# INTRINSIC ULTRACONTRACTIVITY AND PROBABILITY*

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Let $p_t^D(x,y) = p_t(x,y)$ be the Dirichlet heat kernel for $\frac{1}{2}\Delta$ in a domain $D \subset \mathbb{R}^n$, $n \geq 2$. In [4] E.B. Davies and B. Simon define the semigroup connected with the Dirichlet Laplacian to be intrinsically ultracontractive if there is a positive (in $D$) eigenfunction $\phi_0$ for $\frac{1}{2}\Delta$ in $D$ and if for each $t > 0$ there are positive constants $c_t$, $C_t$ depending only on $D$ and $t$ such that

$$(1) \qquad c_t\phi_0(x)\phi_0(y) < p_t(x,y) < C_t\phi_0(x)\phi_0(y), \; x,y \in D.$$

(To be precise, they show (1) is equivalent to their definition.) Here we will just say $D$ is IU when (1) holds. Among many results in their very interesting paper Davies and Simon prove that bounded Lipschitz domains are IU. Recently the investigation of IU has been taken up by a number of probabilists who approached the area via their study of distributions of the lifetimes of Doob's conditioned h-processes, a study which in its modern version started with Cranston and McConnell's solution ([3]) of a conjecture of Chung. Here we are going to give an intuitive sketch of the connection between lifetimes and IU, and describe the results in our paper [5].

The reader will note that Rodrigo Bañuelos' paper in this volume deals with the same topic as this paper. Bañuelos was more diligent than we were and wrote his paper first. Sometimes sloth is rewarded (besides being its own reward), and this is one of those times, for, since Bañuelos has surveyed the area, we do not have to. Accordingly only those papers with the most immediate connection to [5] will be mentioned.

Let $\mathbf{X} = \{X_t, t \geq 0\}$, be standard $n$ dimensional Brownian motion and let $P_x$ stand for probability associated with this motion given $X_0 = x$. The kernel $p_t(x,\cdot)$ has an immediate probabilistic interpretation as the density at time $t$ of $\mathbf{X}$ killed when it leaves $D$, that is, if $A$ is a Borel subset of $D$ and $\tau_D = \inf\{t > 0 \colon X_t \notin D\}$,

$$P_x(X_t \in A, \; t < \tau_D) = \int_A p_t(x,y)dy.$$

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Thus $p_t(x,y)/p_s(x,y)$ gives the ratio of the probabilities of killed Brownian motion being infinitessimally close to $y$ (we will just say "hitting $y$") at times $t$ and $s$, respectively.

Now it can be shown that for $\delta > 0$ fixed,

$$\lim_{\varepsilon\to 0} \frac{P_x(\exists s,\ t < \tau_D: |s-t| > \delta \text{ and } |B_s - y| < \varepsilon,\ |B_t - y| < \varepsilon)}{P_x(\exists t < \tau_D: |B_t - y| < \varepsilon)} = 0, \tag{2}$$

and roughly what this implies is that given $y$ is hit by $\mathbf{X}$ before $\tau_D$, it is hit at only one time, although some work must be done to make this rigorous, since the probability $\mathbf{X}$ ever hits $y$ is zero. Thus intuitively, $p_t(x,y)/p_s(x,y)$ gives the ratio of the probabilities of $\mathbf{X}$ *first* hitting $y$ before $\tau_D$ at times $t$ and $s$, respectively, if $X_0 = x$, given $\mathbf{X}$ does hit $y$ before time $\tau_D$. Equivalently,

$$L_{x,y}(t) = \frac{p_t(x,y)}{\int\limits_0^\infty p_t(x,y)dt},\ t > 0,$$

is the density of the time $\mathbf{X}$ (started at $x$) first hits $y$, given that $y$ is hit before $\tau_D$. The integral in the denominator is finite whenever $x \neq y$ and $D$ has a Green function, assumptions always in force here. We use $P_x^y$ and $E_x^y$ to denote probability and expectation associated with $\mathbf{X}$, started at $x$, conditioned to hit $y$ before $\tau_D$, and use $T$ to denote the time of hitting $y$, so that the density of $T$ under $P_x^y$ is $L_{x,y}(\cdot)$. Note $E_x^y T = \int\limits_0^\infty t L_{x,y}(t)dt$.

Cranston and McConnell's theorem is equivalent to the following.

Theorem. *Let $n = 2$. There is an absolute constant $C$ such that*

$$\int_0^\infty t L_{x,y}(t)dt < C \text{ area } D,\ x \neq y,\ x,y \in D.$$

Thus the Cranston–McConnell theorem can be interpreted as a theorem about the shape of the normalized (in $t$) heat kernel which holds uniformly over all points $x, y$. We will call domains satisfying $\sup\limits_{\substack{x,y\in D\\ x\neq y}} E_x^y T < \infty$ Cranston–McConnell domains. Let $\hat{p}_t(x,\cdot) = p_t(x,\cdot)/\int_D p_t(x,y)dy$. It is shown in [5] that $D$ is IU if and only if there is for each $t > 0$ a positive constant $\alpha_t$ such that $\hat{p}_t(x,y)/\hat{p}_t(z,y) < \alpha_t,\ x,y,z \in D$. The *only if* part is immediate, and the *if* part is easy. Thus IU is equivalent to a different kind of uniformity of the heat kernel, uniformity in the first variable under normalization in the second variable. Since $\hat{p}_t(x,y)$ is the density of killed Brownian motion, started at $x$, conditioned to be alive at time $t$, IU says all such densities are comparable, so it is in some sense a mixing condition for this motion.

Now it is immediate that if $\lambda$ is the eigenvalue associated with the eigenfunction $\phi_0$, both $c_t e^{-\lambda t}$ and $C_t e^{-\lambda t}$ may be chosen to be respectively nondecreasing and nonincreasing in $t$, since

$$\int_D \phi_0(y) p_s(y,z)dy = e^{-\lambda s}\phi_0(z),$$

and

$$\int_D p_t(x,y)p_s(y,z)dy = p_{s+t}(x,z).$$

Thus we immediately get from (1) that $c_{t_0}/C_{t_0} < L_{x,y}(t)/[e^{\lambda(s-t)}L_{x,y}(s)] < C_{t_0}/c_{t_0}$, $t_0 < t < s$, a property that we will call uniformly (in $x,y$) close to exponential decay at rate $\lambda$. Especially we see that this uniform decay immediately implies

$$\sup_{\substack{x,y\in D\\ x\neq y}} \int tL_{x,y}(t)dt < C,$$

that is, IU domains are Cranston–McConnell domains. We note there are planar domains of finite area which are not IU domains, and that both $c_t$ and $C_t$ in (1) can be chosen to converge to 1. See [4] or Bañuelos' paper in this volume.

In [5] we prove

Proposition. *The domain $D$ is IU if there is a compact subset $F$ of $D$ and constants $k_t$, $K_t$ such that both*

(*i*) $$\int_F p_t(x,y)dy \geq k_t \int_D p_t(x,y)dy, \ \ x \in D,$$

*and*

(*ii*) *Given $x \in D$ there is $\hat{x} \in F$ such that $p_t(\hat{x},y) \geq K_t p_t(x,y)$, $y \in D$.*

This proposition is used to prove

Theorem 1. *Let $f$ be a real valued uppersemicontinuous function defined as $\{(x_1, \ldots, x_{n-1})$: $\Sigma x_i^2 < 1\}$ such that $-M < f \leq 0$ for some constant $M < \infty$. Then the domain $D_f = \{(x_1, \ldots, x_n) \in \mathbb{R}^n: \sum_{i=1}^{n-1} x_i^2 < 1,\ f(x_1, \ldots, x_{n-1}) < x_n < 1\}$ is* IU.

We note that Bass and Burdzy independently a few months before our work proved the domains $D_f$ are Cranston–McConnell domains.

Our proof of Theorem 1 is probabilistic. We take $F$ to be $\{(x_1, \ldots, x_n): \sum_{i=1}^{n-1} x_i^2 < \alpha_n,\ \varepsilon_n < x_n < 1 - \varepsilon_n\}$ for certain positive numbers $\alpha_n$ and $\varepsilon_n$ and use Girsanov's theorem and coupling arguments to show first i), then ii) of the proposition. Now in fact, the second of the two conditions of this proposition is redundant. The work of Bass and Burdzy [1] implies that ii) follows from i), so that in fact i) alone implies that D is IU. Bass and Burdzy use this to extend Theorem 1 to functions $f$ in $L^p$, $p < n$. Both [1] and [2] show intrinsic ultracontractivity for other classes of domains, and for other semigroups than that connected with the Dirichlet Laplacian.

We also prove in [5]

Theorem 2. *Let $D = (0,1) \times (0,1) \cup \bigcup_{i=1}^{\infty} R_i$, where $R_i = I_n \times (-a_i, 0]$ and $I_1, I_2, \ldots$ are disjoint open intervals contained in $(1/4,\ 3/4)$ and $a_i \geq 0$. Then $D$ is* IU *if and only if $\lim_{n \to \infty}$ area $R_n = 0$.*

Our proof of the if part of this theorem resembles that of the last theorem but is a little harder. The proof of the only if part uses results of Xu [6] and the author about the mean and variance of $T$ under $P_x^y$ to show that if $x = (1/2, 1/2)$, and if the $y_i$ are close to the bottom of $R_i$, then if lim area $R_n \neq 0$, the lifetime densities $L_{x,y_i}$ do not have a uniformly close to exponential shape which, as has been mentioned, is implied by IU. The if part of Theorem 2 answered affirmatively the question raised in [4] whether there are domains of infinite area which are IU. Subsequently very different examples (although closely related to each other) have been given in [1], [2]. Previously Xu [6] had given an example of a domain of infinite area which is a Cranston–McConnell domain. Theorem 2 gives a view of part of the edge of IU. We hope that a geometric characterization of IU for the class of domains above the graph of a function, in two dimensions, will be proved, a goal that does not seem out of reach. There is a conjecture in [5]. At the least Theorem 2 will provide a test case for investigations into this, at most our method will extend. A more distant goal which may or may not be feasible is a nice geometric characterization of IU for all simply connected planar domains.

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