathcal{B}$. Then for every $z \in \mathbb{C}$ such that $|z| < 1$,*

(2.6) $$H_x(z) = H_{x_1}(z) H_{x_2}(z) \ldots H_{x_m}(z),$$

(2.7) $$F_x(z) = u^t H_x(z) \phi(z) \qquad \forall\, x \neq e,$$

and

(2.8) $$\phi_b(z) = p_{b^{-1}} z + \sum_{a \in \mathcal{B}} z M^{ba}(z) \phi_a(z)$$

where

$$M^{ba}(z) = p_{b^{-1}a} + \sum_{x:bx \notin \mathcal{B}} z p_x H^{ea}_{b^{-1}x^{-1}}(z)$$

for all $a, b \in \mathcal{B} - \{e\}$.

PROOF: To reach x the random walker must first enter $x\mathcal{B}$, then travel from the entry point xb to $x = xe$. Therefore, by the Markov property.

$$F_x(z) = \sum_{b \in \mathcal{B}-\{e\}} H^{eb}_x(z)\phi_b(z) \qquad \forall\, x \neq e.$$

This is equivalent to the matrix equation (2.7).

(NOTE: In fact, $H^{ee}_x \equiv 0$ if $x \neq e$.)

The relation (2.6) follows from the Markov property and the fact that the step distribution $\{p_x\}$ is supported by $\mathcal{B}$. This implies that to reach $x\mathcal{B}$ the random walker must pass through (in order) $x_1\mathcal{B}, x_1x_2\mathcal{B}, \ldots, x_1x_2\ldots x_{m-1}\mathcal{B}$, then enter $x\mathcal{B}$. Hence,

$$H^{ab}_x(z) = \sum_{b_1, b_2, \ldots, b_{m-1} \in \mathcal{B}} H^{ab_1}_{x_1}(z) H^{b_1b_2}_{x_2}(z) \ldots H^{b_{m-1}b}_{x_m}(z).$$

This is equivalent to (2.6).

Finally, if $b \neq e$ then $P^b\{T_e \geq 1\} = 1$ so by the Markov property we may condition on the first step of the random walk to obtain

$$\phi_b(z) = p_{b^{-1}}z + \sum_{x:bx \neq e} z p_x \phi_{bx}(z).$$

The relation (2.8) now follows from (2.7), since $\phi_x(z) = F_{x^{-1}}(z)$. □

Equations (2.6)–(2.8) are fundamental to the analysis that follows. Together with (2.1)–(2.2) they imply that each of the functions $F_x(z)$ and $G_x(z)$ is a rational function of $z, H^{ab}_i(z)$, and $\phi_b(z)$, where $i \in \mathcal{A}$ and $a, b \in \mathcal{B}$. Moreover, (2.8) shows that the functions $\phi_b(z), b \in \mathcal{B}$, are algebraic functions of z and $H^{cd}_j, j \in \mathcal{A}$ and $c, d \in \mathcal{B}$. Observe that (2.8) has the form of a matrix equation

$$\phi^*(z) = zp + zM(z)\phi^*(z) \tag{2.9}$$

where p is the vector of constants $p_{b^{-1}}, b \neq e$; and $M(z)$ is a $(|\mathcal{B}|-1)\times(|\mathcal{B}|-1)$ matrix-valued function of z, analytic in $|z| < 1$. The matrix $M(z)$ has entries which are polynomials in z, H^{cd}_i. If $\varepsilon > 0$ is sufficiently small and $|z| \leq \varepsilon$ then $(I - zM(z))$ is invertible, hence

$$\phi^*(z) = z(I - zM(z))^{-1}p \quad \forall\, |z| < \varepsilon. \tag{2.10}$$

This shows that all the generating functions of interest are algebraic functions of z and the finite collection of functions $H^{cd}_j, j \in \mathcal{A}$ and $c, d \in \mathcal{B}$. As will be seen (Proposition

2.6) the functions H_j^{cd} are themselves interrelated by a system of algebraic equations; the nature of this system will determine the character of the singularities of the various G_x, F_x. Before investigating this sytem, however, we observe that certain of the H_i^{ab} may be constant (0 or 1). If $a \in i\mathcal{B}$ then $P^a\{\tau_k = 0\} = 1$, so either $H_i^{ab} \equiv 0$ or $H_i^{ab} \equiv 1$, according as $a \neq ib$ or $a = ib$. If $a \notin i\mathcal{B}$ (but $a \in \mathcal{B}$) it is still possible that $H_i^{ab} \equiv 0$, because it may be impossible to first enter $i\mathcal{B}$ at ib starting from a. However,

LEMMA 2.4: *If $a \notin i\mathcal{B}$ then there exists at least one $b \in \mathcal{B}$ such that $H_i^{ab}(z)$ is not a constant function.*

PROOF: If $a \notin i\mathcal{B}$ then $P^a\{\tau_i = 0\} = 0$. The irreducibility of the random walk (cf. (1.2)) guarantees that $P^a\{\tau_i < \infty\} > 0$. Since $Z_{\tau(i)} \in i\mathcal{B}$ whenever $\tau_i < \infty$, it follows that there is at least one $b \in \mathcal{B}$ such that $P^a\{1 \leq \tau_i < \infty \text{ and } Z_{\tau(i)} = ib\} > 0$. Consequently, $H_i^{ab}(z) = E^a z^{\tau(i)} 1\{Z_{\tau(i)} = ib\}$ is not a constant function. □

Consider those functions $H_i^{ab}(z)$ that are *not* constant. Clearly, there are only finitely many, and by Lemma 2.4, there is at least one. We may enumerate them as follows:

$$h_1(z), h_2(z), \ldots, h_\nu(z).$$

PROPOSITION 2.5: *There exist polynomials $Q_1(h), Q_2(h), \ldots, Q_\nu(h)$ in the ν variables $h = (h_1, h_2, \ldots, h_\nu)$, each with all coefficients nonnegative and with at least one nonzero term, such that $\forall z \in \mathbb{C}$, if $|z| < 1$ then*

$$h_i(z) = zQ_i(h_1(z), \ldots, h_\nu(z)) \;\; \forall\, i = 1, 2, \ldots, \nu. \tag{2.11}$$

PROOF: These equations follow from the Markov property. Consider one of the functions $H_i^{ab}(z)$ that is nonconstant. Under P^a the random variable τ_i must satisfy $\tau_i \geq 1$ almost surely, otherwise $H_i^{ab}(z)$ would be constant, and $P^a\{Z_{\tau(i)} = ib\} > 0$, otherwise $H_i^{ab}(z) \equiv 0$. Since $\tau_i \geq 1$ we may condition on the first step of the random walk to obtain

$$\begin{aligned} H_i^{ab}(z) &= z \sum_x p_x E^{ax} z^{\tau(i)} 1\{Z_{\tau(i)} = ib\} \\ &= z \sum_x p_x E^a z^{\tau(x^{-1}i)} 1\{Z_{\tau(x^{-1}i)} = x^{-1}ib\} \\ &= z \sum_x p_x H_{x^{-1}i}^{ab}(z). \end{aligned} \tag{2.12}$$

But (2.6) implies that each $H_{x^{-1}i}^{ab}$ is a polynomial in the various H_j^{cd}, where $c, d \in \mathcal{B}$ and $j \in \mathcal{A}$. This proves (2.11). It is obvious that the coefficients of Q_i are nonnegative. At least one term must be nonzero, since $h_i(z)$ is not constant. □

PROPOSITION 2.6: *If $p_i > 0$ for each $i \in \mathcal{A}$ then at least some of the polynomials $Q_i(h)$ in (2.11) have nonzero quadratic terms.*

PROOF: Choose $i, j \in \mathcal{A}$ such that $j \neq i^{\pm 1}$, and let $a = x_1 x_2 \ldots x_K$ be a reduced word with $x_1 = i^{-1}$ and $x_K = j$. Set $c = j^{-1} a j^{-1}$. Then $|a| = |c| = K$, hence $a, c \in \mathcal{B}$, but $a, c \notin i\mathcal{B}$ and $a \notin j\mathcal{B}$. By Lemma 2.4 there exists $b \in \mathcal{B}$ such that $H_i^{cb}(z)$ is not constant.

By hypothesis, $p_{j^{-1}} > 0$, so the one-step transition $a \to aj^{-1}$ has positive probability. Note that $aj^{-1} = jc$. Since $a \notin j\mathcal{B}, P^a\{\tau_j = 0\} = 0$, so $P^a\{\tau_j = 1 \text{ and } Z_{\tau(j)} = jc\} > 0$; it follows that $H_j^{ac}(z)$ is not constant. Thus equation (2.12) above for $H_i^{ab}(z)$ includes the nonzero quadratic term $p_{j^{-1}} H_j^{ac} H_i^{cb}$. Since neither H_j^{ac} nor H_i^{cb} is constant, each is included in $\{h_1, h_2, \ldots, h_\nu\}$. Thus, to complete the proof it suffices to verify that H_i^{ab} is also included in $\{h_1, h_2, \ldots, h_\nu\}$, i.e., that H_i^{ab} is not constant. But this follows from (2.12): all terms in (2.12) are either constant or strictly increasing in z for $z \in (0,1)$, and at least one term, $p_{j^{-1}} H_j^{ac} H_i^{cb}$, is strictly increasing. □

The polynomial equations (2.11) for $h_1, h_2, \ldots, h_\nu$ lead to a hierarchy of higher order (in z) equations obtained by making repeated substitutions in (2.11). Consider the i^{th} equation in (2.11); for each h_j occurring in Q_i, substitute $zQ_j(h_1, h_2, \ldots, h_\nu)$. This yields a polynomial equation

$$h_i(z) = zQ_i^{(2)}(z, h_1(z), \ldots, h_\nu(z)).$$

For each h_j occurring in $Q_i^{(2)}$, substitute $zQ_j(h_1, \ldots, h_\nu)$ to obtain $Q_i^{(3)}(z, h_1, \ldots, h_\nu)$. By induction, there exists for each $n \geq 2$ and $i = 1, 2, \ldots, \nu$ a polynomial $Q_i^{(n)}(z, h_1, \ldots, h_\nu)$ obtained from $Q_i^{(n-1)}$ by substituting $zQ_j(h_1, \ldots, h_\nu)$ for each h_j, and for all $n \geq 2, i = 1, 2, \ldots, \nu$, and $|z| < 1$,

$$h_i(z) = zQ_i^{(n)}(z, h_1(z), \ldots, h_\nu(z)). \tag{2.13}$$

Observe that each $Q_i^{(n)}$ has all its coefficients nonnegative.

PROPOSITION 2.7: *Assume that $p_i > 0$ for each $i \in \mathcal{A}$. Then for each pair $(i, j) \in \{1, 2, \ldots, \nu\}^2$ there exists $n \geq 2$ such that the variable h_j appears as a factor in some term of $Q_i^{(n)}$ with strictly positive coefficient.*

NOTE 1: This fact will be of crucial importance in the analysis of the system (2.11) to be carried out in section 3 and Corollary 2.8 below. It will imply that all the functions h_j have the same radius of convergence and the same type singularity at $z = R$.

NOTE 2: A similar, but simpler, fact holds for the functions $\phi_b(z), b \in \mathcal{B}$. If $p_x > 0$ for all $x \in \mathcal{A}$ then by repeatedly substituting for those ϕ_a such that $a \in \mathcal{B}$ in the equation (2.8) for ϕ_b one may obtain an equation for ϕ_b in which one of the h_i appears as a factor in a nonzero term. Here is the argument:

Let $b = b_1 b_2 \ldots b_l$ where each $b_i \in \mathcal{A}$, and let $b_1 = j$. Then the equation (2.8) for ϕ_b contains the term $zp_j\phi_{bj}(z)$. If $bj \in \mathcal{B}$ then $\phi_{bj}(z)$ may be replaced by the right side of one of the equations (2.8), which will include the term $zp_j\phi_{bjj}(z)$. Similarly, if $bjj \in \mathcal{B}$ then $\phi_{bjj}(z)$ may be replaced by an expression containing the term $zp_j\phi_{bjjj}(z)$. Thus, one may eventually obtain an equation for ϕ_b containing a term $z^n p_j^n \phi_{bj^n}(z)$ where $bj^n \notin \mathcal{B}$. Note that $p_j^n > 0$ because $p_j > 0$. Since $bj^n \notin \mathcal{B}$, if the random walk is started at bj^n then to reach e it must first reach $\mathcal{B}$; consequently,by the Markov property,

$$\phi_{bj^n}(z) = \sum_{a\in\mathcal{B}} H^{ea}_{j^{-n}b^{-1}}(z)\phi_a(z).$$

It now follows from Lemma 2.4 and equation (2.6) that one of the functions $h_i(z)$ is a factor of $\phi_{bj^n}(z)$. Therefore , successive substitutions in (2.8) eventually lead to an equation for ϕ_b in which one of the h_i appears as a factor in a nonzero term.

PROOF *of Proposition 2.7*: The indices i, j in question correspond to two of the generating functions $H^{\cdot\cdot}_{\cdot}$ introduced earlier. We will label these H_i^{ab} and H_j^{cd} respectively (but note – the subscripts i, j now have different meaning, in particular, they are elements of $\mathcal{A}$). Neither H_i^{ab} nor H_j^{cd} is constant; consequently, $a \notin i\mathcal{B}$ and $c \notin j\mathcal{B}$. It follows that a and c have reduced word representations $a = a_1a_2\dots a_K$ and $c = c_1c_2\dots c_K$ with $a_1 \neq i$ and $c_1 \neq j$.

Recall that the equations (2.11) are equivalent to (2.12) (by way of (2.6)) and that equations (2.12) are nothing more than the Markov property. Thus, making repeated substitutions in (2.11) is tantamount to using the Markov property repeatedly, and produces equations $H_i^{ab} = \sum\cdots$ where the terms in the sum have the form

$$z^m p_{x_1}p_{x_2}\dots p_{x_m} H^{ab}_{(x_1x_2\dots x_m)^{-1}i}. \tag{2.14}$$

Consider such a term: if $a \in (x_1x_2\dots x_m)^{-1}i\mathcal{B}$ then no further use of the Markov property (i.e., no more substitutions) can be made in this term, since $P^a\{\tau(x_1x_2\dots x_m)^{-1} = 0\} = 1$ and thus $H^{ab}_{(x_1x_2\dots x_m)^{-1}i}$ is constant. However, if $a \notin (x_1x_2\dots x_m)^{-1}i\mathcal{B}$ then $P^a\{\tau(x_1x_2\dots x_m)^{-1} \geq 1\} = 1$, so the Markov property *can* be used again. (NOTE: In this case it is possible that $H^{ab}_{(x_1x_2\dots x_m)^{-1}i} \equiv 0$, but not $\equiv 1$; if $\equiv 0$, using the Markov property again just replaces 0 by 0.) Consequently, for any finite sequence $x_1, x_2, \dots, x_m$ such that $p_{x_1}p_{x_2}\dots p_{x_m} > 0$, the term (2.14) appears in an equation $H_i^{ab} = \Sigma\cdots$ obtained from (2.11) by substitutions iff $(x_sx_{s+1}\dots x_m)a \notin i\mathcal{B}\ \ \forall\ s = 1, 2, \dots, m$.

We will show that there exist $x_1, x_2, \dots, x_m \in \mathcal{G}$, for some $m \geq 1$, such that

(a) $p_{x_1}p_{x_2}\dots p_{x_m} > 0$;

(b) $(x_sx_{s+1}\dots x_m)a \notin i\mathcal{B}\ \ \forall\ s = 1, 2, \dots, m$;

(c) $(x_1x_2\dots x_m)^{-1} = wjy$ where

- (c_1) $H_w^{ac}(z) > 0\ \ \forall\ z \in (0, 1)$;
- (c_2) $H_y^{da}(z) > 0\ \ \forall\ z \in (0, 1)$;
- (c_3) w has reduced word representation $w = w_1w_2\dots w_\ell$ with $w_\ell \neq j^{-1}$;
- (c_4) y has reduced word representation $y = y_1y_2\dots y_1$ with $y_1 \neq j^{-1}$.

Given this, it will then follow by the preceding paragraph that the term (2.14) appears in some equation $H_i^{ab} = \Sigma \cdots$ obtained from (2.11) by substitutions. By (2.6), the term (2.14) can be rewritten (for $0 < z < 1$) as

$$z^m p_{x_1} p_{x_2} \dots p_{x_m} H^{ab}_{(x_1 x_2 \dots x_m)^{-1} i}$$
$$= z^m p_{x_1} p_{x_2} \dots p_{x_m} H_w^{ac} H_j^{cd} H_y^{da} H_i^{ab} + \text{ nonnegative terms,}$$

showing that H_j^{cd} appears in a term with positive coefficient in some equation for H_i^{ab} obtained from (2.11) by substitutions. It follows that either a "precursor" or a "successor" of this term, including H_j^{cd} as a factor, appears in one of the polynomial equations (2.13).

It remains, then, to prove (a), (b), and (c_1)–(c_4). We will show that these may be achieved with a finite sequence $x_1, x_2, \dots, x_m \in \mathcal{A}$. Recall our hypothesis that $p_\ell > 0$ for every $\ell \in \mathcal{A}$: it follows that (a) holds automatically for any finite sequence $x_1, x_2, \dots, x_m \in \mathcal{A}$. As for (b), observe that $y \in i\mathcal{B}$ only if $|y| \leq K+1$, so as long as $|x_s x_{s+1} \dots x_m a| > K+1$ then $x_s x_{s+1} \dots x_m a \notin i\mathcal{B}$. But $a = a_1 a_2 \dots a_K$ (as a reduced word) with $a_i \neq i$; consequently, if $x_m = x_{m-1} = a_1$ and if the remaining letters $x_1, x_2, \dots, x_{m-2}$ are chosen so that $x_{\ell+1} \neq x_\ell^{-1}$ for every $\ell = 1, 2, \dots, m-1$ then (b) holds. So it remains to show that (c) can be achieved for some sequence $x_1, x_2, \dots, x_m \in \mathcal{A}$, subject only to the constraints $x_m = x_{m-1} = a_1$ and $x_{\ell+1} \neq x_\ell^{-1}$.

To prove that $H_w^{ac}(z) > 0$ for all $0 < z < 1$ it suffices to show that $P^a\{\tau(w) \geq 1$ and $Z_{\tau(w)} = wc\} > 0$. Set

$$w = w_1 w_2 \dots w_\ell = a_1^{-1} a_1^{-1} w_3 w_4 \dots w_s c_K^{-1} c_{K-1}^{-1} \dots c_1^{-1}$$

where the "filler" $w_3 w_4 \dots w_s$ is chosen so that (i) there is no cancellation between adjacent letters (in particular, $w_3 \neq a$, and $w_s \neq c_K$) and (ii) $s \geq K+2$. Observe that $w_\ell = c_1^{-1} \neq j^{-1}$, as required (recall that $c = c_1 c_2 \dots c_K$ with $c_1 \neq j$). Also, since $s \geq K+2$ and $|a| = K$, $P^a\{\tau(w) = 0\} = 0$. Finally, $P^a\{\tau(w) \geq 1$ and $Z_{\tau(w)} = wc\} > 0$ because each of the one-step transitions in the following chain has positive probability:

$$\begin{aligned} a_1 a_2 \dots a_K &\longrightarrow a_1 a_2 \dots a_{K-1} \\ &\longrightarrow a_1 a_2 \dots a_{K-2} \\ &\longrightarrow \dots \\ &\longrightarrow a_1 \\ &\longrightarrow e \\ &\longrightarrow a_1^{-1} \\ &\longrightarrow a_1^{-1} a_1^{-1} \\ &\longrightarrow \dots \\ &\longrightarrow a_1^{-1} a_1^{-1} w_3 \dots w_s. \end{aligned}$$

This proves (c_1) and (c_3). The existence of $y_1, y_2, \dots, y_r \in \mathcal{A}$ satisfying (c_2) and (c_4) follows by a completely similar argument. □

COROLLARY 2.8: *Assume that* $p_i > 0$ *for each* $i \in \mathcal{A}$. *Then each of the power series* $h_i(z), i = 1, 2, \ldots, \nu$ *and* $F_x(z), G_x(z), x \in \mathcal{G}$, *has radius of convergence* R, *and*

$$h_i(R) < \infty \quad \forall\, i = 1, 2, \ldots, \nu; \tag{2.15}$$

$$F_x(R) < \infty \text{ and } G_x(R) < \infty \quad \forall\, x \in \mathcal{G}. \tag{2.16}$$

PROOF: Each of these power series has nonnegative coefficients, so for each one the radius of convergence is a singular point. Also, for a power series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$ the radius of convergence is inf $\{r\colon \sum_{n=0}^{\infty} a_n r^n = \infty\}$. It therefore follows from Proposition 2.7 and equations (2.13) that all of the power series $h_i(z), i = 1, 2, \ldots, \nu$ have the same radius of convergence (recall that all coefficients of $Q_i^{(n)}$ are nonnegative). Call this radius of convergence r.

Recall that each $h_i(z)$ is one of the functions $H_j^{ab}(z)$, and that $H_j^{ab}(z) = E^a z^{\tau(j)} 1 \{Z_{\tau(j)} = jb\} = H_{a^{-1}j}^{eb}(z)$. Now $\tau(x) = T_{xb}$ on $\{Z_{\tau(x)} = xb\}$, so for $z \geq 0$

$$H_x^{eb}(z) \leq F_{xb}(z), \quad \forall\, x \in \mathcal{G}, b \in \mathcal{B}.$$

By Corollary 2.2 each $F_y(z)$ has radius of convergence $\geq R$; consequently $r \geq R$.

Corollary 2.2 also implies that at least one of the power series $\phi_b(z), b \in \mathcal{B}$, has radius of convergence R. It therefore follows from equation (2.10) that the matrix-valued function $(I - zM(z))^{-1}$ has a singularity at $z = R$. Now $M(z)$ is a matrix whose entries are polynomials in z and $h_i (i = 1, 2, \ldots, \nu)$ with nonnegative coefficients, not all of which are zero. Hence, $zM(z)$ is an analytic matrix-valued function of z for $|z| < r$, and each entry of $zM(z)$ is a nondecreasing function of z for $0 < z \leq R$. By the Perron-Frobenius theorem ([Se], Ch. 1), for each $z \in (O, R]$ the matrix $zM(z)$ has a positive eigenvalue λ_z with (nontrivial) nonnegative left- and right-eigenvectors. Elementary arguments show that $z \to \lambda_z$ is nondecreasing and continuous for $z \in (O, R]$.

We claim that $\lambda_R < 1$. For if this were not the case then there would exist $s \in (O, R]$ such that $\lambda_s = 1$, by the intermediate value theorem (observe that $\lim_{z \to 0+} \lambda_z = 0$ because $\lim_{z \to 0+} zM(z) = 0$). But then $(I - zM(z))^{-1} p$ would have a pole at $z = s$, because the vector p has strictly positive entries. This would imply that $\phi_b(z)$ has a pole at $z = s$ for some $b \in \mathcal{B}$, by (2.10), contradicting the fact that $F_x(R) < \infty$ for all $x \in \mathcal{G}$ (Corollary 2.2).

Therefore, $\lambda_R < 1$. This implies that $\sum_{n=0}^{\infty} R^n M(R)^n$ converges, because λ_R is the spectral radius of $RM(R)$. Consequently, since $(I - zM(z))^{-1} p$ has a singularity at $z = R$, one of the entries of $zM(z)$ must have a singularity at $z = R$. But these entries are polynomials in z and $h_i, i = 1, 2, \ldots, \nu$; hence, one of the functions $h_i(z)$ has a singularity at $z = R$. This proves that $r = R$, that *every* $h_i(z), i = 1, 2, \ldots, \nu$, has a singularity at $z = R$, and that $h_i(R) < \infty$ for each i.

Now consider the function ϕ_b, where $b \in \mathcal{B}\backslash\{e\}$. By (2.8), ϕ_b is a polynomial function of z, ϕ_a, and H_y^{ea}, where $a \in \mathcal{B}$ and $y \notin \mathcal{B}$. Recall that we may substitute repeatedly for various ϕ_a on the right-hand side of (2.8) to obtain another equation for ϕ_b that contains a nonzero term with some nonconstant H_y^{ea} as a factor. (See Note 2 following the statement of Proposition 2.7.) But by (2.6), any such H_y^{ea} may be written as a polynomial in $h_1, h_2, \ldots, h_\nu$ with positive coefficients. Thus, ϕ_b may be written as a polynomial in z, ϕ_a, and h_i, where $a \in \mathcal{B}$ and $i = 1, 2, \ldots, \nu$, in which at least one h_i appears as a factor in a nonzero term, and whose coefficients are all nonnegative. Since $h_i(z)$ has radius of convergence R and each ϕ_a has radius of convergence $\geq R$, it follows that ϕ_b has radius of convergence R.

Finally, it follows by a similar argument using (2.7) and (2.6) that each $F_x(z), x \in \mathcal{G}$, has radius of convergence R. It follows from (2.1) and Proposition 2.1 that the same is true for each $G_x(z)$. □

3. Generating Functions: Algebraic Character of Singularities

In this section we prove that each of the generating functions G_x, F_x, and h_i introduced in the previous section is an algebraic function of z. Using this fact we then prove that, at least in the special case where $p_i > 0$ for every $i \in \mathcal{A}, z = R$ is the *only* singularity in the closed disk $|z| \leq R$, and we determine the nature of the singularity. The arguments will be quite general; they apply to any set of function $h_i(z)$ subject to the conclusions of Propositions 2.5, 2.6, 2.7 and Corollary 2.8, and thus may be useful in problems other than the random walk problem of primary interest in this paper.

PROPOSITION 3.1: *Each of the generating functions $h_i(z), i = 1, 2, \ldots, \nu$, is an algebraic function of z, i.e., for each i there exists a polynomial $P_i(z, \xi)$ in two variables over $\mathbb{C}$ such that $h_i(z)$ satisfies the functional equation*

$$P_i(z, h_i(z)) = 0 \quad \forall\, z. \tag{3.1}$$

PROOF: Define

$$\Gamma = \{(z, w_1, w_2, \ldots, w_\nu) \in \mathbb{C}^{1+\nu} \colon |z| < 1 \text{ and } w_i = h_i(z) \ \ \forall\, i\},$$
$$V = \{(z, w_1, w_2, \ldots, w_\nu) \in \mathbb{C}^{1+\nu} \colon |z| < 1 \text{ and } w_i - zQ_i(w) = 0 \ \ \forall\, i\}.$$

Then by Proposition 2.5, $\Gamma \subset V$. Also, Γ contains the origin.

Consider the Jacobian matrix J of the system (2.11), i.e., the $\nu \times \nu$ matrix of partial derivatives with respect to $w_1, w_2, \ldots, w_\nu$. J is continuous in the variables $z, w_1, \ldots, w_\nu$ and at the origin $J =$ identity. Consequently, there is an open neighborhood $\mathcal{N}$ of the origin in $\mathbb{C}^{1+\nu}$ in which J is invertible. Take any $(z; w) \in V \cap \mathcal{N}\backslash\{\text{origin}\}$; since $(\partial/\partial z)(w_i - zQ_i(w))$ is nonzero at such a point, the hypotheses of the (complex) implicit function theorem are satisfied. Thus each w_i is locally an analytic function of z, and so $V \cap \mathcal{N}\backslash\{\text{origin}\}$ is a one-dimensional complex manifold.

But $V \cap \mathcal{N} \backslash \{\text{origin}\}$ is contained in a *minimal* algebraic set W (an algebraic set is defined to be the set of simultaneous zeros of some set of polynomials in the variables $(z, w_1, \ldots, w_\nu)$. By a standard theorem of elementary algebraic geometry ([L], ch. X, Th. 4), W is the union of finitely many *varieties* $V_1, V_2, \ldots, V_k$ (a variety is an irreducible algebraic set). We will show that each V_i has dimension one, i.e., is an algebraic curve. We shall use the following general facts about a variety U of dimension d in $\mathbb{C}^n$: there is a subset $S \subset U$ (the "smooth points") such that (1) U = closure (S); (2) each $p \in S$ has a neighborhood in U homeomorphic to $\mathbb{C}^d$; and (3) d is the degree of transcendence (over $\mathbb{C}$) of the field of rational functions (in $z, w_1, w_2, \ldots, w_\nu$) on U. For proofs of these facts, see [Sh] sections I.6.1, II.1.4, III.3.2, and VII.2.1.

At every point of V_i the equations $w_j = zQ_j(w)$ must hold, otherwise W would not be minimal (it could be replaced by W intersected with the algebraic set determined by the equations $w_j = zQ_j(w)$). Moreover, V_i must contain a point of $V \cap \mathcal{N} \backslash \{\text{origin}\}$, otherwise W would not be minimal. Consequently, there must be a smooth point p of V_i contained in $V \cap \mathcal{N} - \{\text{origin}\}$. Since the equations $w_j = zQ_j(w)$ must hold at all nearby points of V_i, this implies that the topological dimension of V_i at such a smooth point p is ≤ 1. In fact the dimension must $= 1$, because if not each such smooth point p, being also a point of $V \cap \mathcal{N}$, would be a limit point of $\cap_{j \neq i} V_j$, hence $p \in \cap_{j \neq i} V_j$, contradicting the minimality of W.

Finally, it follows that the transcendence degree of each V_i is 1. Since z and w_j are both rational functions on V_i, there is a polynomial $Q_{ij}(z, \xi)$ in two variables such that $Q_{ij}(z, w_j) \equiv 0$ on V_i. Set $P_j(z, \xi) = \prod_{i=1}^{k} Q_{ij}(z, \xi)$; then $P_j(z, w_j) \equiv 0$ on W. Consequently, (3.1) holds for all z satisfying $0 < |z| < \varepsilon$, for some $\varepsilon > 0$; but it then holds for all z in $|z| < 1$ by analytic continuation. □

COROLLARY 3.2: *For each $x \in \mathcal{G}$ the functions $F_x(z)$ and $G_x(z)$ are algebraic functions of z.*

PROOF: By (2.1)–(2.2), (2.7), and (2.9) these functions are algebraic functions of $h_1, h_2, \ldots, h_\nu$ and z. Let k be the field of rational functions on W, where W is the algebraic set introduced in the previous proof, and let K be the extension field obtained by adjoining F_x. Then K is an algebraic extension of k, since F_x is algebraic over k. It follows that the transcendence degree of K over $\mathbb{C}$ is the same as that of k over $\mathbb{C}$ ([L], Ch. X, Th. 1), hence equals 1. Therefore, there is a polynomial relation between z and F_x. The same argument applies to G_x. □

The algebraicity of the Green's functions G_x was also proved in [Ao] and [St].

Our proof of the existence of the polynomials $P_i(z, \xi)$ in Proposition 3.1 appears to be nonconstructive. However, one can give a constructive algorithm for producing the polynomials in (3.1) which might be computationally feasible in certain cases where $|\mathcal{A}|$ and $|\text{support } \{p_x\}|$ are not too large. There are two steps: (1) obtain a basis for the polynomial ideal generated by the polynomials $w_i - zQ_i(w)$ in (2.11); and (2) use elimination on one variable at a time in this basis until only two are left, namely z and w_i. An algorithm for (1) was given by Hilbert (see the discussion in [Ke], pp. 119–120), and (2) may be accomplished using the elementary theory of resultants ([L], Ch. V). Algorithms for (2) are included in some computer programs for symbolic computations, such as *Mathematica.* In practice, one would probably first try the elimination procedure directly on the polynomial equations (2.11).

Our main interest, however, is not in obtaining explicit expressions or functional equations for the generating functions F_γ, G_x, and h_i, but rather in investigating the location and nature of their singularities in $\{z: |z| \leq R\}$. Recall that each of these functions is defined by a power series in z with radius of convergence $R > 1$ (Proposition 2.1 and Corollary 2.8) and nonnegative coefficients, so each has a singularity at $z = R$. This singularity is not a pole, by (2.15)–(2.16), so it must be an algebraic branch point, by Proposiiton 3.1 and Corollary 3.2.

PROPOSITION 3.3: *Assume that $p_i > 0$ for each $i \in \mathcal{A}$. Then each of the functions $F_x(z), G_x(z)$, and $h_i(z)$, where $x \in \mathcal{G}$ and $i \in \{1, 2, \ldots, \nu\}$ has a convergent Puiseux series in a neighborhood of $z = R$, the first two nonzero terms of which are*

(3.2)
$$\begin{aligned} F_x(z) &= a_0^{(x)} + a_1^{(x)}(R - z)^\alpha + \ldots \qquad (x \neq e), \\ G_x(z) &= b_0^{(x)} + b_1^{(x)}(R - z)^\alpha + \ldots, \\ h_i(z) &= c_0^{(i)} + c_1^{(i)}(R - z)^\alpha + \ldots, \end{aligned}$$

where

(3.3)
$$\begin{aligned} -\infty < a_1^{(x)}, b_1^{(x)}, c_1^{(i)} < 0, \\ 0 < a_0^{(x)}, b_0^{(x)}, c_0^{(i)} < \infty, \\ 0 < \alpha < \infty. \end{aligned}$$

The exponent α is rational, and α is the same for all of the functions G_x, F_x, and h_i.

NOTE: $(R - z)^\alpha$ is the positive branch of the α^{th} power. Also, here and in the sequel, ... indicates higher order terms. We are not claiming that the Puiseux series is in integer powers of $(R - z)^\alpha$, only that the lowest order nonconstant term is a (negative) multiple of $(R - z)^\alpha$.

If $p_i > 0$ for each $i \in \mathcal{A}$ then $\alpha = 1/2$: see Proposition 3.5 below.

PROOF: That each of F_x, G_x, and h_i has a convergent Puiseux series at $z = R$ follows from the fact that $z = R$ is an algebraic branch point of each (and thus, ultimately, from Proposition 3.1). Now each of these functions was defined by a power series in z with radius of convergence R and nonnegative coefficients; consequently, each is monotone increasing in z for $0 \leq z \leq R$. Therefore, for each function the first two terms of the Puiseux series must be as in (3.2), with the coefficients satisfying (3.3). (NOTE: $a_0^{(x)} = F_x(R), b_0^{(x)} = G_x(R)$, and $c_0^{(i)} = h_i(R)$ are all finite and *strictly positive*, since otherwise the functions F_x, G_x, h_i would be constant.)

It remains to be proved that α is the same for all of the functions F_x, G_x, and h_i. Consider first the functions $h_i(z), i = 1, 2, \dots, \nu$; Proposition 2.7 implies that the value of α must be the same for all. It now follows from equation (2.8) that the same value of α must hold for all the functions $\phi_b(z)$, and from equation (2.7) that the same value of α holds for all $F_x(z), x \in \mathcal{G}\backslash\{e\}$, and finally from (2.1), (2.2), and (2.5) that the same value of α holds for all $G_x(z), x \in \mathcal{G}$. □

Next we shall identify the exponent α in (3.2). To do so we will rely heavily on the special form of the algebraic system (2.11), especially, the nonnegativity of the coefficients in the polynomials Q_i and the "irreducibility" expressed in the conclusion of Proposition 2.7. Recall that (2.11) is the system

$$h_i - zQ_i(h_1, h_2, \dots, h_\nu) = 0, i = 1, 2, \dots, \nu.$$

Define

$$J_z = \left(\frac{\partial Q_i}{\partial h_j} \right)_{i,j=1,\dots,\nu} \Bigg|_{h_\ell = h_\ell(z)}$$

to be the Jacobian matrix of the system (2.11) evaluated at $(z, h_1(z), \dots, h_\nu(z))$. Observe that the entries of J_z are polynomials in $h_1, h_2, \dots, h_\nu$ with nonnegative coefficients; hence, for $z \geq 0$ the matrix J_z has nonnegative entries and so is subject to the conclusions of the Perron-Frobenius theory of nonnegative matrices ([Se], Ch. 1).

LEMMA 3.4: *Assume that $p_i > 0$ for every $i \in \mathcal{A}$ and also that $p_e > 0$. Then there exists an integer $n \geq 1$ such that for each $z \in (O, R]$ the matrix J_z^n has strictly positive entries.*

PROOF: Recall that repeated substitutions in (2.11) lead to the equations (2.13):

$$h_i - zQ_i^{(n)}(z, h_1, h_2, \dots, h_\nu) = 0.$$

Consider the Jacobian matrix $(\partial Q_i^{(n)} / \partial h_j)$ evaluated at the point $(z, h_1(z), \dots, h_\nu(z))$: since (2.13) is obtained from (2.11) by substitutions, the chain rule implies that this Jacobian matrix is J_z^n. But Proposition 2.7 implies that for some $n \geq 2$ the variable h_j occurs as a factor in a term of $Q_i^{(n)}$ with strictly positive coefficient. It follows that the $(i, j)^{th}$ entry of J_z^n is strictly positive for all $z \in (O, R]$.

We have assumed that $p_e > 0$. This implies that $zQ_j(h_1, h_2, \dots, h_\nu)$ includes the term $p_e z h_j$ (recall that the equations (2.11) derive from the Markov property). Consequently, if $Q_i^{(n)}$ includes a nonzero term with h_j as a factor, then so does $Q_i^{(n+m)}$ for every $m = 1, 2, \dots$. It now follows from the result of the preceding paragraph that J_z^n has strictly positive entries for all n sufficiently large, provided $z \in (O, R]$. □

Assume, then, that $p_i > 0$ for every $i \in \mathcal{A}$ and that $p_e > 0$. Then by Lemma 3.4 there exists $n \geq 1$ such that the matrix J_z^n is irreducible and aperiodic from every $z \in (O, R]$. By the Perron-Frobenius theorem ([Se], Ch. 1), each J_z has a positive eigenvalue λ_z of multiplicity 1, and all other eigenvalues are $< \lambda_z$ in absolute value. Furthermore, nontrivial right and left eigenvectors for the eigenvalue λ_z have strictly positive entries. It follows that λ_z is a simple root of the characteristic polynomial of J_z, hence by the implicit function theorem λ_z is continuous (in fact, real-analytic) in z. Since the entries of J_z are nondecreasing in z, so is λ_z.

PROPOSITION 3.5: *Assume that $p_e > 0$ and that $p_i > 0$ for every $i \in \mathcal{A}$. Then*

$$\lambda_R = 1/R. \tag{3.4}$$

PROOF: Let $s = \inf\{z > 0 : z\lambda_z = 1\}$; we will prove that $s = R$. First, observe that $s \le R$, because if $z\lambda_z < 1$ for every $z \in (O, R]$ then $I - zJ_z$ would be nonsingular at every $z \in (O, R]$ and, in particular, at $z = R$. This would contradict the fact that $z = R$ is a singularity of each $h_i(z)$ (the complex Implicit Function Theorem applied to the system (2.11) at $R, h_1(R), \ldots, h_\nu(R)$ would imply that each of the functions $h_i(z)$ extends analytically to a neighborhood of $z = R$). Second, observe that $s > 0$, because λ_z is nondecreasing in z, so $\lim_{z\to 0+} z\lambda_z = 0$.

Finally, suppose that $0 < s < R$; we will obtain a contradiction. Let $\boldsymbol{v}^t = (v_1, v_2, \ldots, v_\nu)$ be a left eigenvector of J_s corresponding to the eigenvalue $\lambda_s = 1/s$; assume that $v_i > 0$ for each i. Expand the equation

$$\sum_{i=1}^{\nu} v_i h_i - z \sum_{i=1}^{\nu} v_i Q_i(h_1, h_2, \ldots, h_\nu) = 0 \tag{3.5}$$

in a Taylor series around the point $z = s, h_i^* = h_i(s)$ to obtain

$$\begin{aligned}(z-s)\left\{\sum_{i=1}^{\nu} v_i Q_i(h_1^*, h_2^*, \ldots, h_\nu^*) + \ldots\right\} \\ = -\frac{1}{2}s \sum_{i=1}^{\nu}\sum_{j=1}^{\nu}(h_i - h_i^*)(h_j - h_j^*)\left(\frac{\partial^2(\boldsymbol{v}^t Q)}{\partial h_i \partial h_j}\right)_{h=h^*} + \ldots,\end{aligned} \tag{3.6}$$

where ... indicates higher order terms. Notice that there are no linear terms in $(h_i - h_i^*)$ because $\boldsymbol{v}^t(I - sJ_s) = 0$; this is the rationale for choosing $\boldsymbol{v}^t$ to be a left eigenvector of J_s. However, there *is* a nonzero linear term in $z - s$, because $\Sigma v_i Q_i(h_1^*, h_2^*, \ldots, h_\nu^*) > 0$ since each $v_i > 0$. Now since $0 < s < R$, each $h_i(z)$ is analytic at $z = s$ and so may be expanded in a power series around $z = s$. Substituting this power series for each occurrence of h_i in (3.6) yields an equation of the form

$$C_1(z-s) = C_2(z-s)^2 + C_3(z-s)^3 + \ldots$$

with $C_1 = \Sigma v_i Q_i(h_1^*, h_2^*, \ldots, h_\nu^*) > 0$, which is impossible. □

PROPOSITION 3.6: *Assume that $p_e > 0$ and that $p_i > 0$ for each $i \in \mathcal{A}$. Then the value of α in* (3.2) *is $\alpha = 1/2$.*

PROOF: Let $\boldsymbol{v}^t = (v_1, v_2, \ldots, v_\nu)$ be a left eigenvector of J_R corresponding to the eigenvalue $\lambda_R = 1/R$, with $v_i > 0$ for each i. As in the preceding proof, expand (3.5) in a Taylor series around $z = R, h_i^* = h_i(R)$ to obtain (3.6) with $s = R$. Recall that there are no linear terms in $(h_i - h_i^*)$, but that there is a linear term in $(z - R)$. By Proposition 2.6, the polynomials $Q_i(h_1, \ldots, n_\nu)$ have quadratic terms with positive coefficients, so (3.6) includes nonzero quadratic terms in the variables $(h_i - h_i^*)$, $i = 1, 2, \ldots, \nu$. Substituting the Puiseux series (3.2) for $h_i(z)$ at each occurrence of h_i in (3.6) yields

$$(R - z)\{C + \ldots\} = C'(R - z)^{2\alpha} + \ldots$$

where $C > 0, C' > 0$, and on each side $\ldots$ indicates higher order terms. It follows that $2\alpha = 1$. □

PROPOSITION 3.7: *Assume that $p_e > 0$ and that $p_i > 0$ for each $i \in \mathcal{A}$. Then none of the functions $F_x(z), G_x(z), h_i(z)$ has a singularity at any z satisfying $|z| \leq R$ except $z = R$.*

REMARK: D. Cartwright [C] has proved that the Green's function of an aperiodic, irreducible random walk on *any* discrete group has this property.

PROOF: In view of Corollary 2.8, it suffices to consider only points z on the circle $|z| = R$. Moreover, since $F_x(R), G_x(R)$, and $h_i(R)$ are all finite, the power series for F_x, G_x, and h_i converge absolutely on $|z| = R$, so any singularity would necessarily be a branch point.

Suppose z is a singularity of some $h_i, i \in \{1, 2, \ldots, \nu\}$, satisfying $|z| = R$. Then by the complex Implicit Function Theorem (applied to (2.11)) the matrix $I - zJ_z$ is noninvertible. Consider the matrix J_z: its entries are dominated in absolute value by the corresponding entries of J_R, since the coefficients of Q_i are nonnegative and the functions h_j all satisfy $h_j(R) > 0$. Consequently, the spectral radius of J_z is $\leq \lambda_R$, and by Proposition 3.5, $\lambda_R = 1/R$. Therefore, $I - zJ_z$ is noninvertible only if the spectral radius of J_z is equal to $\lambda_R = 1/R$. We will show that if $p_e > 0$ and $p_i > 0$ for all $i \in \mathcal{A}$ then this is impossible.

Fix $j \in \{1, 2, \ldots, \nu\}$. The function $h_j(z)$ is defined by a power series $h_j(z) = \sum_{n=0}^{\infty} q_n z^n$, where q_n is a probability of the form $q_n = P^a\{\tau(x) = n \text{ and } Z_{\tau(x)} = xb\}$. Now we have assumed that $p_e > 0$; hence, for any $n, m \geq 1$,

$$q_{n+m} \geq q_n p_e^m.$$

Also, h_j is nonconstant, so $q_n > 0$ for some $n \geq 1$. It follows that for all z satisfying $|z| = R$ but $z \neq R$,

$$|h_j(z)| < h_j(R).$$

Consider the matrix $J_z = (\partial Q_i / \partial h_j)$. As noted earlier, $J_z^n = (\partial Q_i^{(n)} / \partial h_j)$ where $Q_i^{(n)}$ are the polynomials in (2.13) obtained from (2.11) by substitutions. By Proposition 2.7, for each pair $(i, j) \in \{1, 2, \ldots, \nu\}^2$ there is an $n \geq 2$ such that $Q_i^{(n)}$ contains a term with positive coefficient and having h_j as a factor. Furthermore, since $p_e > 0$, this is also true of each $Q_i^{(n+m)}, m \geq 1$ (see the proof of Lemma 3.4). Consequently, there exists $n \geq 2$

sufficiently large that for each $i \in \{1,2,\ldots,\nu\}$ the polynomial $Q_i^{(n)}$ contains a term with positive coefficient having $(h_1h_2\ldots h_\nu)^2$ as a factor. In view of the result of the preceding paragraph, this implies that each entry of J_z^n is *strictly* smaller in absolute value than the corresponding entry of J_R^n. Therefore, the spectral radius of J_z is strictly less than $\lambda_R = 1/R$. This proves that no $h_j(z)$ has a singularity on $|z| = R$ other than the one at $z = R$.

Recall from the proof of Corollary 2.8 that $(I - zM(z))^{-1}$ has no pole at $z = R$, and since the entries of $M(z)$ are dominated by those of $M(|z|)$ it follows that $(I - zM(z))^{-1}$ has no poles on the circle $|z| = R$. Since the entries of $M(|z|)$ are polynomials in z and h_i, it follows from (2.10) that ϕ_b has no singularities on $|z| = R$ except $z = R$.

It now follows by routine arguments from (2.7), (2.2), and (2.1) that none of the functions $F_x(z), G_x(z)$ has a singularity on $|z| = R$ except for the singularity at $z = R$. □

4. Local Limit Theorem

THEOREM 4.1: *Assume that the random walk Z_n satisfies the irreducibility and aperiodicity hypotheses (1.2) and (1.3). Then there exist positive constants $B_x, x \in \mathcal{G}$, such that for each $x \in \mathcal{G}$, as $n \to \infty$,*

$$p^{*n}(x) \sim \frac{B_x\sqrt{R}}{2\sqrt{\pi}R^n n^{3/2}}. \tag{4.1}$$

Here R is the radius of convergence of the Green's function $G(z)$ (cf. Proposition 2.1).

NOTE: (1) The relations (4.1) clearly do not hold uniformly in x, since $p^{*n}(x) = 0$ if $|x| > nK$. The behavior of $p^{*n}(x)$ when $|x|$ varies linearly with n will be discussed in section 6.

(2) The function $x \to B_x$ is R^{-1}-harmonic, i.e., for each $x \in \mathcal{G}$,

$$B_x = R^{-1}\sum_{y\in\mathcal{G}} B_y p_{y^{-1}x}.$$

This follows immediately from (4.1).

PROOF: Consider first the special case in which $p_e > 0$ and $p_i > 0$ for every $i \in \mathcal{A}$. Then the results of Propositions 3.3, 3.6, and 3.7 are valid. In particular, each $G_x(z)$ has a singularity at $z = R$, no other singularity in $|z| \le R$, and a convergent Puiseux series in a neighborhood of $z = R$:

$$G_x(z) = b_0^{(x)} + b_1^{(x)}(R - z)^{1/2} + \ldots .$$

The series is in integer powers of $(R - z)^\beta$ for some rational β, and the first nonconstant term is a (negative) scalar multiple of $(R - z)^{1/2}$. Consequently, G_x may be written

$$G_x(z) = \sum_{i=1}^{k} B_i(z)(1 - z/R)^{\alpha_i} + C(z)$$

where $\frac{1}{2} = \alpha_1 < \alpha_2 < \ldots < \alpha_k$, each of $B_i(z)$ and $C(z)$ is analytic in a neighborhood of $z = R$, and $B_1(R) = R^{1/2} b_1^{(x)} < 0$. It therefore follows directly from Darboux's method of asymptotic expansion ([B], Th. 4) that as $n \to \infty$

$$p^{*n}(x) \sim -\{B_1(R)/\Gamma(-\alpha_1)\} R^{-n} n^{-1-\alpha_1},$$

which is equivalent to (4.1) with $B_x = -b_1^{(x)}$.

Now assume that (1.2)–(1.3) hold. By Lemma 1.1, there exists an integer $m \geq 1$ such that $p^{*m}(e) > 0$ and $p^{*m}(i) > 0$ for every $i \in \mathcal{A}$. Thus, the result of the preceding paragraph applies to the random walk $Z_{mn}, n \geq 0$; in particular, there exist constants $\tilde{B}_x > 0, x \in \mathcal{G}$, such that

$$p^{*mn}(x) \sim \frac{\tilde{B}_x \sqrt{\tilde{R}}}{2\sqrt{\pi} \tilde{R}^n n^{3/2}} \tag{4.2}$$

as $n \to \infty$, where $\tilde{R}$ is the radius of convergence of $\sum_{n=0}^{\infty} p^{*mn}(e) z^n$. But the Markov property implies that for each $x \in \mathcal{G}$ and any $\ell \in \{1, 2, \ldots, m-1\}$,

$$p^{*(mn+\ell)}(x) = \sum_y p^{*\ell}(y) p^{*mn}(y^{-1}x),$$

where the sum extends over those $y \in \mathcal{G}$ satisfying $|y| \leq \ell K$ (these being the only points of $\mathcal{G}$ that can be reached in ℓ steps from e). Applying (4.2) to each term in the sum yields (4.1) with suitable constants B_x and $R = \tilde{R}^{1/m}$. It then follows that R must be the radius of convergence of the Green's function $G(z) = \Sigma p^{*n}(e) z^n$. □

Darboux's method also gives

THEOREM 4.2: *Assume that $p_e > 0$ and $p_i > 0$ for all $i \in \mathcal{A}$. Then there are positive constants $A_x, x \in \mathcal{G}$, such that for each $x \in \mathcal{G}$*

$$P\{T_x = n\} \sim \frac{A_x \sqrt{R}}{2\sqrt{\pi} R^n n^{3/2}}$$

as $n \to \infty$.

This probability is true under the weaker hypotheses (1.2)–(1.3) , but we do not yet have a complete proof.

5. The Boundary Process

Recall (section 1) that the random walk Z_n is transient, provided the transition probabilities p_x satisfy the irreducibility hypothesis (1.2). Consequently, $Z_n \to Z_\infty$ a.s. as $n \to \infty$ where $Z_\infty \in \Lambda_+$, where Λ_+ is the set of ends of the Cayley graph of $\mathcal{G}$ (equivalently,

Λ_+ is the set of sequences $x_1x_2\ldots$ from the alphabet $\mathcal{A}$ satisfying $x_{n+1} \neq x_n^{-1} \;\; \forall\, n \geq 1$). Thus, the limit point Z_∞ may be written

$$Z_\infty = A_1A_2A_3\cdots$$

where $A_1, A_2, \ldots \in \mathcal{A}$ and $A_{n+1} \neq A_n^{-1}$. In this section we consider the structure of the "boundary process" $A_1, A_2, \cdots$.

Define μ_n to be the distribution under $P = P^e$ of the process $A_n, A_{n+1}, \ldots$, i.e., for any Borel subset $U \subset \Lambda_+$,

$$\mu_n(U) = P\{(A_n, A_{n+1}, \ldots) \in U\}.$$

We will prove that there is a shift-invariant probability measure μ_φ on Λ_+ such that $\mu_n \ll \mu_\varphi$ and $\mu_n \xrightarrow{\mathcal{D}} \mu_\varphi$ as $n \to \infty$. Moreover, we will show that μ_φ is a *Gibbs state* in the sense of [Bo], ch. 1 (see below for the definition) and will identify the potential function φ; it will then follow that the stationary process induced by μ_φ is Bernoulli ([Bo], Th. 1.25).

The crux of the argument consists of obtaining a manageable asymptotic formula for certain hitting probabilities. Observe that $Z_\infty = x_1x_2x_3\ldots$ iff for each $m = 1, 2, \ldots$ the random walk Z_n hits the set $x_1x_2\ldots x_m\mathcal{B}$ and then at some future time exits $x_1x_2\ldots x_m\mathcal{B}$ a final time without "erasing" any of the letters $x_1x_2\ldots x_m$. Therefore, by Proposition 2.3,

$$P\{A_j = x_j \;\; \forall\, 1 \leq j \leq m\} = u^t H_{x_1}(1)H_{x_2}(1)\ldots H_{x_m}(1)v \tag{5.1}$$

where u, v are $|\mathcal{B}| \times 1$ column vectors with entries

$$u_e = 1, u_y = 0 \;\; \forall\, y \in \mathcal{B}\backslash\{e\};$$
$$v_y = v_y^{x_m} = P^y\{A_1 \neq x_m^{-1}\} \;\; \forall\, y \in \mathcal{B}.$$

Notice that u has nonnegative entries, while v has strictly positive entries. It is evident from (5.1) that the finite dimensional distributions of the process A_n are controlled by the matrix products $H_{x_1}H_{x_2}\ldots H_{x_m}$. The next order of business, then, is to study the asymptotic behavior of such products.

Let (X, d) be a compact metric space and let Y be a closed subset of $X \times X$ such that for each $x \in X$ the set $\{y : (x, y) \in Y\}$ is nonempty. Let $x \to M_x$ be a continuous function from X to the space of $N \times N$ matrices with nonnegative entries. Call this function *primitive* (relative to Y) if there exists an integer $m \geq 1$ such that for every m-tuple $(x_1, x_2, \ldots, x_m) \in X^m$ satisfying $(x_i, x_{i+1}) \in Y$ for all $i = 1, 2, \ldots, m-1$, the matrix product $M_{x_1}M_{x_2}\ldots M_{x_m}$ has all entries strictly positive. (See [Se], Ch. 1, for the source of the term "primitive".)

Define $\Sigma = \Sigma_Y$ to be the set of all doubly infinite sequences $\xi = (x_n)_{n=-\infty}^{\infty}$ with entries $x_n \in X$ and satisfying $x_n, x_{n+1}) \in Y$ for all $n \in \mathbb{Z}$. Since Y is closed, Σ is a closed subset of the sequence space $X^{\mathbb{Z}}$, hence is compact in the product topology. Let d_* by the metric on Σ (or $X^{\mathbb{Z}}$) defined by

$$d_*(\xi, \zeta) = \sum_{n=-\infty}^{\infty} d(x_n, y_n)/2^{|n|};$$

the topology induced by d_* is the product topology. Let $\sigma: \Sigma \to \Sigma$ be the forward shift (thus, the n^{th} entry of $\sigma\xi$ is x_{n+1}); note that σ is Lipschitz continuous. For any function $f: \Sigma \to \mathbb{C}$ define

$$S_n f = f + f \circ \sigma + f \circ \sigma^2 + \ldots + f \circ \sigma^{n-1}, n \geq 0.$$

(In the applications considered in this paper, $X = \mathcal{A} \times F$ where $\mathcal{A}$ is the set of generators (and their inverses) of the group $\mathcal{G}$ and F is a compact subset of $\mathbb{C}$. The relation R is defined by $((i,z),(j,z')) \in R$ iff $j \neq i^{-1}$. The mapping $x \to M_x$ is given by $(i,z) \to H_i(z)$.)

PROPOSITION 5.1: *Assume that $x \to M_x$ is primitive and Hölder continuous (for some exponent) on X. Then there exist constants $C < \infty$ and $0 < \alpha < 1$ and Hölder continuous functions $\varphi, \gamma: \Sigma \to \mathbb{R}$ and $V, W: \Sigma \to \mathcal{P}_+ = \{v \in \mathbb{R}^n : v_i > 0 \ \ \forall\, i$ and $\sum_i v_i = 1\}$ such that for every $\xi = (x_n)_{-\infty}^{\infty} \in \Sigma$, and $n = 1, 2, \ldots$*

$$||e^{-S_n\varphi(\xi)} M_{x_1} M_{x_2} \ldots M_{x_n} - \gamma(\sigma^n\xi) V(\xi) W(\sigma^n\xi)^t|| \leq C\alpha^n. \tag{5.2}$$

Also,

$$\gamma(\xi) = 1/W(\xi)^t V(\xi), \tag{5.3}$$

$$M_{x_1} V(\sigma\xi) = e^{\varphi(\xi)} V(\xi), \tag{5.4}$$

and

$$W(\sigma^{-1}\xi)^t M_{x_1} = e^{\varphi(\alpha\xi)} \left(\frac{\gamma(\xi)}{\gamma(\alpha^{-1}\xi)}\right) W(\xi)^t; \tag{5.5}$$

$V(\xi)$ and $\varphi(\xi)$ are functions only of the "forward" coordinates $x_1, x_2, \ldots$; and $W(\xi)$ is a function only of the "backward" coordinates $\ldots, x_{-1}, x_0$.

NOTE: (1) $||\cdot||$ is the usual matrix norm, i.e., $||M|| = \sup_{v\neq 0}(|Mv|/|v|)$.

(2) If $K = 1$ (nearest neighbor random walk) then $V \equiv W \equiv \gamma \equiv 1$ and $\varphi(\xi) = \varphi(x_1)$. In this case the result (5.2) is trivial.

(3) Proposition 5.1 is an extension of the classical Perron-Frobenius theorem, which is the special case where $x \to M_x \equiv M$ is a constant function. Observe that for each n-periodic sequence ξ the vectors $V(\xi)$ and $W(\xi)$ are right and left eigenvectors, respectively, of $M_{x_1} M_{x_2} \ldots M_{x_n}$; the corresponding eigenvalue is $\exp\{S_n\varphi(\xi)\}$.

PROOF: Without loss of generality, we may assume that for each $x \in X$ the entries of M_x are all strictly positive. If $x \to M_x$ does *not* have this property, but *is* primitive, then we may replace X by $\{(x_1, \ldots, x_m) \in X^m: (x_i, x_{i+1}) \in Y \quad \forall\, i\}$ and $x \to M_x$ by $(x_1, \ldots, x_m) \to M_{x_1} M_{x_2} \ldots M_{x_m}$; it is easy to deduce (5.2) for the old map $x \to M_x$ from (5.2) for the new one, and (5.3)–(5.5) follow easily from (5.2).

Assume, then, that each $M_x, x \in X$, has positive entries. Define

$$\begin{aligned}\mathcal{P} &= \{v \in \mathbb{R}^N: v_i \geq 0 \;\; \forall\, i \text{ and } \Sigma v_i = 1\},\\ \mathcal{P}_+ &= \{v \in \mathbb{R}^N: v_i > 0 \;\; \forall\, i \text{ and } \Sigma v_i = 1\},\\ \mathcal{P}_\varepsilon &= \{v \in \mathbb{R}^N: v_i \geq \varepsilon \;\; \forall\, i \text{ and } \Sigma v_i = 1\}, \varepsilon > 0,\end{aligned}$$

and for each $x \in X$ define functions $T_x: \mathcal{P} \to \mathcal{P}_+$ and $T_x^*: \mathcal{P} \to \mathcal{P}_+$ by

$$T_x(v) = \frac{M_x v}{\mathbf{1}^t M_x v} \quad \text{and} \quad T_x^*(v) = \frac{M_x^t v}{\mathbf{1}^t M_x^t v}$$

(here $\mathbf{1}^t = (1, 1, \ldots, 1)$ and superscript t denotes transpose). The functions $T_\cdot(\cdot)$ and $T_\cdot^*(\cdot)$ are both jointly continuous as mappings of $X \times \mathcal{P}$ into $\mathcal{P}_+$; since $X \times \mathcal{P}$ is compact,the images are compact subsets of $\mathcal{P}_+$. Consequently, there exists $\varepsilon > 0$ such that for every $x \in X, T_x(\mathcal{P}) \subset \mathcal{P}_\varepsilon$ and $T_x^*(\mathcal{P}) \subset \mathcal{P}_\varepsilon$.

Let d_p be the projective metric on $\mathcal{P}_+$ defined by $d_p(v, w) = \max_{i,j} \log(v_i w_j / v_j w_i)$. Since $\mathcal{P}_\varepsilon$ is a compact subset of $\mathcal{P}, d_p$ is uniformly Lipschitz equivalent to the usual Euclidean metric d_E on $\mathcal{P}_\varepsilon$, i.e., there exist positive constants C_1, C_2 such that $C_1 d_E \leq d_p \leq C_2 d_E$ on $\mathcal{P}_\varepsilon \times \mathcal{P}_\varepsilon$. It is well known (and easy to prove – see [Se], section 3.1) that, since the entries of M_x are positive, the induced maps T_x and T_x^* are contractive on $\mathcal{P}_+$ relative to d_p. Thus, since $x \to M_x$ is continuous and X, compact, there exists a constant $0 < \alpha < 1$ such that

$$\begin{aligned}d_p(T_x v, T_x w) &\leq \alpha d_p(v, w),\\ d_p(T_x^* v, T_x^* w) &\leq \alpha d_p(v, w)\end{aligned} \tag{5.6}$$

for all $v, w \in \mathcal{P}_\varepsilon$ and $x \in X$. Consequently, since $x \to M_x$ is Hölder continuous (for some exponent) there exist Hölder continuous (for some possibly different exponent) functions $V, W: \Sigma \to \mathcal{P}_\varepsilon$ such that for every $\xi = (x_n)_{n=-\infty}^{\infty} \in \Sigma$,

$$\begin{aligned}\lim_{n\to\infty} T_{x_1} T_{x_2} \ldots T_{x_n} v &= V(\xi),\\ \lim_{n\to\infty} T_{x_0}^* T_{x_{-1}}^* \ldots T_{x_{-n}} w &= W(\sigma^{-1}\xi),\end{aligned} \tag{5.7}$$

uniformly for $v, w \in \mathcal{P}_\varepsilon$ and $\xi \in \Sigma$. It is clear that V and W are functions of the forward and backward coordinates of ξ, respectively. Note that since d_E and d_p are uniformly Lipschitz equivalent on $\mathcal{P}_\varepsilon$, the Hölder continuity of V and W is valid in either metric.

We now *define* $\varphi(\xi)$ and $\gamma(\xi)$ by (5.4) and (5.3), respectively; Hölder continuity of φ and γ follows immediately from the Hölder continuity of V, W, and $x \to M_x$. Note that (5.5) will follow directly from (5.2). Thus, it remains to prove (5.2).

For $\xi = (x_n)_{n=-\infty}^{\infty} \in \Sigma$ and $n = 1, 2, \ldots$ set

$$\Phi_n(\xi) = M_{x_1} M_{x_2} \ldots M_{x_n},$$
$$\Psi_n^*(\xi) = T_{x_n}^* \circ T_{x_{n-1}}^* \circ \ldots \circ T_{x_1}^*.$$

Then

$$e^{-S_n\varphi(\xi)}\Phi_n(\xi)V(\sigma^n\xi) = V(\xi), \tag{5.8}$$

by (5.4) and by (5.6)–(5.7) there exists a constant $C < \infty$ such that for every $w \in \mathcal{P}$ and $n = 1, 2, \ldots,$

$$d_E(\Psi_n^*(\xi)w, W(\sigma^n\xi)) \le C\alpha^{n-1}. \tag{5.9}$$

We will deduce (5.2) from (5.8) and (5.9). First, notice that for every $u \in \mathcal{P}$ the vector $\Psi_n^*(\xi)u$ is a scalar multiple of $\Phi_n(\xi)^t u$. Consequently, if $u^{(1)}, u^{(2)}, \ldots, u^{(N)}$ are the standard unit vectors in $\mathbb{R}^N$ (i.e., $u^{(1)} = (1, 0, 0, \ldots, 0)^t$, etc.) then there exist scalars $a_n^{(i)}$ and vectors $U_n^{(i)}(\xi) \in \mathbb{R}^N$ such that

$$e^{-S_n\varphi(\xi)}\Phi_n(\xi)^t u^{(i)} = a_n^{(i)}\{W(\sigma^n\xi) + U_n^{(i)}(\xi)\} \tag{5.10}$$

and, by (5.9),

$$|U_n^{(i)}(\xi)| \le C'\alpha^n$$

for some constant $C' < \infty$ independent of $n, i,$ and ξ. Next, by (5.8), the i^{th} coordinate of $V(\xi)$ is given by

$$\begin{aligned} &e^{-S_n\varphi(\xi)}V(\sigma^n\xi)^t\Phi_n(\xi)^t u^{(i)} \\ &\quad = a_n^{(i)}\{W(\sigma^n\xi)^t V(\sigma^n\xi) + U_n^{(i)}(\xi)V(\sigma^n\xi)\} \\ &\quad = a_n^{(i)}\gamma(\sigma^n\xi)^{-1} + a_n^{(i)}O(\alpha^n), \end{aligned}$$

where the $O(\alpha^n)$ term is uniform in i and ξ. It now follows that

$$a_n^{(i)} = \gamma(\sigma^n\xi)V(\xi)_i + O(\alpha^n)$$

where the bound implicit in the $O(\alpha^n)$ term is uniform in i and ξ. It is now evident that (5.10) is equivalent to (5.2). □

Unfortunately, Proposition 5.1 cannot be applied directly to the matrix product in equation (5.1) because the function $i \to H_i(1)$, $i \in \mathcal{A}$, is not primitive. To see this, recall (Proposition 2.3) that for any reduced word $x = x_1 x_2 \ldots x_m$ from the alphabet $\mathcal{A}, H_x = H_{x_1} H_{x_2} \ldots H_{x_m}$; consequently, for any $a, b \in \mathcal{B}$ the $(a, b)^{th}$ entry of the product in (5.1) is

$$H_x^{ab}(1) = P^a\{\tau(x) < \infty; Z_{\tau(x)} = xb\}$$

which is zero for certain b, e.g., any $b = b_1 b_2 \ldots b_K$ such that $b_1 \ne x_m^{-1}$ (provided $m \ge 2K$).

Thus, we will need a generalization of Proposition 5.1 for certain *imprimitive* maps $x \to M_x$. We use the same notation and conventions as in Proposition 5.1 and its proof.

PROPOSITION 5.2: *Assume that $x \to M_x$ is a Hölder continuous mapping of X into the space of $N \times N$ matrices with nonnegative entries. Assume further that there exist integers $m \geq 0, r \geq 1$ and a function $X^r \to 2^{\{1,2,\ldots,N\}}$ taking r-tuples $(x_1, x_2, \ldots, x_r)$ to nonempty subsets $B(x_1, x_2, \ldots, x_r)$ of $\{1, 2, \ldots, N\}$ such that for every $n \geq m$ and all $x_1, x_2, \ldots, x_{n+r} \in X$ satisfying $(x_i, x_{i+1}) \in Y \ \ \forall\, i$,*

$$(5.11) \qquad (M_{x_1} M_{x_2} \ldots M_{x_{n+r}})_{ij} > 0 \iff j \in B(x_{n+1}, x_{n+2}, \ldots, x_{n+r}).$$

Then there exist constants $C < \infty$ and $0 < \alpha < 1$ and Hölder continuous functions $\varphi, \gamma: \Sigma \to \mathbb{R}, V: \Sigma \to \mathcal{P}_+$, and $W: \Sigma \to \mathcal{P}$ such that for every $\xi = (x_n)_{n=-\infty}^{\infty} \in \Sigma$ and $n = 1, 2, \ldots$, relations (5.2)-(5.5) are valid. Moreover,

$$(5.12) \qquad W(\xi)_j > 0 \iff j \in B(x_{1-r}, x_{2-r}, \ldots, x_0).$$

PROOF: Without loss of generality we may assume that $r = 1$ and $m = 0$ (if not, replace X by the appropriate subset of X^{m+r} and $x \to M_x$ by $(x_1, x_2, \ldots, x_{m+r}) \to M_{x_1} M_{x_2} \ldots M_{x_{m+r}}$).

Let $\mathcal{P}, \mathcal{P}_+, \mathcal{P}_\varepsilon$ be as in the proof of Proposition 5.1, and for each nonempty subset R of $\{1, 2, \ldots, N\}$ and $\varepsilon > 0$, define

$$\mathcal{P}_+(R) = \{v \in \mathcal{P}: v_i > 0 \iff i \in R\},$$
$$\mathcal{P}_\varepsilon(R) = \{v \in \mathcal{P}_+(R): v_i \geq \varepsilon \ \ \forall\, i \in R\}.$$

For each $x \in X$ define functions $T_x: \mathcal{P}_+ \to \mathcal{P}_+$ and $T_x^*: \mathcal{P} \to \mathcal{P}_+(B(x))$ by

$$T_x v = \frac{M_x v}{\mathbf{1}^t M_x v} \text{ and } T_x^* v = \frac{M_x^t v}{\mathbf{1}^t M_x^t v};$$

that T_x and T_x^* are well-defined and take values in $\mathcal{P}_+$ and $\mathcal{P}_+(B(x))$, respectively, follows from the hypothesis (5.11), the standing assumption $r = 1, m = 0$, and the fact that $B(x) \neq \phi$ (to assure that the denominators are nonzero). As in the proof of Proposition 5.1, the compactness of X implies that there exists $\varepsilon > 0$ such that $T_x(\mathcal{P}_+) \subset \mathcal{P}_\varepsilon$ and $T_x^*(\mathcal{P}_\varepsilon(B(x)))$.

Each $T_x: \mathcal{P}_\varepsilon \to \mathcal{P}_\varepsilon$ is contractive in the projective metric on $\mathcal{P}_+$. Consequently, by compactness of X and $\mathcal{P}_\varepsilon$, there exists a Hölder continuous $V: \Sigma \to \mathcal{P}_\varepsilon$ such that for every $\xi = (x_n)_{-\infty}^{\infty} \in \Sigma$,

$$\lim_{n \to \infty} T_{x_1} T_{x_2} \ldots T_{x_n} v = V(\xi)$$

uniformly for $v \in \mathcal{P}_\varepsilon$ and $\xi \in \Sigma$.

For each nonempty $R \subset \{1, 2, \ldots, N\}$ there is a projective metric on $\mathcal{P}_+(R)$, which is uniformly Lipschitz equivalent to the Euclidean metric on $\mathcal{P}_\varepsilon(R)$. For any finite sequence

$x_1, x_2, \ldots, x_n$ from X such that $B(x_1) = B(x_n), T_{x_1} T_{x_2} \ldots T_{x_n}$ maps $\mathcal{P}_+(B(x_1))$ into itself and is contractive in the projective metric. Note that there are only $2^N - 1$ possibilities for $B(x_i)$; hence, for any sequence $x_1 x_2 \ldots x_n$ there exist $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ with $m \geq n/2^N$ such that $B(x_{i_1}) = B(x_{i_2}) = \ldots = B(x_{i_m})$. Consequently, by compactness of X and $\mathcal{P}_\varepsilon(R)$, there exist a Hölder continuous function $W: \Sigma \to \mathcal{P}$ and constants $C' < \infty, 0 < \alpha < 1$ such that for every $\xi = (x_n)_{-\infty}^{\infty} \in \Sigma$ and $w \in \mathcal{P}$,

$$d_E(T^*_{x_0} T^*_{x_{-1}} \cdots T^*_{x_{-n}} w, W(\sigma^{-1}\xi)) \leq C' \alpha^n$$

for all $n = 1, 2, \cdots$. Observe that $W(\xi) \in \mathcal{P}_\varepsilon(B(x_0))$ for every $\xi \in \Sigma$.

The proof of (5.2)–(5.5) may now be completed by the same argument used in the proof of Proposition 5.1. □

We turn again to the matrix product in equation (5.1). We will show that, although the assignment $i \to H_i(1)$ is imprimitive, nevertheless the hypothesis (5.11) of Proposition 5.2 is satisfied. Take $X = \mathcal{A} \times F$, where F is a compact subset of $(O, R]$, and $Y = \{((i, z), (j, z')) \in X^2 : j \neq i^{-1}\}$; let $x \to M_x$ be the assignment $(i, z) \to H_i(z)$. For each $r \geq 0$ let $\mathcal{G}_r = \{y \in \mathcal{G} : |y| = r\}$.

LEMMA 5.3: *There exists an integer $r \geq 1$ and a function $\mathcal{G}_r \to 2^{\mathcal{B}}$ taking each $y = y_1 y_2 \ldots y_r \in \mathcal{G}_r$ to a nonempty subset $B(y)$ of $\mathcal{B}$, such that the following is true. For any $n \geq 0$ and $y = y_1 y_2 \ldots y_{n+r} \in \mathcal{G}_{n+r}$, and any choice of $z_1, z_2, \ldots, z_{n+r} \in (O, R]$,*

$$(H_{y_1}(z_1) H_{y_2}(z_2) \ldots H_{y_{n+r}}(z_{n+r}))^{ab} > 0 \iff b \in B(y_{n+1} y_{n+2} \ldots y_{n+r}). \tag{5.13}$$

PROOF: Recall that $H_x^{ab}(z) = E^a z^{\tau(x)} 1\{Z_{\tau(x)} = xb; \tau(x) < \infty\}$. It is clear from this that either $H_x^{ab}(z) > 0$ for all $z \in (O, R]$ or $H_x^{ab}(z) = 0$ for all $z \in (O, R]$. Thus, it suffices to prove (5.13) for $z_1 = z_2 = \ldots = z_{n+r} = 1$. Now by Proposition 2.3,

$$\begin{aligned}(H_{y_1}(1) H_{y_2}(1) \ldots H_{y_{n+r}}(1))^{ab} &= H_y^{ab}(1) \\ &= H_{a^{-1}y}^{eb}(1) \\ &= P\{\tau(a^{-1}y) < \infty; Z_{\tau(a^{-1}y)} = a^{-1}yb\}.\end{aligned}$$

This is positive iff there is a positive probability path from e to $a^{-1}yb$ that does not enter $a^{-1}y\mathcal{B}$ until the last step.

Recall the standing assumption (1.2) that the random walk Z_n may visit any $x \in \mathcal{G}$ with positive probability. Hence, for each $i \in \mathcal{A}$ there is a positive probability path from e to i. Since there are only finitely many elements of $\mathcal{A}$ it follows that there exists $s > 0$ such that for every $i \in \mathcal{A}$ there are positive probability paths from e to i and from i to e that stay entirely in $\cup_{n \leq s} \mathcal{G}_n$. Consequently, for any $y \in \mathcal{G}$ such that $|y| > 2K + s$,

$$H_y^{eb}(1) > 0 \iff H_y^{ib}(1) > 0 \iff H_{i^{-1}y}^{eb} > 0.$$

The last inequality allows us to remove the letters $y_1, y_2, \dots$ from y, one at a time, until the word size is reduced to $2K+s$. Therefore, whether $H_y^{eb}(1) > 0$ or $H_y^{eb}(1) = 0$ depends only on b and the last $2K+s$ letters in y. The lemma now follows with $r = 3K+s$. □

Lemma 5.3 implies that the conclusions of Proposition 5.2 are applicable to $i \to H_i(1)$, where $i \in \mathcal{A}$, and $Y = \{(i,j) \in \mathcal{A} \times \mathcal{A} : j \neq i^{-1}\}$. Let Λ be the set of all doubly infinite sequences $\xi = (x_n)_{n=-\infty}^{\infty}$ with entries $x_n \in \mathcal{A}$ satisfying $(x_n, x_{n+1}) \in Y$ for all $n \in \mathbb{Z}$. Then by Proposition 5.2 there exist constants $C < \infty$ and $0 < \alpha < 1$ and Hölder continuous functions $\varphi, \gamma : \Lambda \to \mathbb{R}, V : \Lambda \to \mathcal{P}_+$, and $W : \Lambda \to \mathcal{P}$ (where $\mathcal{P} = \{v \in \mathbb{R}^{\mathcal{B}} : v_b \geq 0 \ \ \forall\, b$ and $\sum_{b \in \mathcal{B}} v_b = 1\}$ and $\mathcal{P}_+ = \{v \in \mathcal{P} : v_b > 0 \ \ \forall\, b \in \mathcal{B}\}$) such that

$$||e^{-S_n\varphi(\xi)} H_{x_1}(1) H_{x_2}(1) \cdots H_{x_n}(1) - \gamma(\sigma^n\xi) V(\xi) W(\sigma^n\xi)^t|| \leq C\alpha^n$$

for all $\xi \in \Lambda$ and $n = 0, 1, 2, \dots$. Applying this to equation (5.1) gives

$$P\{A_j = x_j \ \ \forall\, 1 \leq j \leq m\} = C(\xi)\tilde{C}(\sigma^m\xi)e^{S_m\varphi(\xi)}(1 + O(\alpha^m)) \tag{5.14}$$

for every $\xi = (x_n)_{-\infty}^{\infty} \in \Lambda$ and $m \geq 1$, where the bound implicit in the $O(\alpha^m)$ term is uniform in ξ. Here

$$C(\xi) = u^t V(\xi) = V(\xi)_e > 0,$$
$$\tilde{C}(\xi) = \gamma(\xi) W(\xi)^t v^{x_0} > 0,$$

because V has all entries positive and so does v^{x_0} (see (5.1)).

According to [Bo], Th. 1.4, there is a unique shift-invariant probability measure μ_φ on Λ with the following property: there exist constants $0 < C_1 < C_2 < \infty$ and $P(\varphi) \in \mathbb{R}$ such that for every cylinder set

$$\Lambda(x_1 x_2 \dots x_m) = \{(y_n)_{-\infty}^{\infty} \in \Lambda : y_n = x_n \ \ \forall\, 1 \leq n \leq m\}$$

we have

$$C_1 \leq \frac{\mu_\varphi(\Lambda(x_1 x_2 \dots x_m))}{\exp\{S_m\varphi(\xi) - mP(\varphi)\}} \leq C_2 \tag{5.15}$$

for every $\xi \in \Lambda(x_1 x_2 \dots x_m)$. The measure μ_φ is called the *Gibbs state* with potential function φ, and the constant $P(\varphi)$ is called the thermodynamic pressure of φ. It should be noted that μ_φ is the distribution of a k-step Markov chain if φ is a function of the first k coordinates $x_1, x_2, \dots, x_k$ of ξ.

THEOREM 5.4: *For all $n \geq 1$ the measure μ_n is absolutely continuous with respect to μ_φ, and $\mu_n \xrightarrow{\mathcal{D}} \mu_\varphi$ as $n \to \infty$. Moreover, $P(\varphi) = 0$.*

PROOF: For each $m \geq 1$ the sum over all reduced words $x_1 x_2 \dots x_m$ of the probabilities $P\{A_n = x_n \ \ \forall\, 1 \leq n \leq m\}$ is 1. Since $C(\xi), C'(\xi)$ are bounded away from 0 and ∞ and μ_ϕ is a probability measure, it follows from (5.14) and (5.15) that $P(\varphi) = 0$. Moreover, since (5.14)–(5.15) hold for *all* cylinder sets and since the cylinder sets generate the Borel σ-algebra, it follows that $\mu_1 \ll \mu_\varphi$ and that the Radon-Nikodym derivative $h = \left(\frac{d\mu_1}{d\mu_\varphi}\right)$ is bounded away from 0 and ∞. Now consider the restrictions of μ_1, μ_φ to the σ-algebra $\mathcal{F}_n$ generated by the coordinate functions $\xi_j, j \geq n$; one has

$$\left(\frac{d\mu_1|\mathcal{F}_n}{d\mu_\varphi|\mathcal{F}_n}\right) = E_{\mu_\varphi}(h|\mathcal{F}_n) \longrightarrow E_{\mu_\varphi}(h|\mathcal{F}_\infty) = 1$$

because the tail field $\mathcal{F}_\infty = \bigcap_{n\geq 1} \mathcal{F}_n$ is 0–1 under μ_φ (μ_φ is mixing – see [Bo], Ch. 1). It follows that $\mu_n \ll \mu_\varphi$ and $\mu_n \xrightarrow{\mathcal{D}} \mu_\varphi$. □

NOTE: When $K = 1$ (nearest neighbor random walk) $\varphi(\xi) = \varphi(x_0)$ and the boundary process A_n is a Markov chain on the state space $\mathcal{A} = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_L, a_L^{-1}\}$. When $K > 1$ it appears that in general the process induced by μ_φ is non-Markovian.

6. Saddlepoint Approximations

The Local Limit Theorem (section 4) gives asymptotic approximations as $n \to \infty$ for the transition probabilities $p^{*n}(x)$ for fixed $x \in \mathcal{G}$. These do not, however, hold uniformly for $x \in \mathcal{G}$, and thus give no information about $p^{*n}(x)$ as $n \to \infty$ and $|x| \to \infty$ simultaneously.

To derive approximations suitable for this case we will use the saddlepoint method ([dB], Ch. 5). The strategy is the same as in the nearest neighbor case [La], to wit, to analyze the exact formula

$$p^{*n}(x) = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} G_x(re^{i\theta}) e^{-in\theta} d\theta \tag{6.1}$$

for an appropriate value of $r \in (O, R)$. (Note: (6.1) is a consequence of the Fourier inversion formula, and is valid for all $n \geq 0, x \in \mathcal{G}$, and $0 < r \leq R$.) But the analysis of the generating function(s) $G_x(z)$ is significantly different in the finite range case. Recall that by (2.1) and (2.6)–(2.7),

$$G_x(z) = F_x(z) G(z), \tag{6.2}$$

$$F_x(z) = u^t H_{x_1}(z) H_{x_2}(z) \ldots H_{x_m}(z) \phi(z), \tag{6.3}$$

where $x = x_1 x_2 \ldots x_m$ is the reduced word representation of $x, u_e = 1$ and $u_b = 0 \ \forall\, b \in \mathcal{B}\backslash\{e\}$, and $\phi(z) = (\phi_b(z))_{b\in\mathcal{B}}$ (and recall that $\phi_b(z) = F_{b^{-1}}(z)$). In the case of nearest neighbor random walk the matrix product in (6.3) can be reduced to a product of scalar-valued generating functions, but in the finite-range case the matrix product cannot be avoided.

By Lemma 5.3 the results of Proposition 5.2 are valid for the assignment $(i, z) \to H_i(z)$ where $(i, z) \in \mathcal{A} \times F$ and F is any compact subset of $(O, R]$. Thus, there exist Hölder continuous functions φ, γ, V, W on $\Lambda \times F$ such that for each $z \in F$ the relations (5.2)–(5.5) are valid for $\varphi(\xi) = \varphi(\xi, z), \gamma(\xi) = \gamma(\xi, z)$, etc., and $M_i = H_i(z)$. (NOTE: φ is different from ϕ.) Our first task will be to show that $\varphi(\xi, z), \gamma(\xi, z), V(\xi, z)$, and $W(\xi, z)$ are analytic in z for z in some region containing (O, R) and that (5.2)–(5.5) remain valid for complex z sufficiently near (O, R).

For each $\xi = (x_n)_{n=-\infty}^{\infty} \in \Lambda$ and $n = 1, 2, \ldots$, define

$$\Phi_n(\xi, z) = H_{x_1}(z) H_{x_2}(z) \ldots H_{x_n}(z).$$

PROPOSITION 6.1: *There exists a connected open neighborhood $\mathcal{N}$ of $(O, R]$ in the closed disk $\{|z| \leq R\}$ such that for each $\xi \in \Lambda$ the functions $\varphi(\xi, z), \gamma(\xi, z), V(\xi, z)$, and $W(\xi, z)$ extend continuously to $z \in \mathcal{N}$ and analytically to $z \in \mathcal{N} \cap \{|z| < R\}$. For each compact subset $\mathcal{K} \subset \mathcal{N}$ there are constants $C < \infty$ and $0 < \alpha < 1$ such that for all $\xi \in \Lambda, z \in \mathcal{K}$, and $n = 1, 2, \ldots$,*

$$\| \exp\{-S_n \varphi(\xi, z)\} \Phi_n(\xi, z) - \gamma(\sigma^n \xi, z) V(\xi, z) W(\sigma^n \xi, z)^t \| \leq C \alpha^n; \tag{6.4}$$

$$\Phi_1(\xi, z) V(\sigma \xi, z) = e^{\varphi(\xi, z)} V(\xi, z); \tag{6.5}$$

$$W(\sigma^{-1}\xi, z)^t \Phi_1(\xi, z) = e^{\varphi(\xi, z)} \left\{ \frac{\gamma(\xi, z)}{\gamma(\sigma^{-1}\xi, z)} \right\} W(\xi, z)^t; \tag{6.6}$$

and

$$\gamma(\xi, z) = \frac{1}{W(\xi, z)^t V(\xi, z)} \neq 0 \tag{6.7}$$

The vectors V, W satisfy $\mathbf{1}^t V(\xi, z) = \mathbf{1}^t W(\xi, z) = 1$, and $V(\xi, z)$ has all entries nonzero. The functions φ, γ, V, and W are all jointly continuous in ξ and z, and for each fixed z are Hölder continuous in ξ. The functions $V(\xi)$ and $\varphi(\xi)$ are functions only of the "forward" coordinates $x_1, x_2, \ldots$; and $W(\xi)$ is a function only of the "backward" coordinates $\ldots, x_{-1}, x_0$.

PROOF: We will show that as $n \to \infty$,

$$\frac{\Phi_n(\xi, z)\mathbf{1}}{\mathbf{1}^t \Phi_n(\xi, z)\mathbf{1}} \longrightarrow V(\xi, z)$$

and

$$\frac{\Phi_n(\sigma^{-n}\xi, z)^t \mathbf{1}}{\mathbf{1}^t \Phi_n(\sigma^{-n}\xi, z)^t \mathbf{1}} \longrightarrow W(\xi, z)$$

uniformly for $\xi \in \Lambda$ and z in (sufficiently small) closed disks centered at points on $(O, R]$, and that the errors are exponentially small in n. Since for each $n \geq 1$ the functions $\Phi_n(\xi, z)$ and $\Phi_n(\sigma^{-n}\xi, z)^t$ are jointly continuous in (ξ, z) and analytic in z, it will then follow that V and W are continuous in (ξ, z) and analytic in z, and consequently, by (6.5) and (6.7), that φ and γ are also continuous in (ξ, z) and analytic in z. Hölder continuity will follow from the exponential rate of convergence.

To establish (6.4) and the convergences indicated in the previous paragraph, we will transfer certain of the ideas and arguments used in proving Propositions 5.1–5.2 to the setting of a suitable *complex* projective space. Let $N = |\mathcal{B}|$; we will identify $\mathbb{R}^N$ with $\mathbb{R}^{\mathcal{B}}$ and $\mathbb{C}^N$ with $\mathbb{C}^{\mathcal{B}}$ wherever convenient. (Keep in mind that $H_x(z)$ and therefore $\Phi_n(\xi, z)$ are $N \times N$ matrices.) Let $\mathcal{P}, \mathcal{P}_+$, and $\mathcal{P}_\varepsilon$ be as in the proof of Prop. 5.1, and for each nonempty $B \subset \mathcal{B}$ let $\mathcal{P}_+(B)$ and $\mathcal{P}_\varepsilon(B)$ be as in the proof of Prop. 5.2. Define analogous spaces of complex vectors as follows:

$$
\begin{aligned}
&\overline{\mathcal{P}} = \{v \in \mathbb{C}^N : \mathbf{1}^t v = 1\};\\
&\overline{\mathcal{P}}_\varepsilon = \{v \in \mathcal{P} : |v_i| \geq \varepsilon \ \ \forall\, i\};\\
&\overline{\mathcal{P}}(B) = \{v \in \overline{\mathcal{P}} : v_i \neq 0 \text{ iff } i \in B\};\\
&\overline{\mathcal{P}}(B) = \{v \in \overline{\mathcal{P}}(B) : |v_i| \geq \varepsilon \ \ \forall\, i \in B\}.
\end{aligned}
$$

Note that $\mathcal{P}, \mathcal{P}_\varepsilon$ are compact subsets of $\overline{\mathcal{P}}, \overline{\mathcal{P}}_\varepsilon$, respectively, and that $\mathcal{P}_\varepsilon(B)$ is a compact subset of $\overline{\mathcal{P}}_\varepsilon(B)$; also, $\mathcal{P}_+(B) \subset \overline{\mathcal{P}}(B)$ but neither is compact.

For $\xi \in \Lambda$ and z such that $|z| \leq R$ define mappings $\Psi_n(\xi, z)$ and $\Psi_n^*(\xi, z)$ on $\overline{\mathcal{P}}$ by

$$
\Psi_n(\xi, z)v = \frac{\Phi_n(\xi, z)v}{\mathbf{1}^t \Phi_n(\xi, z)v} \text{ and } \Psi_n^*(\xi, z)v = \frac{\Phi_n(\sigma^{-n}\xi, z)^t v}{\mathbf{1}^t \Phi_n(\sigma^{-n}\xi, z)^t v}.
$$

These are the analogues of T_x, T_x^* in the proofs of Propositions 5.1–5.2. But notice that they are not well-defined for all $v \in \overline{\mathcal{P}}$ and $|z| \leq R$, because the denominators may be zero. We will show that they *are* well-defined for $v \in \mathcal{P}$ sufficiently close to $\mathcal{P}_\varepsilon$ and z near $(O, R]$.

Fix $t \in (O, R]$, and let r be as in the statement of Lemma 5.3. By Lemma 5.3 and [Se], Lemma 3.1, there exists $\varepsilon > 0$ such that for *every* $\xi \in \Lambda$ the function $\Psi_r(\xi, t)$ is well-defined on $\mathcal{P}_+$ and maps $\mathcal{P}_+$ into $\mathcal{P}_\varepsilon$. Moreover, $\Psi_r(\xi, t)$ is contractive on $\mathcal{P}_\varepsilon$ relative to the Birkhoff projective metric (see the proof of Prop. 5.1). Since the projective metric and the Euclidean metric are uniformly Lipschitz equivalent on $\mathcal{P}_\varepsilon$, there exists $n \geq 1$ such that $\Psi_{nr}(\xi, t)$ is a strict contraction on $\mathcal{P}_\varepsilon$ relative to the Euclidean metric. Consequently, there is an open neighborhood $\mathcal{U}$ of $\mathcal{P}_\varepsilon$ in $\overline{\mathcal{P}}$ and an integer $m \geq 1$ such that for every $\xi \in \Lambda$ the function $\Psi_{mr}(\xi, t)$ maps closure $(\mathcal{U})$ into $\mathcal{U}$ and acts as a strict contraction on closure $(\mathcal{U})$. But $\Psi_n(\xi, z)$ is jointly continuous in (ξ, z), so there exist $\delta > 0$ and $0 < \alpha < 1$ such that for all $\xi \in \Lambda$ and (complex) z satisfying $|z - t| \leq \delta$ and $|z| \leq R$,

$$
\Psi_{mr}(\xi, z)(\text{closure } (\mathcal{U})) \subset \mathcal{U}
$$

and

$$
d(\Psi_{mr}(\xi, z)u, \Psi_{mr}(\xi, z)v) \leq \alpha d(u, v).
$$

(Note: This also uses the fact that Λ is compact. Here d is the Euclidean metric on $\overline{\mathcal{P}}$.)

As in the proof of Proposition 5.1, the last inequality implies the existence of vectors $V(\xi, z) \in \mathcal{U}$ for $\xi \in \Lambda$ and $|z-t| \leq \delta$ such that for suitable constants $C < \infty$ and $0 < \alpha < 1$ (independent of z and ξ)

$$d(\Psi_n(\xi, z)v, V(\xi, z)) \leq C\alpha^n \tag{6.8}$$

for every $v \in$ closure $(\mathcal{U})$ and $n = 1, 2, \ldots$.

A similar argument shows that the mappings $\Psi_n^*(\xi, z)$ are well-defined in a neighborhood $\mathcal{U}^*$ of $\mathcal{P}$ in $\overline{\mathcal{P}}$ and map closure $(\mathcal{U}^*)$ contractively into $\mathcal{U}^*$ for $|z-t| \leq \delta^*$. It then follows that there exist vectors $W(\xi, z) \in \mathcal{U}^*$ for $\xi \in \Lambda$ and $|z-t| \leq \delta^*$ such that

$$d(\Psi_n^*(\xi, z)v, W(\xi, z)) \leq C^*(\alpha^*)^n \tag{6.9}$$

for all $v \in$ closure $(\mathcal{U}^*)$ and $n = 1, 2, \ldots$. Now φ may be defined by (6.5) and γ by (6.7). If δ, δ^* have been taken sufficiently small then the neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ can be taken to be "close" to $\mathcal{P}_\varepsilon$ and $\mathcal{P}$, and so $W(\xi, z)^t V(\xi, z)$ will stay bounded away from 0, and $V(\xi, z)$ will have nonzero entries.

The inequalities (6.8)–(6.9) may now be used to prove (6.4) by the same argument used to prove (5.2) from (5.9). The relation (6.6) follows from (6.4) by an easy argument.□

The convergence (6.4) and the analyticity in z of φ, V, W, and γ will ultimately enable us to analyze the integrand $G_x(re^{i\theta})$ in (6.1) for θ close to zero. However, Proposition 6.1 gives us no control over $G_x(re^{i\theta})$ for θ away from zero. For this we need the following:

PROPOSITION 6.2: *For every $\delta > 0$ there are constants $C < \infty$ and $\alpha \in (0,1)$ such that for all $r \in [\delta, R], \theta \in [-\pi, \pi] \backslash [-\delta, \delta]$, and all $x \in \mathcal{G}$,*

$$\left| \frac{G_x(re^{i\theta})}{G_x(r)} \right| \leq C\alpha^{|x|}. \tag{6.10}$$

PROOF: Recall that $G_x(z) = G(z)(u^t H_x(z)\phi(z))$ and that $G(z), \phi_b(z), H_x^{ab}(z)$ are all defined by power series with nonnegative coefficients. Consequently, each of these functions attains its maximum modulus on the circle $|z| = r$ at $z = r$. Recall also (2.6) that if x has reduced word representation $x = x_1 x_2 \ldots x_m$ then $H_x = H_{x_1} H_{x_2} \ldots H_{x_m}$, and that each entry of $H_i(z)$, for any $i \in \mathcal{A}$, is either 0, 1, or one of $h_1(z), h_2(z), \ldots, h_\nu(z)$.

ASSUME now that $p_e > 0$. For each $j = 1, 2, \ldots, \nu$ the function $h_j(z)$ is defined by a power series $h_j(z) = \sum_{n=1}^{\infty} q_n z^n$ whose coefficients q_n are probabilities of the form $q_n = P^a\{\tau(y) = n; Z_{\tau(y)} = yb\}$. Since $p_e > 0$, it follows that $q_{n+m} \geq q_n p_e^m$ for all $m, n \geq 1$, and since $h_j(z)$ is not a constant function, at least one coefficient q_n is positive. Therefore, for all $r \in (O, R]$ and $\theta \in [-\pi, \pi] \backslash \{0\}$,

$$|h_j(re^{i\theta})| < h_j(r).$$

Now consider $H_x(z)$ for $|x|$ at least as large as the integer r in Lemma 5.3. Lemma 5.3 implies that every row of $H_x(z)$ has at least one entry that is positive for all $z \in (O, R]$. Moreover, if $|x|$ is sufficiently large, say $|x| \geq k$, then any nonzero entry must have at least one h_i as a factor in some term (otherwise the entry would be 1, implying that the random walk visits $x\mathcal{B}$ with probability one, which is not the case if $|x| \geq k$). Consequently, by the result of the previous paragraph, for each $\delta > 0$ there exists $\beta \in (0,1)$ such that for all $r \in [\delta, R] and \theta \in [-\pi, \pi] \backslash [-\delta, \delta]$,

$$||H_x(re^{i\theta})|| \leq \beta ||H_x(r)|| \qquad \forall\, x \in \mathcal{G} \text{ with } |x| = k$$
$$\Longrightarrow ||H_x(re^{i\theta})|| \leq \beta^n ||H_x(r)|| \qquad \forall\, x \in \mathcal{G} \text{ with } |x| = nk.$$

The inequality (6.10) now follows from (6.2)–(6.3).

It remains to show that the assumption $p_e > 0$ is extraneous. Here we use (1.2)–(1.3) which guarantee that $p^{*m}(e) > 0$ for some $m \geq 1$. We can express the Green's functions $G_x(z)$ in terms of the Green's functions for the "m-step" random walk Z_{mn} as follows:

$$G_x(z) = \sum_{y:|y^{-1}x| \leq mK} \tilde{G}_y(z) \Gamma_{y^{-1}x}(z) \tag{6.11}$$

where

$$\tilde{G}_y(z) = \sum_{n=0}^{\infty} p^{*nm}(y) z^{nm},$$
$$\Gamma_w(z) = \sum_{n=0}^{m-1} p^{*n}(w) z^n.$$

Since each $\Gamma_w(z)$ is a polynomial with nonnegative coefficients, (6.10) for G_x follows from (6.10) for $\tilde{G}_y$. □

Relations (6.2)–(6.4) allow us to approximate $G_x(z)$ by $(\exp\{S_m\varphi(\xi, z)\})(\cdots)$ where $\cdots$ indicates terms that do not change much as $m \to \infty$. Here $x = x_1 x_2 \ldots x_m$ and $\xi \in \Lambda$ is any sequence whose first m entries are $x_1 x_2 \ldots x_m$. Consequently, to analyze the integral (6.1) by the saddlepoint method we need some control over the first two derivatives of $S_m\varphi(\xi, z)$ for $z \in (O, R]$.

Let $\mathcal{I}$ denote the set of ergodic, σ-invariant probability measures on Λ. For each n-periodic sequence $\xi \in \Lambda$ there is a unique ergodic, σ-invariant probability measure ν_ξ supported by $\{\xi, \sigma\xi, \sigma^2\xi, \ldots, \sigma^{n-1}\xi\}$: ν_ξ attaches mass $1/n$ to each $\sigma^i\xi$. Let $\mathcal{I}_0 = \{\nu_\xi : \xi \in \Lambda$ is periodic$\}$; then $\mathcal{I}_0$ is weak-* dense in $\mathcal{I}$.

For $s \in (-\infty, \log R)$ and $\xi \in \Lambda$ define

$$\psi(\xi, s) = \varphi(\xi, e^s),$$
$$\dot{\psi}(\xi, s) = \frac{d}{ds}\psi(\xi, s),$$
$$\ddot{\psi}(\xi, s) = \frac{d^2}{ds^2}\psi(\xi, s).$$

PROPOSITION 6.3: *For every $\mu \in \mathcal{I}$ the integral $\int_\Lambda \dot{\psi}(\xi, s) d\mu(\xi)$ is a continuous, strictly increasing function of $s \in (-\infty, \log R)$, and*

$$\int_\Lambda \dot{\psi}(\xi, s) d\mu(\xi) \geq \frac{1}{K} \quad \forall\, s \in (-\infty, \log R). \tag{6.12}$$

Moreover,

$$\lim_{s \to \log R} \inf_{\mu \in \mathcal{I}} \int_\Lambda \ddot{\psi}(\xi, s) d\mu(\xi) = \infty. \tag{6.13}$$

If $p_x > 0$ for all $x \in \mathcal{G}$ such that $|x| = k$, where $k \leq K$, then

$$\lim_{s \to -\infty} \sup_{\mu \in \mathcal{I}} \int_\Lambda \ddot{\psi}(\xi, s) d\mu(\xi) \leq \frac{1}{k}. \tag{6.14}$$

PROOF: The functions $\dot{\psi}, \ddot{\psi}$ are jointly continuous in (ξ, s), by Proposition 6.1 and the Cauchy integral formulas for derivatives. Since Λ is compact, the integrals $\int \dot{\psi}(\xi, s) d\mu(\xi)$ and $\int \ddot{\psi}(\xi, s) d\mu(\xi)$ are well-defined and continuous in s, and the latter is the derivative of the former. In Proposition 6.4 below we will prove that $\int \ddot{\psi}(\xi, s) d\mu(\xi) > 0$; it will then follow that $\int \dot{\psi}(\xi, s) d\mu(\xi)$ is strictly increasing in s.

Since $\mathcal{I}_0$ is deuse in $\mathcal{I}$ in the weak-* topology we may replace $\mathcal{I}$ by $\mathcal{I}_0$ in each of (6.12)–(6.14). Let $\zeta \in \Lambda$ be periodic with period $m \geq 1$; (6.4) implies that

$$\lim_{n \to \infty} \text{trace } (\exp\{S_{mn}\varphi(\zeta, z)\} \Phi_{mn}(\zeta, z)) = 1 \tag{6.15}$$

uniformly for z in any compact subset of $\mathcal{N}$. Consequently, by the Cauchy integral formula, the derivative with respect to z converges to zero. Thus,

$$\begin{aligned} \int \varphi'(\xi, z) d\nu_\zeta(\xi) &= \lim_{n \to \infty} \frac{1}{mn} \text{trace } (e^{-S_{mn}\varphi(\zeta, z)} \Phi'_{mn}(\zeta, z)), \\ \int \dot{\psi}(\xi, s) d\nu_\zeta(\xi) &= e^s \int \varphi'(\xi, e^s) d\nu_\zeta(\xi) \end{aligned} \tag{6.16}$$

where $'$ indicates $\frac{d}{dz}$. Now $\Phi_n(\xi, z) = H_x(z)$ where $x = x_1 x_2 \ldots x_n$ and $x_1, x_2, \ldots, x_n$ are the first n entries of ξ, so trace $\Phi_n(\xi, z)$ is a sum of terms of the form $\prod_{j=1}^{\ell} h_{i_j}(z)$. Lemma 5.3 guarantees that there is at least one such term provided $n \geq r$. Moreover, for each such term $\ell \geq (n-1)/K$, because $x = x_1 x_2 \ldots x_n$ cannot be reached from e in less than $(n-1)/K$ steps. The derivative of $\prod_{j=1}^{\ell} h_{i_j}(z)$ with respect to z is

$$\frac{d}{dz} \prod_{j=1}^{\ell} h_{i_j}(z) = \sum_{j=1}^{\ell} \frac{h'_{i_j}(z)}{h_{i_j}(z)} \prod_{d_*=1}^{\ell} h_{i_{j_i}}(z). \tag{6.17}$$

Each $h_i(z), i = 1, 2, \cdots, \nu$, is a power series in z with nonnegative coefficients and no constant term, so for each i and each $z \in (O, R)$ we have $h_i'(z)/h_i(z) \geq 1/z$. Consequently, for all $z \in (O, R), \xi \in \Lambda$, and $n \geq 1$,

$$\frac{1}{n}\frac{\text{trace } \Phi_n'(\xi, z)}{\text{trace } \Phi_n(\xi, z)} \geq \frac{n-1}{nKz}.$$

This together with (6.15)–(6.16) proves (6.1) for $\mu - \nu_\zeta$, and therefore for all $\mu \in \mathcal{I}$.

Assume now that $p_y > 0$ for every $y \in \mathcal{G}$ satisfying $|y| = k$. Then for each $x \in \mathcal{G}$ there exists a positive probability path from e to x with no more than $|x|/k + C$ steps, where $C < \infty$ is a constant independent of x. It follows that for each $x \in \mathcal{G}$, trace $(H_x(z))$ contains a term $p_{y_1} p_{y_2} \dots p_{y_\ell} z^\ell$ with $p_{y_i} > 0$ for all i and $\ell \leq |x|/k + C'$. Now each entry $H_x^{ab}(z)$ of $H_x(z)$ is a power series in z with nonnegative coefficients, all bounded above by 1; thus, we may write trace $(H_x(z)) = \sum_{j=1}^{\infty} b_j^x z^j$ where $0 \leq b_j^x \leq |\mathcal{B}|$. Consequently, the contribution to trace $(H_x'(z))$ from those terms of the series indexed by $j \geq |x|(1/k + \varepsilon)$ is for $0 < z < 1$, bounded above by

$$|\mathcal{B}| \left\{ \frac{N z^{N-1}}{1-z} + \frac{z^N}{(1-z)^2} \right\}$$

where $N = [|x|(\varepsilon + 1/k)]$($[\cdot]$ indicates greatest integer). But trace $(H_x(z))$ has a term $b_\ell^x z^\ell$ with $\ell \leq |x|/k + C'$ and $b_\ell^x \geq p_*^\ell$, where $p_* = \min\{p_b : b \in \mathcal{B} \text{ and } p_b > 0\}$. If z is sufficiently small that $p_*^\ell \gg z^{|x|\varepsilon}$ then this term contributes more to the trace than those terms indexed by $j \geq N$ contribute to the derivative. Finally, terms $b_j^x z^j$ with $\ell < j \leq N$ contribute $j b_j^x z^{j-1}$ to the derivative, and $j/|x| \leq 1/k + \varepsilon$. We conclude that for any $\varepsilon > 0$ there exists $z_\varepsilon > 0$ such that for all $z \in (0, z_\varepsilon)$,

$$\lim_{z \to 0+} \frac{z \text{ trace } H_x'(z)}{|x| \text{ trace } H_x(z)} \leq \frac{1}{k} + \varepsilon$$

for all $x \in \mathcal{G}$ satisfying $|x| \geq n_\varepsilon$, some $n_\varepsilon < \infty$. The result (6.14) now follows from (6.15)–(6.16).

Finally, consider (6.13). Take $\mu = \nu_\zeta$ where $\zeta \in \Lambda$ is periodic; then $\int \dot{\psi}(\xi, s) d\mu(\xi)$ is given by (6.16). Now for any $\xi \in \Lambda$ and $0 < z < R$ the trace of $\Phi_n(\xi, z)$ is a sum of terms of the form $\prod_{j=1}^{\ell} h_{i_j}(z)$ with $\ell \geq (n-1)/K$, and the trace of $\Phi_n'(\xi, z)$ is a sum of terms of the form (6.17). If $h_*(z) = \min_i (h_i'(z)/h_i(z))$ then certainly

$$\frac{1}{n}\frac{\text{trace } \Phi_n'(\xi, z)}{\text{trace } \Phi_n(\xi, z)} \geq \frac{n-1}{nK} h_*(z).$$

Assume that $p_j > 0$ for each $j \in \mathcal{A}$; then by Propositions 3.3 and 3.6, $\lim_{z \to R-}(h_i'(z)/h_i(z)) = \infty$ for each $i = 1, 2, \dots, \nu$, and hence $h_*(z) \to \infty$ as $z \to R-$. Since (6.15)–(6.16) exhibit $\int \dot{\psi}(\xi, s) d\mu(\xi)$ as the limit of a ratio of traces, this proves that if $p_j > 0$ for each $j \in \mathcal{A}$ then

$$\lim_{z \to R-} \inf_{\nu_\zeta \in \mathcal{I}_0} \int \dot{\psi}(\xi, s) d\nu_\zeta(\xi) = \infty,$$

and (6.13) follows.

It remains to show that the assumption $p_j > 0 \ \ \forall \ j \in \mathcal{A}$ is extraneous. Again we use the standing hypotheses (1.2)–(1.3). By Lemma 1.1 these guarantee that for some $m \geq 1, p^{*m}(i) > 0$ for every $i \in \mathcal{A}$. Hence (6.13) is true for the random walk with 1-step transition probabilities $p^{*m}(x)$. Unfortunately, this doesn't yet prove (6.13) for the original random walk because the φ-functions may be different. To get around this difficulty, we will express the integral in (6.13) in terms of the Green's functions and then appeal to (6.11).

Fix $\xi = (x_n)_{n=-\infty}^{\infty} \in \Lambda$ and for each $n \geq 1$ set $x^{(n)} = x_1 x_2 \ldots x_n$. If ξ is k-periodic then (6.2)–(6.4) imply that

$$\lim_{n\to\infty} e^{-S_{nk}\varphi(\xi,z)} \left\{ \frac{G_{x^{(nk)}}(z)}{G(z)} \right\} = \eta(\xi, z)$$

where

$$\eta(\xi, z) = u^t V(\xi, z) W(\xi, z)^t \phi(z).$$

This holds uniformly for z in any compact subset of $\mathcal{N}$, and the limit $\eta(\xi, z)$ is bounded away from zero. Consequently, the derivatives also converge, yielding

$$\int \varphi'(\zeta, z) d\nu_\xi(\zeta) = \lim_{n\to\infty} \frac{1}{nk} \left\{ \frac{G'_{x^{(nk)}}(z)}{G_{x^{(nk)}}(z)} \right\}.$$

This, together with (6.11), implies that if (6.13) holds for the random walk with transition probabilities $p^{*m}(x)$ then it must hold for the original random walk as well. □

PROPOSITION 6.4: *For every* $a > -\infty$

$$\text{(6.18)} \qquad \inf_{a \leq s < \log R} \inf_{\mu \in \mathcal{I}} \int \ddot{\psi}(\xi, s) d\mu(\xi) > 0.$$

PROOF: Let $\zeta \in \Lambda$ be m-periodic and $z \in (O, R)$ be fixed. For some $b \in \mathcal{B}$ the b^{th} entries of the vectors $V(\zeta, z)$ and $W(\zeta, z)$ are both positive (the entries of $V(\xi, z)$ are all positive, the entries of $W(\xi, z)$ are nonnegative, and $W(\xi, z)^t \mathbf{1} = 1$). Consequently, by (6.4), as $n \to \infty$

$$\exp\{-S_{mn}\varphi(\zeta, z)\} \Phi^{bb}_{mn}(\zeta, z) \longrightarrow \frac{V(\zeta, z)_b W(\zeta, z)_b}{\gamma(\zeta, z)}$$

and in fact this holds uniformly in a neighborhood of z. Observe that the limit is strictly positive. Hence, by the Cauchy integral formulas for derivatives (with $\cdot$ indicating d/ds),

$$\int \dot{\psi}(\xi, s) d\nu_\zeta(\xi) = \lim_{n\to\infty} \frac{1}{mn} \left\{ \frac{\dot{\Phi}^{bb}_{mn}(\zeta, e^s)}{\Phi^{bb}_{mn}(\zeta, e^s)} \right\}$$

and

$$\int \ddot{\psi}(\xi,s)d\nu_\zeta(\xi) = \lim_{n\to\infty}\frac{1}{mn}\left\{\frac{\ddot{\Phi}^{bb}_{mn}(\zeta,e^s)\Phi^{bb}_{mn}(\zeta,e^s)-\dot{\Phi}^{bb}_{mn}(\zeta,e^s)^2}{\Phi^{bb}_{mn}(\zeta,e^s)^2}\right\}.$$

Recall that $\Phi^{bb}_{mn}(\zeta,z) = H^{bb}_x(z) = E^b z^{\tau(x)}1\{\tau(x)<\infty; Z_{\tau(x)} = xb\}$ where $x = x_1x_2\dots x_{mn}$ and ζ has $x_1x_2\dots x_{mn}$ as its first mn entries. Consequently, $\{\ddot{H}^{bb}_x(e^s)H^{bb}_x(e^s) - \dot{H}^{bb}_x(e^s)^2\}/H^{bb}_x(e^s)$ is the variance of $\tau(x)$ under a certain probability measure, and as such is no smaller than the expectation of the *conditional* variance given $Z_{\tau(x_1)}, Z_{\tau(x_1x_2)},\dots,Z_{\tau(x)}$. Now recall that $H^{bb}_x(e^s)$ is a sum of terms of the form $\prod_{j=1}^{\ell} h_{i_j}(e^s)$ with $\ell \geq (|x|-1)/K$; conditioning on the values of $Z_{\tau(x_1)}, Z_{\tau(x_1x_2)},\dots$ is the same as specifying one of the terms $\prod h_{i_j}(e^s)$. Consequently, the conditional variance is

$$\sum_{j=1}^{\ell}\left\{\frac{\ddot{h}_{i_j}(e^s)h_{i_j}(e^s)-\dot{h}_{i_j}(e^s)^2}{h_{i_j}(e^s)^2}\right\}$$

for some sequence $i_1,i_2,\dots,i_\ell$ from $\{1,2,\dots,\nu\}$.

Assume now that $p_e > 0$. Then each $h_i(z)$ is a power series $\sum_{n=1}^{\infty} q_n z^n$ with $q_{n+m}\geq q_n p_e^m$ and some $q_n > 0$, and hence $\{\ddot{h}_i(e^s)h_i(e^s) - \dot{h}_i(e^s)^2\} > 0$ and is bounded away from 0 for $a \leq s < \log R$. It follows that the terms in the last displayed sum are bounded away from 0, say by $\varepsilon > 0$, so the conditional variance is at least $\ell\varepsilon$. Since $\ell \geq (|x|-1)/K$, it now follows by the two preceding paragraphs that for all $s\in[a,\log R)$ and all periodic $\zeta\in\Lambda$,

$$\int \ddot{\psi}(\xi,s)d\nu_\zeta(\xi) \geq \varepsilon/K.$$

It remains to prove that the assumption $p_e > 0$ was extraneous. By (1.2) there exists $m\geq 1$ such that $p^{*m}(e) > 0$; hence (6.18) is true for the random walk with 1-step transition probabilities $p^{*m}(x)$. By an argument like that used in proving (6.13), if $\zeta = (x_n)_{n=-\infty}^{\infty}\in\Lambda$ is k-periodic then

$$\int \ddot{\psi}(\xi,s)d\nu_\zeta(\xi) = \lim_{n\to\infty}\frac{1}{nk}\left\{\frac{\ddot{G}_{x^{(nk)}}(e^s)G_{x^{(nk)}}(e^s)-\dot{G}_{x^{(nk)}}(e^s)^2}{G_{x^{(nk)}}(e^s)^2}\right\}.$$

where $x^{(n)} = x_1x_2\dots x_n$. Therefore, by (6.11), if (6.18) holds for the random walk with 1-step transition probabilities $p^{*m}(x)$ then it must hold also for the original random walk.□

Let $\mu\in\mathcal{I}$ and $q\geq 0$. Define

$$\beta(\mu,q) = \inf_{-\infty<s\leq\log R}\{q\int_\Lambda \psi(\xi,s)d\mu(\xi) - s\};$$

this is the Legendre transform of $s \to \int \psi(\xi, s)d\mu(\xi)$. Since $\int \psi(\xi, s)d\mu(\xi)$ is a strictly convex function of s, by Proposition 6.4, the inf is uniquely attained at some $s \in [-\infty, \log R]$. If $q = 0$ then the inf is attained at $s = s(\mu, 0) = \log R$, and in general the inf is attained at an interior point

$$s = s(\mu, q) \in (-\infty, \log R)$$

iff $q \int \dot{\psi}(\xi, s)d\mu(\xi) = 1$. By Proposition 6.3, $\int \dot{\psi}(\xi, s)d\mu(\xi)$ is a positive, strictly increasing function of s always exceeding $1/K$, so we may define

$$I = \lim_{s \to -\infty} \inf_{\mu \in \mathcal{I}} \int \dot{\psi}(\xi, s)d\mu(\xi) \geq 1/K.$$

By (6.13), if $p_x > 0$ for every $x \in \mathcal{G}$ satisfying $|x| = k$ then $I \leq 1/k$. Since $\int \dot{\psi}(\xi, s)d\mu(\xi) \to \infty$ as $s \to \log R$, it now follows that $s(\mu, q)$ is a well-defined interior point of $(-\infty, \log R)$ whenever $q \in (0, 1/I)$. Moreover, $s(\mu, q)$ is a strictly decreasing function of q.

If $\xi \in \Lambda$ is periodic and ν_ξ is the σ-invariant probability measure supported by the orbit $\{\xi, \sigma\xi, \sigma^2\xi, \ldots\}$ of ξ, then we will write

$$s(\xi, q) = s(\nu_\xi, q),$$
$$\beta(\xi, q) = \beta(\nu_\xi, q).$$

For each finite reduced word $x = x_1 x_2 \ldots x_m \in \mathcal{G}$ let $\xi_x \in \Lambda$ be a periodic sequence of minimal period containing $x_1 x_2 \ldots x_m$ in its smallest periodic block. (If $x_m \neq x_1^{-1}$ then $\xi_x = \ldots x_1 x_2 \ldots x_m x_1 x_2 \ldots x_m \ldots$. In general ξ_x isn't uniquely determined, but the choice of ξ_x doesn't matter.)

THEOREM 6.5: *Let $x \in \mathcal{G}$ be such that $|x| = m$. Set $r = e^{s(\xi_x, m/n)}$. Then*

$$(6.19) \qquad p^{*n}(x) \sim \frac{\exp\{n\beta(\xi_x, \frac{m}{n})\}}{\sqrt{2\pi S_m \ddot{\psi}(\xi_x, s(\xi_x, \frac{m}{n}))}} \{G(r)\gamma(\sigma^m \xi_x, r)V_e(\xi_x, r)W(\sigma^m \xi_x, r)^t \phi(r)\}$$

and

$$(6.20) \qquad P\{T_x = n\} \sim p^{*n}(x)G(r)$$

as $m, n \to \infty$ in such a way that $\frac{m}{n}$ stays in the interior of the interval $(0, 1/I)$. Furthermore, these relations hold uniformly in x for $\frac{m}{n}$ in any compact subset of $(0, 1/I)$.

PROOF: We use the Fourier inversion formula (6.1) with $r = \exp\{s(\xi_x, \frac{m}{n})\}$. Relations (6.2)–(6.4) allow us to re-express the integrand $G_x(re^{i\theta})$ in terms of φ for $|\theta| < \delta, \delta > 0$ small. Proposition 6.2 implies that the integral over $\theta \in [-\pi, \pi] \setminus [-\delta, \delta]$ is of exponentially smaller magnitude than the integral over $[-\delta, \delta]$. Hence, by (6.2)–(6.4),

$$p^{*n}(x) \sim (2\pi r^n)^{-1} \int_{-\delta}^{\delta} \exp\{S_m \psi(\xi_x, s + i\theta) - in\theta\} C(re^{i\theta}) d\theta$$

where

$$C(z) = G(z)\gamma(\sigma^m \xi_x, z)V_e(\xi_x, z)W(\xi_x, z)^t \phi(z)$$

and $s = s(\xi_x, \frac{m}{n}) = \log r$. Now s has been chosen so that $\theta = 0$ is a saddlepoint of the function $S_m\{\psi(\xi_x, s + i\theta) - in\theta$; thus, when expanded in a Taylor series, its linear term vanishes, leaving $S_m\psi(\xi_x, s) - S_m\ddot{\psi}(\xi_x, s)\theta^2/2 + \ldots$. Therefore, the integral may be analyzed by an entirely routine application of Laplace's method of asymptotic expansion ([dB], Ch. 4), which gives (6.19). A similar analysis gives (6.20). □

It should be observed that this proof only establishes the uniformity of (6.19)–(6.20) for $\frac{m}{n}$ bounded away from zero. However, for $\frac{m}{n} \approx 0$ the approximation (6.19) agrees formally with (4.1), so it is likely that (6.19) is uniform for $\frac{m}{n} \in [0, (1/I) - \varepsilon]$, any $\varepsilon > 0$. See [La], Proposition 5 for a more complete discussion of this point in the nearest neighbor case.

7. Asymptotic Behavior of the Word Length Functional

Guivarch showed [G] that $|Z_n|$, the word length of Z_n, obeys a strong law of large numbers: there is a constant $\beta \in (0, \infty)$ such that $|Z_n|/n \to \beta$ a.s. as $n \to \infty$. His proof used Kingman's subadditive ergodic theorem. Later, Sawyer and Steger [SS] showed that there is also a central limit theorem: $(|Z_n| - n\beta)/\sqrt{n} \xrightarrow{\mathcal{D}}$ Normal $(0, \sigma^2)$ as $n \to \infty$ for some $\sigma^2 \geq 0$, but did not prove that σ^2 is always > 0. In this section we will show how the results of section 6, in combination with certain results about Ruelle's Perron-Frobenius operators ([Bo], Ch. 1) may be used to derive even sharper results about the distribution of $|Z_n|$ for large n, and in particular will show that $\sigma^2 > 0$ if (1.1)–(1.3) hold.

Recall that Λ is the set of doubly-infinite sequences $\xi = (x_n)_{n=-\infty}^{\infty}$ with $x_n \in \mathcal{A}$ and $x_{n+1} \neq x_n^{-1}$ $\forall\, n$. Define Λ_+ to be the set of one-sided sequences $\xi = (x_n)_{n=1}^{\infty}$ with $x_n \in \mathcal{A}$ and $x_{n+1} \neq x_n^{-1}$ $\forall\, n \geq 1$. As earlier let $\sigma: \Lambda \to \Lambda$ and $\sigma: \Lambda_+ \to \Lambda_+$ be the forward shift operator, and for any $\mathbb{R}$- or $\mathbb{C}$-valued function f on Λ or Λ_+ define $S_n f = f + f \circ \sigma + \ldots + f \circ \sigma^{n-1}$. For any Hölder continuous functions $f, g: \Lambda_+ \to \mathbb{C}$ define

$$(\mathcal{L}_f g)(\xi) = \sum_{\zeta: \sigma\zeta = \xi} e^{f(\zeta)} g(\zeta) \quad \forall\, \xi \in \Lambda_+.$$

The operators $\mathcal{L}_f$, called "Ruelle operators", are bounded linear operators on the space of Hölder continuous functions on Λ_+ with given exponent – see [Bo], Ch. 1 or [Po] for greater detail. Observe that

$$(\mathcal{L}_f^n g)(\xi) = \sum_{\zeta: \sigma^n \zeta = \xi} e^{S_n f(\zeta)} g(\zeta).$$

For real-valued f, the operator $\mathcal{L}_f$ enjoys certain spectral properties similar to those of Perron-Frobenius matrices. In particular, there exist a constant $P(f) \in \mathbb{R}$ (the "pressure"), a strictly positive, Hölder continuous function $h_f: \Lambda_+ \to (0, \infty)$, and a Borel probability measure ν_f on Λ_+ satisfying $\int h_f d\nu_f = 1$, such that

$$\mathcal{L}_f h_f = e^{P(f)} h_f \text{ and } \mathcal{L}_f^* \nu_f = e^{P(f)} \nu_f$$

(here $\mathcal{L}^*$ denotes the adjoint of $\mathcal{L}$). Moreover, the rest of the spectrum of $\mathcal{L}_f$ is contained in a disc centered at 0 of radius $< e^{P(f)}$. The measure μ_f defined by $\frac{d\mu_f}{d\nu_f} = h_f$ is σ-invariant and therefore extends to Λ; it is the "Gibbs state" associated with f.

Now let $\{\varphi_z(\xi): z \in \mathcal{N}, \xi \in \Lambda_+\}$ be a family of Hölder continuous functions on Λ_+ indexed by $z \in \mathcal{N}$ for some domain $\mathcal{N} \subset \mathbb{C}$ such that $z \in \varphi_z$ is analytic with respect to one of the Hölder norms. We have in mind $\varphi_z(\xi) = \varphi(\xi, z)$ or $\psi_s(\xi) = \psi(\xi, s)$ where φ, ψ are as in section 6. Assume that for $z \in \mathbb{R} \cap \mathcal{N}$ the function φ_z is real-valued. Then the results concerning the spectrum of $\mathcal{L}_{\varphi_z}$ described above are applicable for each fixed $z \in \mathbb{R} \cap \mathcal{N}$. Furthermore, standard results from regular perturbation theory (cf. [Ka], Ch. IV, sec. 3 and Ch. VII, sec. 1) imply that $P(z), h_z, \nu_z$, etc. extend analytically to a neighborhood of $\mathbb{R}$ in $\mathcal{N}$, where the relations $\mathcal{L}_z h_z = e^{P(z)} h_z$, $\mathcal{L}_z^* \nu_z = e^{P(z)} \nu_z$, and $\int h_z d\nu_z = 1$ remain valid. (Notational convention: we will write $\mathcal{L}_z, P(z)$, etc., instead of $\mathcal{L}_{\varphi_z}, P(\varphi_z)$, etc.) In addition, for each real z_* there exists $\varepsilon > 0$ such that for all $z \in \mathcal{N}$ satisfying $|z - z_*| < \varepsilon$,

$$\text{spectrum } (\mathcal{L}_z) - \{e^{P(z)}\} \subset \{w \in \mathbb{C}: |w| \leq e^{P(z_*)} - \varepsilon\}.$$

Denote d/dz by $'$.

LEMMA 7.1: *For each* $z \in \mathbb{R} \cap \mathcal{N}$,

$$P'(z) = \int \varphi_z'(\xi) d\mu_z(\xi), \tag{7.1}$$

$$P''(z) = \int \varphi_z''(\xi) d\mu_z(\xi) + \lim_{n\to\infty} \frac{1}{n} \int \{S_n\varphi_z'(\xi) - nP'(z)\}^2 d\mu_z(\xi). \tag{7.2}$$

PROOF: Differentiate both sides of the equation $(e^{-P(z)}\mathcal{L}_z)^n h_z = h_z$ with respect to z, divide by n, integrate against ν_z, and let $n \to \infty$ to obtain

$$\lim_{n\to\infty} \int (e^{-P(z)}\mathcal{L}_z)^n \left\{ \left(\frac{S_n\varphi_z'}{n} - P'(z) \right) h_z \right\} (\xi) d\nu_z(\xi) = 0.$$

Now use the equations $((e^{-P(z)}\mathcal{L}_z)^*)^n \nu_z = \nu_z, (d\mu_z/d\nu_z) = h_z$, and $\int h_z d\nu_z = 1$ to rewrite this as

$$\lim_{n\to\infty} \int \frac{S_n\varphi_z'}{n} d\mu_z = P'(z).$$

Since μ_z is σ-invariant, this proves (7.1).

To prove (7.2), differentiate both sides of $(e^{-P(z)}\mathcal{L}_z)^n h_z = h_z$ *twice* with respect to z, then divide by n, integrate against ν_z, and let $n \to \infty$ to obtain

$$\lim_{n\to\infty} \frac{1}{n} \int (e^{-P(z)}\mathcal{L}_z)^n \{(S_n\varphi_z'' - nP''(z) - (S_n\varphi_z' - nP'(z))^2) h_z\} d\nu_z = 0.$$

Since $(e^{-P(z)}\mathcal{L}_z^*)^n \nu_z = \nu_z$ and $d\mu_z = h_z d\nu_z$, this implies (7.2). □

We may apply this result to the assignment $s \to \psi_s$ where $\psi_s(\xi) = \psi(\xi, s)$ is as in section 6. Together with Proposition 6.3, Lemma 7.1 implies that $\dot{P}(s) \geq 1/K$ for all $s \in (-\infty, \log R)$, that $\lim_{s \to \log R} \dot{P}(s) = \infty$, and that if $p_x > 0$ for all $x \in \mathcal{G}$ satisfying $|x| = k$ then $\lim_{s \to -\infty} \dot{P}(s) \leq 1/k$. (Note: superscript $\dot{}$ indicates d/ds, as in section 6.) Together with Proposition 6.4, Lemma 7.1 implies that $\ddot{P}(s) > 0$ and $\ddot{P}(s)$ is bounded away from 0 for $s \in [a, \log R]$, for any $a > -\infty$; hence $P(s)$ is strictly convex, and $\dot{P}(s)$ is strictly increasing, for $s \in (-\infty, \log R)$. Set

$$I_P = \inf_{-\infty < s < \log R} \dot{P}(s).$$

Denote by $B(q)$ the Legendre transform of $P(s)$, i.e.,

$$B(q) = \inf_{-\infty < s \leq \log R} \{qP(s) - s\}.$$

Since $P(s)$ is strictly convex, the inf is attained uniquely at some $s = s(q) \in [-\infty, \log R]$. If $q = 0$ then $s(0) = \log R$; if $q \in (0, I_P)$ then $s(q) \in (-\infty, \log R)$; and if $q \geq I_P$ then $s(q) = -\infty$. Note that $s(q) = s \in (-\infty, \log R)$ iff $\dot{P}(s) = 1/q$. Clearly, $s(q)$ is a decreasing function. By the chain rule

$$\frac{d}{dq} B(q) = P(s(q)) \text{ and } \frac{d^2}{dq^2} B(q) = \frac{-\dot{P}(s(q))^2}{q\ddot{P}(s(q))} < 0$$

so $B(q)$ is strictly concave on $(0, I_P)$.

According to Theorem 5.4, $P(0) = 0$. Consequently, $B(q)$ attains its maximum value uniquely at $q = 1/\dot{P}(0)$, and this value is 0. Note that $\dot{P}(0) = \int \dot{\psi}(\xi, 0) d\mu_0(\xi)$ by Lemma 7.1, and that μ_0 coincides with μ_φ in Theorem 5.4.

THEOREM 7.2: *There are positive constants $C(q), q \in (0, I_P)$, such that as $n \to \infty$ and $m \to \infty$*

$$P\{|Z_n| = m\} \sim \frac{\exp\{nB(\frac{m}{n})\}}{\sqrt{2\pi m \ddot{P}(s(\frac{m}{n}))}} C(\frac{m}{n}) \tag{7.3}$$

uniformly for $\frac{m}{n}$ in any compact subset of $(0, I_P)$. Also,

$$\left(\frac{d^2}{dq^2} B(q) \right)_{q = 1/\dot{P}(0)} = \frac{C(1/\dot{P}(0))^2 \dot{P}(0)}{\ddot{P}(0)}. \tag{7.4}$$

REMARKS: (7.3)–(7.4) imply the central limit theorem for $|Z_n|$ first obtained by [SS], and show that the limiting variance is strictly positive. The result (7.3) shows that in fact a *local* central limit theorem is true. Theorem 7.2 also allows us to identify the constant β in Guivarch's strong law $|Z_n|/n \xrightarrow{a.s.} \beta$:

$$\beta = \frac{1}{\dot{P}(0)} = \frac{1}{\int \dot{\psi}(\xi, 0) d\mu_0}$$

where $\mu_0 = \mu_\varphi$ is the Gibbs state figuring in Theorem 5.4.

PROOF *of Theorem 7.2*: By the Fourier inversion formula,

$$P\{|Z_n| = m\} = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} \sum_{x:|x|=m} G_x(re^{i\theta}) e^{-in\theta} d\theta \tag{7.5}$$

for any $r \in (O, R]$. We will analyze this integral by choosing r so that the integrand has a saddlepoint at r, then using Laplace's method of asymptotic expansion as in the proof of Theorem 6.5. Proposition 6.2 guarantees that asymptotically (as $n, m \to \infty$) the contribution to the integrand from $\theta \in [-\pi, \pi] \backslash [-\delta, \delta]$ will be of smaller order of magnitude than the contribution from $\theta \in [-\delta, \delta]$. Hence, we may focus on the behavior of the integrand in the range $\theta \in [-\delta, \delta]$ where $\delta > 0$ is chosen sufficiently small that $re^{i\theta} \in \mathcal{N}$ whenever $|\theta| \leq \delta$ and $\mathcal{N}$ is as in Proposition 6.1.

So consider $\sum_{x:|x|=m} G_x(z)$ for $z \in \mathcal{N}$. Recall (6.1)-(6.2) that $G_x(z) = F_x(z)G(z)$ and that $F_x(z)$ has a representation as a matrix product. The behavior of this product is governed by (6.4) and, most important, (6.4) holds *uniformly* for $\xi \in \Lambda$ and z in compact subsets of $\mathcal{N}$. Thus, if $|x| = m$,

$$G_x(z) \sim e^{S_m \varphi(\xi_x, z)} \eta_+(\xi_x, z) \eta_-(\sigma^m \xi_x, z) G(z) \tag{7.6}$$

where

$$\eta_+(\xi, z) = u^t V(\xi, z),$$
$$\eta_-(\xi, z) = \gamma(\xi, z) W(\xi, z)^t \phi(z),$$

and ξ_x is *any* sequence in Λ whose first m entries are $x_1, x_2, \dots, x_m$ where $x = x_1 x_2 x_3 \dots x_m$. The approximation is uniformly accurate as $m \to \infty$ for $|x| = m$ and z in any compact subset of $\mathcal{N}$.

Recall that $\varphi(\xi, z)$ and $V(\xi, z)$ are functions only of the "forward" coordinates $x_1, x_2, \dots$ of ξ; hence $\varphi(\xi, z)$ and $\eta_+(\xi, z)$ may be considered functions on Λ_+. This is not true of $\eta_-(\xi, z)$, however. The presence of η_- in (7.6) complicates the analysis somewhat for this reason. To keep this added complication from obscuring the main idea behind the analysis of $\sum_{x:|x|=m} G_x(z)$ we will first show how we would proceed if the factor η_- were *not* present in (7.6). Using the notation $x = x_1 x_2 \dots x_m$, let $\xi^{(i)}, \xi_x^{(i)} \in \Lambda_+$ be defined for each $i \in \mathcal{A}, i \neq x_m^{-1}$ by

$$\xi^{(i)} = iii\dots$$
$$\xi_x^{(i)} = x_1 x_2 \dots x_m iii\dots .$$

Then by (7.6),

$$\sum_{x:|x|=m} G_x(z) \sim \frac{1}{2|\mathcal{A}|-1} \sum_{i\in\mathcal{A}} \sum_{x_1x_2\ldots x_m:x_m^{-1}\neq i} e^{S_m\varphi(\xi_x^{(i)},z)}\eta_+(\xi_x^{(i)},z)\eta_-(\sigma^m\xi_x^{(i)},z)G(z).$$

But

$$\sum_{i\in\mathcal{A}} \sum_{x_1x_2\ldots x_m:x_m^{-1}\neq i} e^{S_m\varphi(\xi_x^{(i)},z)}\eta_+(\xi_x^{(i)},z)$$
$$=\sum_{i\in\mathcal{A}}(\mathcal{L}_z^m\eta_+(\cdot,z))(\xi^{(i)})$$

where $\mathcal{L}_z = \mathcal{L}_{\varphi(\cdot,z)}$ is the Ruelle operator introduced earlier. For z sufficiently close to the real axis, the spectrum of $\mathcal{L}_z$ has an eigenvalue $e^{P(z)}$ near the positive axis, and the rest of the spectrum is in a smaller disc centered at 0. Hence,

$$\sum_{i\in\mathcal{A}}(\mathcal{L}_z^m\eta_+(\cdot,z))(\xi^{(i)}) \sim e^{mP(z)}\sum_{i\in\mathcal{A}} h_z(\xi^{(i)})\int_{\Lambda_+}\eta_+(\xi,z)d\nu_z(\xi).$$

Consequently, if the factor η_- were not present in (7.6), we would be justified in substituting $C(z)e^{mP(z)}$ for $\sum_{x:|x|=m} G_x(z)$ in (7.5), where $C(z)$ is a nonzero constant (not depending on m). Asymptotic analysis of (7.5) based on the Laplace method would then give (7.3) (the details are very similar to those in the proof of Theorem 6.5).

The function $\eta_-(\sigma^m\xi_x^{(i)},z)$ depends only in i and $x_m, x_{m-1},\ldots$. Since it is Hölder continuous, it can be well-approximated by a function depending only on $x_m, x_{m-1},\ldots,$ x_{m-k+1} and i; the approximation may be made arbitrarily close by letting $k\to\infty$. (In fact the error is $O(e^{-\alpha k})$ for suitable $\alpha > 0$.) Now if $\eta_-(\sigma^m\xi_x^{(i)},z)$ were only a function of $x_m, x_{m-1},\ldots, x_{x-k+1}$ and i then we could write

$$\sum_{x_1x_2\ldots x_{m+k}:x_{m+k}^{-1}\neq i} e^{S_{m+k}\varphi(\xi_x^{(i)},z)}\eta_+(\xi_x^{(i)},z)\eta_-(\sigma^{m+k}\xi_x^{(i)},z)$$
$$= \sum_{x_{m+1}x_{m+2}\ldots x_{m+k}:x_{m+k}^{-1}\neq i} \mathcal{L}_z^m(e^{S_k\varphi}\eta_+)(\sigma^m\xi_x^{(i)})\eta_-(\sigma^{m+k}\xi_x^{(i)},z)$$

(the dependence of φ and η_+ on z is suppressed). The spectral theory of $\mathcal{L}_z$ could then be applied as in the previous paragraph. Using these approximations and letting $k\to\infty$ gives, finally,

$$\sum_{x:|x|=m} G_x(z) \sim e^{mP(z)}C(z)$$

for a constant $C(z)\neq 0$ (different from the $C(z)$ of the previous paragraph). Therefore, (7.5) may in fact be analyzed by the Laplace method, proving (7.3).

It remains to prove (7.4). In principle, this could be done by keeping careful track of the constants in the preceding argument, but this would be tedious and possibly difficult.

However, (7.4) must follow from (7.3) and Guivarch's strong law, because the probabilities in (7.4) for $(m/n\beta) \in [-\varepsilon, \varepsilon]$ must add up to (almost) one. □

REFERENCES

[Ao] AOMOTO, K. (1984). Spectral theory on a free group and algebraic curves. *J. Fac. Sci. Univ. Tokyo* IA **31**, 297–317.

[B] BENDER, E. (1974). Asymptotic methods in enumeration. *SIAM Reveiw*, **16**, 485–513.

[Bo] BOWEN, R. (1974). *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Springer LNM, **470**.

[dB] DE BRUIJN, N. (1958). *Asymptotic Methods in Analysis.* Dover, NY.

[C] CARTWRIGHT, D. (1991). Singularities of the Green's function of a random walk on a discrete group. Preprint.

[CS] CARTWRIGHT, D. and SAWYER, S. (1989). The Martin boundary for general isotropic random walks in a tree. Preprint.

[De] DERRIENIC, Y. (1975). Marche aléatoire sur le groupe libre et frontière de Martin. *Z. Warsch.*, **32**, 261–276.

[De_2] DERRIENNIC, Y. (1980). Quelques applications due théorème ergodique sousadditif. *Asterisque* **74**, 183–201.

[DM] DYNKIN, E. and MALYUTOV, M. (1961). Random walks on groups with a finite number of generators. *Sov. Math.*, **2**, 399–402.

[Fe] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, vol. 2, 2nd ed. Wiley, NY.

[FP] FIGÀ-TALAMANCA, A. and PICARDELLO, M. (1983). *Harmonic Analysis on Free Groups.* Dekker, NY.

[GW] GERL, P. and WOESS, W. (1986). Local limits and harmonic functions for non-isotropic random walks on free groups. *Prob. Th. Rel. Fields*, **71**, 341–355.

[G] GUIVARĆH, Y. (1980). Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. *Asterisque*, **74**, 47–98.

[KV] KAIMANOVICH, K. and VERSIK, A. (1983). Random walks on discrete groups: boundary and entropy. *Ann. Probab.*, **11**, 457–490.

[Ka] KATO, T. (1980). *Perturbation Theory for Linear Operators.* Springer-Verlag, New York.

[Ke] KENDIG, K. (1977). *Elementary Algebraic Geometry.* Springer, NY.

[K_1] KESTEN, H. (1959). Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, **92**, 336–354.

[K_2] KESTEN, H. (1959). Full Banach mean values on countable groups. *Math. Scand.*, **7**, 146–156.

[La] LALLEY, S. (1990). Saddlepoint approximations and space-time Martin boundary for nearest neighbor random walk on a homogeneous tree. Preprint.

[L] LANG, S. (1971). *Algebra.* Addison-Wesley, Reading.

[N] NUMMELIN, E. (1984). *General Irreducible Markov Chains and Nonnegative Operators.* Cambridge U. Press, Cambridge.

[Pi] PICARDELLO, M. (198). Spherical functions and local central limit theorems on free groups.

[Po] POLLICOTT, M. (1984). A complex Ruelle-Perron-Frobenius theorem and two counterexamples. *Ergodic Theory Dynamical Systems*, **4**, 135–146.

[Sa] SAWYER, S. (1978). Isotropic random walks on a tree. *Z. Warsch.*, **42**, 279–292.

[SS] SAWYER, S. and STEGER, T. (1987). The rate of escape for anisotropic random walks in a tree. *Probab. Th. Rel. Fields*, **76**, 207–230.

[Se] SENETA, E. (1981). *Nonnegative Matrices and Markov Chains*, 2nd ed., Springer, NY.

[Sh] SHAFAREVICH, I. (1974). *Basic Algebraic Geometry.* Springer, NY.

[St] STEGER, T. (1986). *Anisotropic Harmonic Analysis for Homogeneous Trees.* Ph.D. Thesis, Washington Univ., St. Louis.