A Remark on Stochastic Differential Equations with Markov Solutions

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ABSTRACT

When the solution of a stochastic differential equation is Markov, one can deduce that the driving semimartingale has independent increments.

Let Z be a semimartingale on a filtered Probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and f a Borel function such that the equation

$$dX_t = f(X_{t-})dZ_t, X_0 = X \tag{1}$$

has for each x, a unique strong solution. It is then well known (cf. [2] or [3]) that if Z is a Lévy process (i.e., Z has independent and stationary increments), then the processes X^x are time-homogeneous Markov processes, with a transition semigroup that does not depend on x.

This well-known fact has a somewhat surprising converse, which has a simple proof:

Theorem 1. Suppose f is never zero. If the processes X^x are homogeneous Markov with the same transition semigroup for all x, then Z is a Lévy process.

Proof: Let Ω' be the path space of right continuous functions with left limits (càdlàg) on $[0, \infty)$, and let X' be the canonical process, $(\mathcal{F}'_t)_{t\geq 0}$ its canonical filtration, and θ'_t be the semigroup of shift operators. If P'_x denotes the law of X^x, then our hypotheses imply that $(\Omega', (\mathcal{F}'_t)'_{t\geq 0}, (\theta'_t)_{t\geq 0}, X', P'_x)$ is a Dynkin realization of a Markov process, as in the books of Blumenthal and Getoor [1] or Sharpe [4].

Since f is never zero, we can write from (1),

$$Z_t = Z_0 + \int_0^t f(X^x_{s-})^{-1} dX^x_s. \tag{2}$$

Thus we have the existence on (Ω', P'_x) of the stochastic integral

$$Z'_t = \int_0^t f(X'_{s-})^{-1} dX'_s, \tag{3}$$

and moreover:

(4) the law of Z' under P'_x is the law of the process $Z - Z_0$.

On the other hand, Z' is an additive functional (see [2]). For each positive Borel function g, the Markov property together with (4) implies

$$\begin{aligned} E'_x\{g(Z'_{t+s} - Z'_t)|\mathcal{F}'_t\} &= E'_x\{g(Z'_s) \circ \theta'_t|\mathcal{F}'_t\} \\ &= E'_{X'_t}\{g(Z'_s)\} = E\{g(Z_s - Z_0)\}. \end{aligned}$$

Therefore $Z'_{t+s} - Z'_t$ is P'_x-independent of $\mathcal{F}'_t$, hence also independent of Z'_r, all $r \leq t$. Moreover, the law of $Z'_{t+s} - Z'_t$ under P'_x is the same as the law of $Z_s - Z_0$. Applying (4) again yields the result. $\triangle$

Remark: Consider the case where f is identically one. Then (1) becomes

$$X_t^x = x + Z_t - Z_0, \tag{5}$$

and Theorem 1 states that if X^x are all homogeneous Markov with the same semigroup, then Z (and hence also X^x) are Lévy processes. This would appear to imply that all homogeneous Markov processes are Lévy processes! The subtle hypothesis involved here is that (5) can be rewritten

$$X_t^x = X_0^x + Z_t - Z_0$$

and then by hypothesis the law of Z does not depend on $X_0^x = x$; the only homogeneous Markov processes X such that $X_t - X_0$ has a law independent of X_0 are the Lévy processes.

One can do a little more along the same lines; indeed the next result is even more elementary. Suppose $(\Omega, \mathcal{F})$ is equipped with a family P_z of probabilities under which Z is a semimartingale with $P_z(Z_0 = z) = 1$. If Z is homogeneous Markov for each P_z, with transition probabilities independent of z, it is well known (e.g., [2] or [3]) that the vector process (Z, X^x), with X^x given by (1), is homogeneous Markov with transition probabilities independent of (z, x). The following is a converse:

Theorem 2: If under each P_z and for each x the vector process (Z, X^x) is homogeneous Markov with transition probabilities independent of (z, x), then the process Z is itself homogeneous Markov under each P_z (with transition probabilities independent of z).

Proof: Let $(Q_t)_{t \geq 0}$ be the transition semigroup of (Z, X^x). Then $Q_t(z, x; A \times \mathbb{R}) = P_z(Z_t \in A | Z_0 = z \text{ and } X_0^x = x) = P_z(Z_t \in A)$, whereas $Q_t(z, x; A \times \mathbb{R}) = R_t(z, A)$ does not depend on x. It is then immediate that Z itself is homogeneous Markov with transitions $(R_t)_{t \geq 0}$. $\square$

Note that Theorem 2 does not really use equation (1), and hence there is no hypothesis on f! We use only the fact that the probabilities P_z do not depend on x. Thus, a bit paradoxically, Theorem 2 is more elementary than Theorem 1 (which is however false without some hypothesis on f: consider the case where f is identically zero).

References

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Special Note: This article also has a version in French which has been submitted for publication in the Séminaire de Probabilités.