SADDLE–POINT APPROXIMATIONS AND SPACE–TIME MARTIN BOUNDARY FOR NEAREST NEIGHBOR RANDOM WALK ON A HOMOGENEOUS TREE

by

Steven P. Lalley
Purdue University

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Department of Statistics
Purdue University

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Abstract

Let X_n be a nearest neighbor random walk on the group $\mathcal{G}$ = free product of L copies of $\mathbb{Z}_2$. Explicit saddlepoint approximations for $P(X_n = x)$ are given; these hold uniformly for those $x \in \mathcal{G}$ that can be reached in n steps. The saddlepoint approximations are used to identify the space–time Martin boundary of the random walk as $(\Lambda \times [0, R]) \cup (\mathcal{G} \times \{R\})$, where Λ is the space of ends of the Cayley graph of $\mathcal{G}$ and $R^{-1} > 0$ is the spectral radius of the transition operator. The extreme points of the space–time Martin boundary are shown to be $\Lambda \times [0, R]$.

0. Introduction

The purpose of this note is to identify the space–time Martin boundary for a nearest–neighbor random walk on a homogeneous tree. For simplicity we shall only consider trees of a certain type, to wit, Cayley graphs of free products of the two–element group $\mathbb{Z}_2$. However, the techniques developed here are applicable also to certain other trees, e.g., Cayley graphs of finitely generated free groups, and the space–time Martin boundaries in such cases admit similar descriptions.

Let $\mathcal{G}$ be the free product of $L \geq 3$ copies of $\mathbb{Z}_2$, i.e., $\mathcal{G}$ has generators $\{a_1, a_2, \ldots, a_L\}$ and relations $a_i^2 = e =$ identity. Each $x \in \mathcal{G}$ may be written uniquely as a reduced word in the generators: $x = a_{i_1} a_{i_2} \ldots a_{i_m}$, where $i_j \neq i_{j+1}$ for $j = 1, 2, \ldots, m-1$. The Cayley graph of $\mathcal{G}$ has vertex set $\mathcal{G}$ and edge set $\{(x, xa_i) : x \in \mathcal{G},\ i = 1, 2, \ldots, L\}$. A nearest–neighbor random walk $(X_n)_{n \geq 0}$ on $\mathcal{G}$ is a Markov chain with 1–step transition probabilities

$$P(X_{n+1} = xa_i | X_n = x) = p_i,$$

$$P(X_{n+1} = x | X_n = x) = p_e,$$

where $p_e > 0$, $p_i > 0$ for each $i = 1, 2, \ldots, L$, and $p_e + \sum_{i=1}^{L} p_i = 1$. (The reason for assuming $p_e > 0$ is to make the Markov chain aperiodic. This is only for convenience, however; our results could easily be extended to allow $p_e = 0$.) When not otherwise specified, we will assume that $P(X_0 = e) = 1$.

It is known [C], [D], [DM] that a nearest–neighbor random walk on $\mathcal{G}$ is transient, and that its Martin boundary coincides with the space Λ of ends of $\mathcal{G}$, defined to be the set of infinite words $\omega = a_{i_1} a_{i_2} \ldots$ in which the same letter a_i does not appear twice in succession, i.e., $i_j \neq i_{j+1} \forall j \geq 1$. The space $\mathcal{G} \cup \Lambda$ is endowed with the natural "word topology": each $\{x\}$, $x \in \mathcal{G}$, is open and closed; Λ has the product topology (the topology of coordinatewise convergence); and $x_n \to \omega$, with $x_n \in \mathcal{G}$ and $\omega \in \Lambda$, iff the words representing x_n converge coordinatewise to ω.

The space–time Markov chain associated with the nearest–neighbor random walk is the Markov chain Y_n with transition probabilities

$$P(Y_{n+1} = (m+1, xa_i)|Y_n = (m,x)) = p_i,$$
$$P(Y_{n+1} = (m+1, x)|Y_n = (m,x)) = p_e$$

and state space $\Gamma = \{(m,x): m \in Z_+,\ x \in \mathcal{G}, \text{ and } |x| \leq m\}$. Here $|x|$ denotes the word–length of x, i.e., if $x = a_{i_1} \dots a_{i_k}$ then $|x| = k$; also $Z_+ = \{0, 1, 2, \dots\}$. The state space Γ should be visualized as the vertex set of a directed tree with root $(0, e)$ and arrows from (m, x) to $(m+1, x)$ and from (m, x) to $(m+1,\ xa_i)$, $i = 1, 2, \dots, L$. The transition probabilities of Y_n are related to those of X_n as follows:

$$\begin{aligned} &q_n((m,x),\ (m+n, x')) \\ &\quad \triangleq P(Y_{k+n} = (m+n, x')|Y_k = (m,x)) \\ &\quad = P(X_{k+n} = x'|X_k = x) \\ &\quad \triangleq p_n(x, x') = p_n(e, x^{-1}x') \triangleq p^{*n}(x^{-1}x'). \end{aligned}$$

We shall prove (Theorem 2) that the Martin boundary of the space–time process Y_n (the "space–time Martin boundary") is

$$(\Lambda \times [0, R]) \cup (\mathcal{G} \times \{R\})$$

for a certain real number $R > 0$. The topology is as follows: $\Lambda \times [0, R]$ has the product topology, and Λ has the "word topology"; also $(\Lambda \cup \mathcal{G}) \times \{R\}$ has the "word topology", so $(x_n, R) \to (\omega, R)$ iff $x_n \to \omega$ letter by letter. We shall also give an explicit formula for the Martin kernel. Finally, we shall identify the "space of exits" B_e (the extreme points of the Martin boundary) as

$$B_e = \Lambda \times [0, R].$$

Thus, the space–time random process Y_n is an example of a transient Markov chain whose space of exits is strictly smaller than its Martin boundary (see [KSK], sec. 10–13, ex. 4 for another example which, however, is more artificial).

The identification of the space–time boundary entails the development of sharp asymptotic estimates for the transition probabilities $p^{*n}(x)$. Local limit theorems for $p^{*n}(x)$ have been obtained by [GW] (see also [S], [FP], [CS] for local limit theorems in similar settings); these state that for any fixed $x \in \mathcal{G}$

$$p^{*n}(x) \sim C_x R^{-n} n^{-3/2}$$

as $n \to \infty$. These relations clearly cannot hold uniformly for $x \in \mathcal{G}$. Let $\xi = \xi(n, x)$ be the vector $(\xi_1, \xi_2, \ldots, \xi_L)$ of nonnegative reals such that $n\xi_i$ is the number of times the letter a_i occurs in the word representing x. We will prove (Theorem 1) that for certain functions $\varphi(\xi)$, $\beta_n(\xi)$,

$$p^{*n}(x) \sim \beta_n(\xi(n,x)) \exp\{n\varphi(\xi(n,x))\}$$

as $n \to \infty$, *uniformly* for $x \in \mathcal{G}$ such that $|x| \leq n$, where $n^{-1} \log \beta_n(\xi) \to 0$ uniformly for ξ satisfying $\Sigma\xi_i \leq 1$. Explicit asymptotic formulas for $\beta_n(\xi)$ will also be given (Propositions 4–5). These saddlepoint approximations are not entirely routine, because $\beta_n(\xi)$ makes a transition from $Cn^{-3/2}$ to $C'n^{-1/2}$ as $\Sigma\xi_i$ varies from 0 to ε, then another transition from $C''n^{-1/2}$ to C''' as $\Sigma\xi_i$ varies from $1 - \varepsilon$ to 1.

1. Generating Functions

The saddlepoint approximations are based upon analyticity properties of certain generating functions of a type also used in [GW], [SS], and [A]. We begin by studying these, following the method of [GW]. For $x \in \mathcal{G}$, $i \in \{1, 2, \ldots, L\}$, and $z \in \mathbb{C}$, $|z| < 1$, define

$$\begin{aligned} T(x) &= \inf\{n \geq 1 : X_n = x\}, \\ F(x,z) &= Ez^{T(x)} 1\{T(x) < \infty\}, \\ F_i(z) &= F(a_i, z), \\ G(x,z) &= \sum_{n=0}^{\infty} p^{*n}(x) z^n, \\ G(z) &= G(e,z), \\ H(z) &= \sum_{i=1}^{L} p_i F_i(z) + p_e. \end{aligned}$$

The functions $F(x,z)$, $G(x,z)$ are evidently analytic in $|z| < 1$. In fact they are algebraic functions, as we will show next.

First, notice that if $x = a_{i_1} a_{i_2} \dots a_{i_m}$ then to reach x the random walk must visit in sequence a_{i_1}, $a_{i_1} a_{i_2}, \dots, a_{i_1} a_{i_2} \dots a_{i_{m-1}}$, x. Furthermore, the passage time from $a_{i_1} a_{i_2} \dots a_{i_k}$ to $a_{i_1} a_{i_2} \dots a_{i_{k+1}}$ has the same law as that from e to $a_{i_{k+1}}$, by the translation invariance of the random walk. Hence, by the Markov property,

$$F(x, z) = \prod_{j=1}^{m} F_{i_j}(z),$$
$$G(x, z) = F(x, z) G(z).$$

Second, conditioning on the first step X_1 of the random walk leads to the equations

$$G(z) = 1 + zH(z)G(z),$$
$$F_i(z) = p_i z + zH(z)F_i(z) - p_i z F_i(z)^2.$$

The latter is a quadratic equation in $F_i(z)$, which may be solved in terms of z and $H(z)$, and hence z and $G(z)$, by the first equation. The positive $\sqrt{\ }$ in the quadratic formula must be used because $z = 0$ is a regular point of $F_i(z)$. Thus,

(1.1)
$$F_i(z) = \frac{-1 + \sqrt{1 + 4p_i^2 z^2 G(z)^2}}{2p_i z G(z)}.$$

Multiplying by p_i, summing on i, and substituting for $H(z)$ in terms of $G(z)$ yields the algebraic equation

(1.2)
$$G(z) = P(zG(z)),$$

where

$$P(t) = 1 + p_e t + \sum_{i=1}^{L} \{-1 + \sqrt{1 + 4p_i^2 t^2}\}.$$

PROPOSITION 1: *Let t_* be the unique positive solution of the equation $P(t) = tP'(t)$. Then each of the power series $G(z)$ and $F_i(z)$, $1 \le i \le L$, has radius of convergence $R = 1/P'(t_*)$, and for each of these functions the argument $z = R$ is the only singularity in $\{|z| \le R\}$. Moreover, there exist functions $H_i(z)$, $K_i(z)$ for $0 \le i \le L$, each analytic in $\{|z| < R + \delta\}$, for some $\delta > 0$ such that*

$$G(z) = H_0(z) + \sqrt{R - z}\, K_0(z),$$
$$F_i(z) = H_i(z) + \sqrt{R - z}\, K_i(z) \qquad \forall |z| < R;$$

and

$$\begin{aligned} G(R) &= H_0(R) = P(t_*), \\ F_i(R) &= H_i(R) = (-1 + \{1 + 4p_i^2 t_*^2\}^{1/2})/2p_i t_* < 1 \quad \forall\, 1 \le i \le R, \\ K_0(R) &= -\{2P(t_*)P'(t_*)^3/P''(t_*)\}^{1/2} < 0, \\ K_i(R) &= \{K_0(R)H_i(R)/G(R)\}\{1 + 4p_i^2 t_*^2\}^{-1/2} < 0 \quad \forall\, 1 \le i \le R. \end{aligned}$$

NOTE: This is adapted from [GW], with minor changes. For completeness we shall give a proof, most of which is taken from [GW].

LEMMA 1: *Each of the functions $G(x,z)$ has an analytic continuation to all $z \in \mathbb{C}\backslash\mathbb{R}$; thus, all singularities of $G(x,z)$ are real.*

PROOF: Consider the bounded linear operator $M: \ell^2(\mathcal{G}) \to \ell^2(\mathcal{G})$ defined by $Mf(x) = p_e f(x) + \sum_{i=1}^{L} p_i f(xa_i)$. Clearly, $\|M\| \le 1$, and M is Hermitian. Consequently, its spectrum lies entirely in $[-1,1]$, and its resolvent $R_z = (zI - M)^{-1}$ is analytic for $z \in \mathbb{C}\backslash[-1,1]$. But $z^{-1}R_{z^{-1}} = \sum_{n=0}^{\infty} z^n M^n$ for $|z| < 1$, so

$$\begin{aligned} z^{-1}R_{z^{-1}}f(x) &= \sum_{n=0}^{\infty} \sum_{y \in \mathcal{G}} z^n p^{*n}(x^{-1}y) f(y) \\ &= \sum_{y \in \mathcal{G}} G(x^{-1}y, z) f(y) \end{aligned}$$

for $|z| < 1$. Applying this to $f = 1_{\{e\}}$ we see that $G(x^{-1}, z)$ extends analytically to all of $\mathbb{C}\backslash\mathbb{R}$. ///

PROOF *of Prop. 1*: Each of the power series $G(x,z)$, $F_i(z)$ has nonnegative coefficients, so its radius of convergence coincides with its smallest positive singularity. We will use the functional equations (1.1)–(1.2) to determine the singularities.

Consider the function $P(t)$. Routine calculations show that $P(0) > 0$, $P'(0) > 0$, $P''(t) > 0$ for all $t > 0$, and that $y = P(t)$ approaches the asymptote $y = (p_e + 2\sum_{i=1}^{L} p_i)t - L + 1$ as $t \to \infty$. For each $t \ge 0$ the line tangent to the graph of P at $(t, P(t))$ intersects the y–axis at a point $(0, y_t)$; the function $t \to y_t$ is continuous and strictly decreasing, since $P'' > 0$. But $y_0 = P(0)$ and $y_t \to -L + 1 < 0$ as $t \to \infty$, so there is a unique $t = t_* > 0$ where $y_t = 0$, equivalently $P(t) = tP'(t)$.

The functional equation $G(z) = P(zG(z))$ implies that for $0 < z < 1$ the value $G(z)$ is the y–coordinate of the (first) point of intersection of the line $y = t/z$ with the curve $y = P(t)$. The results of the previous paragraph imply that this intersection is transversal for all $0 < z < 1/P'(t_*)$. Consequently, by the Implicit Function Theorem, $G(z)$ has an analytic continuation along the segment $0 < z < 1/P'(t_*)$; this proves that the radius of convergence of G is at least $1/P'(t_*)$. Furthermore, as $z \uparrow 1/P'(t_*)$, $G(z) \uparrow P(t_*) < \infty$.

To see that $z = 1/P'(t_*)$ is in fact a singularity of G, rewrite (1.2) as $f(z,\omega) = 0$, where $f(z,\omega) = \omega - P(z\omega)$ and $\omega = G(z)$. The value t_* has been chosen so that $f_\omega = 0$ at $z_* = 1/P'(t_*)$, $\omega_* = P(t_*)$. Hence, when $f(z,\omega)$ is expanded in a Taylor series around (z_*,ω_*), it becomes locally quadratic in $(\omega - \omega_*)$. It follows that $G(z) = a_0 + a_1(z_* - z)^{1/2} + \ldots$ for z near z_*, $z < z_*$, with $a_1 \neq 0$, so $z_* = R$ is a singularity. Routine calculations show that $G(z) = H_0(z) + (R-z)^{1/2}K_0(z)$ where $H_0(R)$, $K_0(R)$ are as advertised.

Lemma 1 implies that G has no singularities on $|z| = R$ except $z = R$ and possibly $z = -R$. Now $|G(z)| \leq G(|z|)$ for all $|z| \leq R$, with equality iff $z = |z|$ (because $p_e > 0$, and so $p^{*n}(e) > 0$ for all $n \geq 0$). From the functional equation $G(z) = P(zG(z))$ it follows that $z = -R$ is a singularity only if $P(-RG(-R)) = -RG(-R)P'(-RG(-R))$, by the Implicit Function Theorem. But

$$|P(t) - tP'(t)| = |1 - L + \sum_{i=1}^{L}\{1 + 4p_i^2t^2\}^{1/2}| > 0$$

if $|t| < t_*$. Since $|RG(-R)| < RG(R)$ it follows that $z = -R$ is *not* a singularity of G.

Finally, consider the function $F_i(z)$. Since F_i is defined by a power series with nonnegative coefficients, its radius of convergence coincides with its smallest positive singularity. The only singularities of $F_i(z)$ are real, by Lemma 1, since $G(a_i,z) = F_i(z)G(z)$. The functional equation (1.1) expresses $F_i(z)$ as a function of $zG(z)$, and for real z this function is analytic. It follows that $z = R$ is a singularity of $F_i(z)$, but $z = -R$ is not. Routine calculations show that $F_i(z) = H_i(z) + (R-z)^{1/2}K_i(z)$ with $H_i(R)$, $K_i(R)$ as in the statement of the proposition. ///

NOTE: It can be shown directly that $R > 1$; however, this is not needed for any of the subsequent results.

For $-\infty \le s \le \log R$ define

$$\psi_0(s) = \log G(e^s)$$
$$\psi_i(s) = \log F_i(e^s), \; i = 1, 2, \ldots, L.$$

PROPOSITION 2: *The functions $\psi_i(s)$ are strictly increasing, strictly convex functions of $s \in (-\infty, \log R]$ satisfying*

$$\psi_i''(s) > 0, \quad i = 0, 1, \ldots, L,$$
$$\lim_{s \uparrow \log R} \psi_i'(s) = \infty, \quad i = 0, 1, \ldots, L,$$
$$\lim_{s \downarrow -\infty} \psi_i'(s) = 1, \quad i = 1, 2, \ldots, L.$$

PROOF: First take $i = 1, 2, \ldots, L$ and differentiate both sides of $Ee^{s\tau_i} 1\{\tau_i < \infty\} = e^{\psi_i(s)}$, where τ_i is the first time the random walk visits a_i. This is allowable because F_i is analytic. We obtain

$$\psi_i'(s) = E\tau_i e^{s\tau_i} 1\{\tau_i < \infty\} / e^{\psi_i(s)}$$
$$\psi_i''(s) = E\tau_i^2 e^{s\tau_i} 1\{\tau_i < \infty\} / e^{\psi_i(s)}$$
$$- (E\tau_i e^{s\tau_i} 1\{\tau_i < \infty\} / e^{\psi_i(s)})^2.$$

It follows that $\psi_i''(s) > 0$ because the right hand side of the last equation is a variance of a nonconstant random variable. Next, $\psi_i'(s) = e^s F_i'(e^s)/F_i(e^s) \uparrow \infty$ as $s \uparrow \log R$, because $F_i(z)$ has an algebraic singularity at $z = R$. Finally, it follows from the expansion $F_i(z) = p_i z + z^2 \tilde{F}_i(z)$ near $z = 0$ that $\psi_i'(s) \to 1$ as $s \to -\infty$.

The corresponding statements for ψ_0 follow by similar arguments. ///

2. Saddlepoint Approximations

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_L)$ be any vector of nonnegative real numbers satisfying $\Sigma \xi_i \le 1$; define

$$\varphi(\xi) = \inf_{-\infty < s \le \log R} \left(\sum_{i=1}^{L} \xi_i \psi_i(s) - s \right). \tag{2.1}$$

Since each $\psi_i(s)$ is a strictly convex function of $s \in \mathrm{R}$, the infimum is attained at precisely one $s \in [-\infty, \log R]$. If $\xi_1 = \xi_2 = \ldots = \xi_L = 0$, the infimum is attained at $s = \log R$. If

$0 < \Sigma\xi_i < 1$, the infimum is attained at the unique $s = s(\xi) \in (-\infty, \log R)$ satisfying

$$\sum_{i=1}^{L} \xi_i \psi_i'(s) = 1 \tag{2.2}$$

(recall that $\psi_i'(s)$ is a strictly increasing function such that $\psi_i'(s) \to 1$ as $s \to -\infty$ and $\psi_i'(s) \to \infty$ as $s \to \log R$). If $\Sigma\xi_i = 1$, the infimum is attained at $s = -\infty$, and in this case

$$\varphi(\xi) = \sum_{i=1}^{L} \xi_i \log p_i \tag{2.3}$$

(since $e^{\psi_i(s)} = p_i e^s + 0(e^{2s})$ as $s \to -\infty$).

Now let $x = a_{i_1} a_{i_2} \dots a_{i_m}$ be a reduced word in $\mathcal{G}$. For $n \geq m$, define $\xi(n, x) = \xi = (\xi_1, \xi_2, \dots, \xi_L)$ to be the vector of nonnegative real numbers such that $n\xi_i$ is the number of occurrences of the letter a_i in the word $a_{i_1} a_{i_2} \dots a_{i_m}$.

THEOREM 1: *For each reduced word $x = a_{i_1} a_{i_2} \dots a_{i_m}$, if $n \geq m$ then*

$$p^{*n}(x) = \beta_n(\xi(n, x)) \exp\{n\varphi(\xi(n, x))\},$$

where

$$\lim_{n \to \infty} \sup_{\xi} \tfrac{1}{n} |\log \beta_n(\xi)| = 0,$$

the supremum being taken over all nonnegative vectors $\xi = (\xi_1, \xi_2, \dots, \xi_L)$ satisfying $\Sigma\xi_i \leq 1$.

Theorem 1 will follow from Props. 4–5 below, and the detailed asymptotic behavior of the factors $\beta_n(\xi)$ will be described. The main point of the theorem is that the function $\varphi(\xi)$ controls the exponential decay of the probabilities $p^{*n}(x)$. This should be compared with the local limit theorem of [GW], which states that for each *fixed* x (not varying with n)

$$p^{*n}(x) \sim C_x n^{-3/2} R^{-n}$$

as $n \to \infty$. (See also Corollary 1 below.) This is consistent with Theorem 1, because for fixed x the word length m is constant and so $\xi(n, x) \to 0$ which implies $\varphi(\xi(n, x)) \to -\log R$. But Theorem 1 also indicates that (at least for long words x) the exponential rate

$\frac{1}{n}\log p^{*n}(x)$ undergoes a long, gradual change before finally settling down at $-\log R$ when n gets much larger than the word length of x. It is interesting that this gradual trend in $\frac{1}{n}\log p^{*n}(x)$ is an *increasing* trend, as the next result shows.

PROPOSITION 3: *For each nonnegative vector $\xi = (\xi_1, \xi_2, \ldots, \xi_L)$ such that $\Sigma\xi_i = 1$, the function $t \to \varphi(t\xi)$ is strictly increasing in t.*

PROOF: If $0 < \Sigma\xi_i < 1$ then the infimum in (2.1) is attained uniquely at $s = s(\xi)$ satisfying $\Sigma\xi_i\psi_i'(s) = 1$. Consequently, if $0 < \Sigma\xi_i < 1$ then $\varphi(\xi) = \Sigma\xi_i\psi_i(s(\xi)) - s(\xi)$ and so

$$\begin{aligned}\frac{\partial\varphi}{\partial\xi_i} &= \psi_i(s(\xi)) + \left(\sum_j \xi_j\psi_j'(s(\xi))\right)\frac{\partial s}{\partial\xi_i} - \frac{\partial s}{\partial\xi_i}\\ &= \psi_i(s(\xi))\\ &< 0,\end{aligned}$$

because $F_i(z) \leq F_i(R) < 1$ for $0 \leq z \leq R$. Consequently, if $\Sigma\xi_i = 1$ then for $0 < t < 1$

$$\frac{d}{dt}\varphi(t\xi) = \Sigma\xi_i\frac{\partial\varphi}{\partial\xi_i}(t\xi) < 0. \qquad ///$$

PROPOSITION 4: *For any $0 < \varepsilon < 1$ the asymptotic behavior of $\beta_n(\xi)$ as $n \to \infty$ for nonnegative vectors $\xi = (\xi_1, \ldots, \xi_L)$ in the range $\varepsilon \leq \Sigma\xi_i \leq 1$ is as follows: with $k = n(1 - \Sigma\xi_i)$,*

$$\beta_n(\xi) \sim G(e^{s(\xi)})\{2\pi n\sum_i \xi_i\psi_i''(s(\xi))\}^{-1/2} \quad \text{if } \Sigma\xi_i \leq 1 - n^{-3/4}, \tag{2.4}$$

$$\beta_n(\xi) \sim k^k e^{-k}/k! \quad \text{if } 1 \geq \Sigma\xi_i \geq 1 - n^{-3/4}. \tag{2.5}$$

These relations hold uniformly for nonnegative vectors ξ satisfying $\varepsilon \leq \Sigma\xi_i \leq 1$.

NOTE: For large k, $k^k e^{-k}/k! \sim (2\pi k)^{-1/2}$. Consequently, the two asymptotic formulas "merge" at the crossover point $1 - n^{-3/4}$, because $\{2\pi n\Sigma\xi_i\psi_i''(s(\xi))\}^{-1/2} \sim \{2\pi k\}^{-1/2}$ for $\Sigma\xi_i \approx 1 - n^{-3/4}$, as will be seen in the proof. Also, there is nothing special about the exponent $3/4$; any exponent $\frac{1}{2} < \alpha < 1$ would work just as well.

PROOF: By the Cauchy Integral Formula,

$$
\begin{aligned}
p^{*n}(x) &= (2\pi i)^{-1} \int_{\Gamma_r} G(x,z) z^{-n-1} dz \\
&= (2\pi i)^{-1} \int_{\Gamma_r} \prod_{j=1}^{L} F_j(z)^{m_j} G(z) z^{-n-1} dz
\end{aligned}
\tag{2.6}
$$

where Γ_r is the circle of radius r with center 0, oriented counterclockwise, $0 < r \leq R$, and $m_j = n\xi_j$ is the number of occurrences of the letter a_j in the reduced word x. Set $r = e^{s(\xi)}$; then $z = r$ is a "saddlepoint" for the integrand, in particular, for small $|\theta|$,

$$
\begin{aligned}
&\Big(\prod_{j=1}^{L} F_j(re^{i\theta})^{m_j}\Big)(re^{i\theta})^{-n} \\
&\qquad = \exp\left\{\sum_{j=1}^{L} m_j \psi_j(s(\xi) + i\theta) - ns(\xi) - in\theta\right\} \\
&\qquad = \exp\{n\varphi(\xi(n,x))\} \exp\left\{-n \sum_{j=1}^{L} \xi_j \psi_j''(s(\xi))\theta^2/2 + 0(n\theta^2)\right\}
\end{aligned}
\tag{2.7}
$$

since $\Sigma m_j \psi_j'(s(\xi)) = n\Sigma\xi_j\psi_j'(s(\xi)) = n$. Consequently, the integral may be analyzed by Laplace's method of asymptotic expansion ([E], sec. 2.4), which (at least formally) gives the result (2.4) for $0 < \Sigma\xi_j < 1$.

To justify the use of Laplace's method we must verify that the major contribution to the integral comes from those $z \in \Gamma_r$ near r. This, however, follows easily from the fact that for each $j = 1, 2, \ldots, L$ the ratio $|F_j(re^{i\theta})|/|F_j(r)|$ is uniformly bounded away from 1 for $0 < \delta \leq |\theta| \leq \pi$ and $\varepsilon \leq r \leq R - \varepsilon$, $\varepsilon > 0$. Furthermore, the two–term Taylor series (in θ) for $\psi_j(s + i\theta)$ is *uniformly* accurate for $0 \leq |\theta| \leq \delta$ and s bounded away from $-\infty$ and $\log R$. Therefore, the Laplace expansion (2.4) is valid uniformly for $\xi = (\xi_1, \ldots, \xi_L)$ in the range $\xi \leq \Sigma\xi_i \leq 1 - \varepsilon$, for any $\varepsilon > 0$.

It remains to investigate the behavior of the integral (2.6) when $\Sigma\xi_i$ is near 1. This requires more care, because $n\Sigma\xi_j\psi_j''(s(\xi))$ is *not* large when $n - n\Sigma\xi_i$ bounded, and so the integral is not dominated by those $z = re^{i\theta}$ with $|\theta|$ small.

Routine computations show that for each $i = 1, 2, \ldots, L$

$$\psi_i'(s) = 1 + p_e e^s + 0(e^{2s}),$$
$$\psi_i''(s) = p_e e^s + 0(e^{2s}),$$
$$\psi_i'''(s) = p_e e^s + 0(e^{2s})$$

for small e^s. Since $s(\xi) = \log r$ is the unique solution of $\Sigma \xi_i \psi_i'(s) = 1$ it follows that

$$r = e^{s(\xi)} = (1 - \Sigma\xi_i)/p_e + 0((1 - \Sigma\xi_i)^2)$$

for $\Sigma\xi_i$ near 1. Now consider vectors ξ in the range $1 - \varepsilon \le \Sigma\xi_j \le 1 - n^{-3/4}$ with $\varepsilon > 0$ small. For $r = e^{s(\xi)}$,

$$\left|\frac{F_j(re^{i\theta})}{F_j(r)}\right|^{m_j} = \left|1 - p_e r(1 - e^{i\theta}) + 0(r^2)\right|^{m_j},$$

and since $r \ge C_1 n^{-3/4}$ it follows that the integral (2.6) is dominated by those $re^{i\theta}$ with $0 \le |\theta| \le \delta$, $\delta > 0$ small, uniformly for $1 - \varepsilon \le \Sigma\xi_j \le 1 - n^{-3/4}$. Moreover, if $\Sigma\xi_j \le 1 - n^{-3/4}$ then $n\Sigma\xi_j\psi_j''(s(\xi)) \ge C_2 n^{1/4} \to \infty$ and $\psi_j'''(s(\xi)) \le C_3\psi_j''(s(\xi))$, so the logarithm of the integrand (2.7) is, for small $|\theta|$, nearly quadratic in θ, with a large coefficient. Thus Laplace's method applies again, yielding (2.4) uniformly for $1 - \varepsilon \le \Sigma\xi_j \le 1 - n^{-3/4}$.

Finally, consider vectors ξ in the range $\Sigma\xi_j \ge 1 - n^{-3/4}$. Laplace's method breaks down in this regime, because $n\Sigma\xi_j\psi_j''(s(\xi))$ does not converge to ∞ uniformly. However, for each $j = 1, 2, \ldots, L$ and $r > 0$ small,

$$\frac{F_j(re^{i\theta})}{F_j(r)e^{i\theta}} = 1 - p_e r(1 - e^{i\theta}) + 0(r^2)$$

is the characteristic function of a nonnegative, integer–valued random variable $X^{(j)}$ such that

$$P(X^{(j)} = 0) = 1 - p_e r + 0(r^2),$$
$$P(X^{(j)} = 1) = p_e r + 0(r^2),$$
$$P(X^{(j)} \ge 2) = 0(r^2).$$

Observe that if $\Sigma\xi_j \ge 1 - n^{-3/4}$ then $r \le Cn^{-3/4}$ and $r^2 \le C'n^{-3/2}$.

Now

$$\begin{aligned}
&\exp\{-n\varphi(\xi(n,x))\}p^{*n}(x)\\
&= (2\pi)^{-1}\int_{-\pi}^{\pi}\prod_{j=1}^{L}\left\{\frac{F_j(re^{i\theta})}{F_j(r)}\right\}^{m_j} G(re^{i\theta})e^{-in\theta}d\theta\\
&= (2\pi)^{-1}\int_{-\pi}^{\pi}\prod_{j=1}^{L}\left\{\frac{F_j(re^{i\theta})}{F_j(r)e^{i\theta}}\right\}^{m_j} e^{-ik\theta}d\theta + 0(r)
\end{aligned}$$

(since $G(0) = 1$). This last integral equals

$$P\left\{\sum_{j=1}^{L}\sum_{i=1}^{m_j} X_i^{(j)} = k\right\},$$

where the random variables $X_i^{(j)}$ are independent and each $X_i^{(j)}$, $i = 1,\dots,m_j$ has the same distribution as $X^{(j)}$. The random variables $X_i^{(j)}$ may be constructed on a probability space also equipped with independent Poisson r.v.s $\hat{X}_i^{(j)}$ satisfying

$$E\hat{X}_i^{(j)} = p_e r$$

and

$$P\left(\sum_{j=1}^{L}\sum_{i=1}^{m_j} X_i^{(j)} \neq \sum_{j=1}^{L}\sum_{i=1}^{m_j}\hat{X}_i^{(j)}\right) = 0(r^2) = 0(n^{-3/2})$$

uniformly in the range $\Sigma\xi_j \geq 1 - n^{-3/4}$. Now $\sum_{j=1}^{L} m_j \sim n$ and $E\hat{X}_i^{(j)} = p_e r = (1-\Sigma\xi_j) + 0(r^2)$, so

$$\begin{aligned}
E\sum_{j=1}^{L}\sum_{i=1}^{m_j}\hat{X}_i^{(j)} &= n(1-\sum_{j=1}^{L}\xi_j) + n0(r^2)\\
&= k + 0(n^{-1/2}).
\end{aligned}$$

Therefore,

$$\frac{P\left(\sum_{j=1}^{L}\sum_{i=1}^{m_j}\hat{X}_i^{(j)} = k\right)}{\{k^k e^{-k}/k!\}} \sim 1$$

uniformly in $\Sigma\xi_j \geq 1 - n^{-3/4}$. Since $k = n(1-\Sigma\xi_j) \leq n^{1/4}$ and $k^k e^{-k}/k! \sim (2\pi k)^{-1/2}$, it follows that the $0(r) = 0(n^{-3/4})$ errors made in the analysis above are asymptotically negligible.

Observe that as $s(\xi) \to -\infty$, $\psi_j''(s(\xi)) \sim p_e e^{s(\xi)}$, so $\{2\pi n \Sigma \xi_j \psi_j''(s(\xi))\}^{-1/2} \sim \{2\pi n \Sigma \xi_j (1 - \Sigma \xi_i)\}^{-1/2} \sim \{2\pi k\}^{-1/2}$, as noted earlier, so the asymptotic formulas (2.4)–(2.5) do indeed merge at the crossover point. ///

The behavior of $\beta_n(\xi)$ in the regime $\Sigma \xi_i = o(1)$ is only slightly more complicated. Recall (Prop. 1) that

$$G(z) = H_0(z) + \sqrt{R-z}\, K_0(z), \tag{2.8}$$

$$F_i(z) = H_i(z) + \sqrt{R-z}\, K_i(z) \tag{2.9}$$

where $H_i(z)$ and $K_i(z)$ are analytic in $|z| < R + \delta$, and $K_i(R) < 0 < H_i(R)$ for each $i = 0, 1, \ldots, L$.

PROPOSITION 5: *For any $0 < \varepsilon < 1$ the asymptotic behavior of $\beta_n(\xi)$ for nonnegative vectors $\xi = (\xi_1, \xi_2, \ldots, \xi_L)$ in the range $0 \le \Sigma \xi_i \le \varepsilon$ is as follows:*

$$\beta_n(\xi) \sim G(e^{s(\xi)})\{2\pi n \sum_{i=1}^{L} \xi_i \psi_i''(s(\xi))\}^{-1/2} \quad \textit{if } \Sigma \xi_i \ge n^{-1/4}, \tag{2.10}$$

$$\beta_n(\xi) \sim \{R/4\pi n^3\}^{1/2}\{-G(R) n \sum_{i=1}^{L} \xi_i (K_i(R)/F_i(R)) - K_0(R)\} \quad \textit{if } \Sigma \xi_i \le n^{-1/4}. \tag{2.11}$$

These relations hold uniformly for ξ in the range $0 \le \Sigma \xi_i \le \varepsilon$ as $n \to \infty$.

Before proving Prop. 5 we will show that it implies the local limit theorem (Th. 2) of [GW].

COROLLARY 1: *Let $x = a_{i_1} a_{i_2} \ldots a_{i_m} \in \mathcal{G}$ be fixed. Then as $n \to \infty$,*

$$p^{*n}(x) \sim b_x (2\sqrt{\pi})^{-1} \sqrt{R}\, R^{-n} n^{-3/2} \tag{2.12}$$

where

$$b_x = \left\{ \prod_{j=1}^{m} F_{ij}(R) \right\} \left\{ -G(R) \sum_{j=1}^{m} (K_{i_j}(R)/F_{i_j}(R)) - K_0(R) \right\}. \tag{2.13}$$

PROOF: By Th. 1, $p^{*n}(x) = \beta_n(\xi(n,x)) \exp\{n\varphi(\xi(n,x))\}$ where β_n satisfies (2.11). Consider first the exponential factor. The vector $\xi(n,x) \to 0$ as $n \to \infty$, because x is fixed;

furthermore, $\varphi(0) = -\log R$ and $\partial\varphi/\partial\xi_i = \psi_i(s(\xi))$ (see the proof of Prop. 3). Consequently, by Taylor's theorem,

$$n\varphi(\xi(n,x)) = -n\log R + n\sum_{i=1}^{L}\xi(n,x)_i\psi_i(s(\tilde{\xi}))$$
$$= -n\log R + \sum_{j=1}^{m}\psi_{i_j}(s(\tilde{\xi}))$$

where $\tilde{\xi}$ is a point on the line segment from 0 to $\xi(n,x)$. But $\xi(n,x) \to 0$ as $n \to \infty$, so

$$\exp\{n\varphi(\xi(n,x))\} \sim R^{-n}\prod_{j=1}^{m}F_{i_j}(R).$$

The result now follows from (2.11), because

$$n\sum_{i=1}^{L}\xi(n,x)_i(K_i(R)/F_i(R)) = \sum_{j=1}^{m}(K_{i_j}(R)/H_{i_j}(R)).$$

PROOF *of Prop. 5*: We begin again with the formula (2.6) for $p^{*n}(x)$. As in the proof of Prop. 4 we set $r = e^{s(\xi)}$ and observe that the main contribution to the integral comes from $|\theta| < \delta$, where $\delta > 0$ is small. (The errors made in discarding those $|\theta| \geq \delta$ are small uniformly in ξ, because $|F_j(re^{i\theta})|/F_j(r)$ are bounded away from 1 for $|\theta| \geq \delta$ and $r \leq R$.) For $|\theta| < \delta$ we can again use the Taylor expansion (2.7). The only difficulty in applying the Laplace method now is that the $0(n\theta^2)$ error term in the final exponential in (2.7) does not stay small compared to the main quadratic term $n\Sigma\xi_j\psi_j''(s(\xi))\theta^2/2$ when $\Sigma\xi_j \to 0$. But routine (and tedious) calculations show that the error term is uniformly small compared to $n\Sigma\xi_j\psi_j''(s(\xi))\theta^2/2$ provided $\Sigma\xi_j >> n^{-1/2}$, e.g., if $\Sigma\xi_j \geq n^{-1/4}$. Thus, the Laplace expansion (2.10) is valid uniformly in this range.

To obtain an asymptotic expansion for $p^{*n}(x)$ valid in the range $0 \leq \Sigma\xi_j \leq n^{-1/4}$ we must make a transformation in the integral (2.6). This is because as $\Sigma\xi_j \downarrow 0$ the "saddlepoint" $r = e^{s(\xi)} \uparrow R$, a singularity of the integrand, and this causes the Laplace method to break down. Equations (2.8)–(2.9) suggest the substitution $\varsigma = \{R - z\}^{1/2}$ for $Re\ z \leq R$ (the positive branch); then (2.6) becomes

$$p^{*n}(x) = (\pi i)^{-1}\oint_{\Gamma}\left\{\prod_{j=1}^{L}(H_j(R-\varsigma^2) + \varsigma K_j(R-\varsigma^2))^{m_j}\right\} \tag{2.14}$$
$$\cdot(H_0(R-\varsigma^2) + \varsigma K_0(R-\varsigma^2))(R-\varsigma^2)^{-n-1}\varsigma d\varsigma$$

where Γ is any contour that winds once around the point R and lies entirely in the domain $\{\varsigma: |R-\varsigma^2| < R+\delta\}$ (recall that $H_j(z)$ and $K_j(z)$ are analytic in $|z| < R+\delta$). The contour $\Gamma = \Gamma_\xi$ we will use consists of a vertical line segment $\{s+i\theta: |\theta| \le \alpha\}$, where $\alpha > 0$ and

$$s = -R\sum_{j=1}^{L} \xi_j K_j(R)/2H_j(R),$$

together with an arc of the circle of radius $\sqrt{R'}$ centered at $\sqrt{R}$, where $R < R' < R+\delta$ and R' does not depend on ξ.

Observe that Γ makes its closest approach to R at the point $\varsigma = s$; consequently, $|R-\varsigma^2|$ is smallest at $\varsigma = s$. Keep in mind that $\Sigma\xi_j \le n^{-1/4}$, so $\Sigma m_j = n\Sigma\xi_j \le n^{3/4}$, and so despite the fact that $|H_j(R-\varsigma^2)+\varsigma K_j(R-\varsigma^2)|$ may *not* take its maximum value on Γ at $\varsigma = s$, the major contribution to the integral comes from those $\varsigma \in \Gamma$ near s. The error made in replacing Γ by the vertical segment $\{s+i\theta: |\theta| \le \alpha\}$ is uniformly small for ξ in the range $0 \le \Sigma\xi_i \le n^{-1/4}$.

The reason for choosing $s = -R\Sigma\xi_j K_j(R)/2H_j(R)$ is that s is very near a saddlepoint for the integral (2.14). The minimum of $\prod\{H_j(R-\varsigma^2)+\varsigma K_j(R-\varsigma^2)\}^{m_j}(R-\varsigma^2)^{-n}$ for $0 \le \varsigma \le R^{1/2}$ is $\exp\{n\varphi(\xi(n,x))\}$, and is assumed at $\varsigma = s + 0(s^2)$; at this point the derivative of $\Sigma m_j \log(H_j(R-\varsigma^2)+\varsigma K_j(R-\varsigma^2)) - n\log(R-\varsigma^2)$ must be zero. Consequently,

$$\begin{aligned}&\left\{\prod_{j=1}^{L}(H_j(R-\varsigma^2)+\varsigma K_j(R-\varsigma^2))^{m_j}\right\}(R-\varsigma^2)^{-n}\\ &= \exp\{n\varphi(\xi(n,x))\}\exp\{-n((\theta^2/R)+0(s\theta^2+s^2\theta)\}.\end{aligned}$$

Hence, (2.14) may be rewritten as

$$\begin{aligned}&p^{*n}(x)\exp\{n\varphi(\xi(n,x))\}\\ &\sim \int_{\theta=-\alpha}^{\alpha} e^{-n(\theta^2/R)}\{H_0(R)+(s+i\theta)K_0(R)\}(s+i\theta)d\theta/\pi R\end{aligned}$$

uniformly for $0 \le \Sigma\xi_i \le n^{-1/4}$. The formula (2.11) now follows by routine arguments based on the substitution $u = \sqrt{2n/R}\,\theta$. ///

3. The Space–Time Martin Boundary

Recall that Λ is the set of infinite reduced words $\omega = a_{i_1}a_{i_2}\ldots$ from the alphabet $\{a_1, a_2, \ldots, a_L\}$ ("reduced" means that no letter appears more than once in succession). For $x = a_{i_1}a_{i_2}\ldots a_{i_m} \in \mathcal{G}$ and $\omega = a_{j_1}a_{j_2}\ldots \in \Lambda$, define $N(x,\omega)$ to be the maximum k such that $a_{i_\ell} = a_{j_\ell} \forall \ell = 1, 2, \ldots, k$ (note that $k = 0$ if $a_{i_1} \neq a_{j_1}$), and for $0 < \lambda \leq R$ define

$$Q_\lambda(x,\omega) = \left\{ \prod_{\ell=N(x,\omega)+1}^{m} F_{i_\ell}(\lambda) \right\} \Big/ \left\{ \prod_{\ell=1}^{N(x,\omega)} F_{i_\ell}(\lambda) \right\}. \tag{3.1}$$

Recall that the space–time process Y_n has state space $\Gamma = \{(n,x): 0 \leq |x| \leq n\}$, and that only transitions of the form $(n,x) \to (n+1,x')$ are permissible. Hence, the root $(0,e)$ is the only state from which all other states are accessible, so we will use the unit point mass at $(0,e)$ as the "reference measure" ([Dy], sec. 8) for the calculation of the Martin kernel. For (m,x), $(n,y) \in \Gamma$ define

$$K_{(n,y)}(m,x) = p_{n-m}(x,y)/p_n(e,y); \tag{3.2}$$

this is the Martin kernel (note that for $n < m$, it is zero). For each fixed $(n,y) \in \Gamma$, $K_{(n,y)}$ is a nonnegative superharmonic function. The Martin space $\mathcal{M}$ is defined to be the set of all functions K_ς which are (pointwise) limits of sequences of $K_{(n,y)}$, $(n,y) \in \Gamma$, and the Martin boundary is $\mathcal{M} \backslash \{K_{(n,y)}: (n,y) \in \Gamma\}$.

PROPOSITION 6: *Let $(n_j, y_j) \in \Gamma$ be such that $n_j \to \infty$, $y_j \to \omega \in \Lambda$, and $\xi(n_j, y_j) \to \xi = (\xi_1, \xi_2, \ldots, \xi_L)$ where $\sum_{i=1}^{L} \xi_i < 1$. Then for each $(m,x) \in \Gamma$,*

$$\lim_{j\to\infty} K_{(n_j,y_j)}(m,x) = \lambda^m Q_\lambda(x,\omega) \tag{3.3}$$

where

$$\lambda = e^{s(\xi)}.$$

PROOF: Write $\xi^{(j)} = \xi(n_j, y_j)$ and $\hat{\xi}^{(j)} = \xi(n_j - m, x^{-1}y_j)$. By Theorem 1,

$$K_{(n_j,y_j)}(m,x) = \frac{\beta_{n_j-m}(\hat{\xi}^{(j)})}{\beta_{n_j}(\xi^{(j)})} \cdot \frac{\exp\{(n_j - m)\varphi(\hat{\xi}^{(j)})\}}{\exp\{n_j\varphi(\xi^{(j)})\}}. \tag{3.4}$$

Note that $\xi^{(j)} \to \xi$ and $\hat{\xi}^{(j)} \to \xi$; consequently, if $0 < \Sigma\xi_i < 1$ then Prop. 4 (specifically, (2.4)) implies that

$$\lim_{j\to\infty} \frac{\beta_{n_j-m}(\hat{\xi}^{(j)})}{\beta_{n_j}(\xi_j)} = 1. \tag{3.5}$$

Now suppose $\Sigma\xi_i = 0$. Since $y_j \to \omega \in \Lambda$, the word length $|y_j|$ converges to ∞. But $|y_j| = n_j \sum_{i=1}^{L} \xi_i^{(j)}$, so both $n_j \sum_{i=1}^{L} \xi_i^{(j)}$ and $(n_j - m) \sum_{i=1}^{L} \hat{\xi}_i^{(j)}$ converge to ∞, and

$$\frac{\sum_{i=1}^{L} \hat{\xi}_i^{(j)}(K_i(R)/H_i(R))}{\sum_{i=1}^{L} \xi_i^{(j)}(K_i(R)/H_i(R))} \to 1.$$

Hence, Prop. 5 (both (2.10) and (2.11) are needed) implies that (3.5) is valid even when $\Sigma\xi_i = 0$.

Consider the difference between the vectors $\xi^{(j)}$ and $\hat{\xi}^{(j)}$. Recall that $n_j\xi_i^{(j)}$ and $(n_j - m)\hat{\xi}_i^{(j)}$ are the numbers of times the letter a_i appears in the reduced words y_j and $x^{-1}y_j$, respectively. Consequently, if $x = a_{i_1}a_{i_2}\ldots a_{i_m}$ then for sufficiently large j,

$$(n_j - m)\hat{\xi}_i^{(j)} - n\xi_i^{(j)} = \sum_{\ell=N(x,\omega)+1}^{m} 1\{i_\ell = i\} - \sum_{\ell=1}^{N(x,\omega)} 1\{i_\ell = i\}.$$

Recall from the proof of Prop. 3 that $(\partial\varphi/\partial\xi_i) = \psi_i(s(\xi))$; it follows that

$$\begin{aligned}
&(n_j - m)\varphi(\hat{\xi}^{(j)}) - n_j\varphi(\xi^{(j)}) \\
&\quad = -m\varphi(\xi^{(j)}) + \sum_{i=1}^{L}(n_j - m)(\hat{\xi}_i^{(j)} - \xi_i^{(j)})\psi_i(s(\xi^{(j)})) + 0(1) \\
&\quad = -m\varphi(\xi) + m\sum_{i=1}^{L} \xi_i\psi_i(s(\xi)) \\
&\qquad + \sum_{i=1}^{L} \left\{ \sum_{\ell=N(x,\omega)+1}^{m} 1\{i_\ell = i\} - \sum_{\ell=1}^{N(x,\omega)} 1\{i_\ell = i\} \right\} \psi_i(s(\xi)) \\
&\qquad + o(1) \\
&\quad = ms(\xi) + \sum_{i=1}^{L} \left\{ \sum_{\ell=N(x,\omega)+1}^{m} 1\{i_\ell = i\} - \sum_{\ell=1}^{N(x,\omega)} 1\{i_\ell = i\} \right\} \psi_i(s(\xi)) + o(1)
\end{aligned}$$

since $\varphi(\xi) = \Sigma \xi_i \psi_i(s(\xi)) - s(\xi)$. The result (3.3) now follows from (3.4), in view of (3.5). ///

PROPOSITION 7: *Let* $(n_j, y_j) \in \Gamma$ *be such that* $n_j \to \infty$, $y_j \to \omega \in \Lambda$, *and* $\xi(n_j, y_j) \to \xi = (\xi_1, \xi_2, \ldots, \xi_L)$ *where* $\sum_{i=1}^{L} \xi_i = 1$. *Then for each* $(m, x) \in \Gamma$, *with* $|x| = \nu$ *and* $x = a_{i_1} a_{i_2} \ldots a_{i_\nu}$,

$$\lim_{j\to\infty} K_{(n_j, y_j)}(m, x) = 0 \text{ unless } m = \nu = N(x, \omega); \tag{3.6}$$

$$\lim_{j\to\infty} K_{(n_j, y_j)}(m, x) = (p_{i_1} p_{i_2} \ldots p_{i_\nu})^{-1} \text{ if } m = \nu = N(x, \omega). \tag{3.7}$$

PROOF: As in the proof of Prop. 6, write $\xi^{(j)} = \xi(n_j, y_j)$ and $\hat{\xi}^{(j)} = \xi(n_j - m, x^{-1} y_j)$; also, set $k_j = n_j(1 - \sum_{i=1}^{L} \xi_i^{(j)})$ and $\hat{k}_j = (n_j - m)(1 - \sum_{i=1}^{L} \hat{\xi}_i^{(j)})$. Observe that $k_j \to \infty$ iff $\hat{k}_j \to \infty$. There will be two cases to consider: (i) k_j, $\hat{k}_j \to \infty$, and (ii) $k_j \to k$ and $\hat{k}_j \to \hat{k}$. (If one of these doesn't hold, we pass to a subsequence.)

By Prop. 4 (both (2.4) and (2.5) are needed),

$$\lim_{j\to\infty} \frac{\beta_{n_j - m}(\hat{\xi}^{(j)})}{\beta_{n_j}(\xi^{(j)})} = 1 \text{ if } k_j, \hat{k}_j \to \infty;$$

and

$$\lim_{j\to\infty} \frac{\beta_{n_j - m}(\hat{\xi}^{(j)})}{\beta_{n_j}(\xi^{(j)})} = \frac{\hat{k}^{\hat{k}} e^{-\hat{k}} k!}{k^k e^{-k} \hat{k}!} \text{ if } k_j \to k,\ \hat{k}_j \to \hat{k}.$$

By Theorem 1,

$$K_{(n_j, y_j)}(m, x) = \frac{\beta_{n_j - m}(\hat{\xi}^{(j)})}{\beta_{n_j}(\xi^{(j)})} \cdot \frac{\exp\{(n_j - m)\varphi(\hat{\xi}^{(j)})\}}{\exp\{n_j \varphi(\xi^{(j)})\}}$$

and by Taylor's theorem,

$$(n_j - m)\varphi(\hat{\xi}^{(j)}) - n_j \varphi(\xi^{(j)})$$
$$= -m\varphi(\hat{\xi}^{(j)}) + n_j \sum_{i=1}^{L} (\hat{\xi}_i^{(j)} - \xi_i^{(j)}) \psi_i(s(\tilde{\xi}^{(j)}))$$

where $\tilde{\xi}^{(j)}$ is a point on the line segment from $\xi^{(j)}$ to $\hat{\xi}^{(j)}$. Now $\hat{\xi}^{(j)} \to \xi$, and $\varphi(\xi) = \Sigma \xi_i \log p_i$, by (2.3). Moreover, $s(\tilde{\xi}^{(j)}) \to -\infty$ and so (recall that $F_i(z) = p_i z + 0(|z|^2)$ as $|z| \to 0$)

$$\psi_i(s(\tilde{\xi}^{(j)})) = s(\tilde{\xi}^{(j)}) + \log p_i + o(1).$$

Now consider the differences $\hat{\xi}^{(j)} - \xi^{(j)}$. If $\nu < m$ or if $N(x,\omega) < \nu$ then (at least for large j) $|x^{-1}y_j| > |y_j| - m$, and consequently $n_j \sum_{i=1}^{L} (\hat{\xi}_i^{(j)} - \xi_i^{(j)}) \geq 1$; it follows that in this case

$$n_j \sum_{i=1}^{L} (\hat{\xi}_i^{(j)} - \xi_i^{(j)}) \psi_i(s(\tilde{\xi}^{(j)})) \to -\infty.$$

This proves (3.6). If, on the other hand, $\nu = m = N(x,\omega)$ then (for large j) $|x^{-1}y_j| = |y_j| - m$ and so $\sum_i \hat{\xi}_i^{(j)} = \sum_i \xi_i^{(j)}$. Consequently, in this case

$$\begin{aligned}
&(n_j - m)\varphi(\hat{\xi}^{(j)}) - n_j \varphi(\xi^{(j)}) \\
&\quad = -m\varphi(\hat{\xi}^{(j)}) + n_j \sum_{i=1}^{L} (\hat{\xi}_i^{(j)} - \xi_i^{(j)})(\psi_i(s(\tilde{\xi}^{(j)})) - s(\tilde{\xi}^{(j)})) \\
&\quad = -\sum_{\ell=1}^{\nu} \log p_{i_\ell} + o(1).
\end{aligned}$$

To complete the proof of (3.7) we need only note that if $\nu = m = N(x,\omega)$ and $k_j \to k < \infty$ then $k_j = \hat{k}_j$ eventually. ///

For $x, y \in \mathcal{G}$ define

$$Q_R(x,y) = b_{x^{-1}y} / b_y$$

where $b_{x^{-1}y}$ and b_y are as in (2.13).

PROPOSITION 8: *Let* $(n_j, y) \in \Gamma$ *where* $y \in \mathcal{G}$ *and* $n_j \to \infty$. *Then for each* $(m,x) \in \Gamma$

$$\lim_{j \to \infty} K_{(n_j,y)}(m,x) = R^m Q_R(x,y).$$

PROOF: This is an immediate consequence of Cor. 1.

Propositions 6–8 identify all possible limits of the functions $K_{(n,y)}$ where $(n,y) \in \Gamma$. The limit functions are as follows:

(3.8) $$K_{(\lambda,\omega)}(m,x) \triangleq \lambda^m Q_\lambda(x,\omega) \qquad , 0 < \lambda \leq R,\ \omega \in \Lambda;$$

(3.9) $$K_{(R,y)}(m,x) \triangleq R^m Q_R(x,y) \qquad , y \in \mathcal{G};$$

and for $\omega \in \Lambda$,

(3.10) $$K_{(0,\omega)}(m,x) \triangleq \begin{cases} (p_{i_1} p_{i_2} \cdots p_{i_m})^{-1} & \text{if } m = |x| = N(x,\omega) \\ & \text{and } x = a_{i_1} a_{i_2} \ldots a_{i_m}; \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 9: *Each of the functions $K_{(\lambda,\omega)}$, $K_{(R,y)}$, $K_{(0,\omega)}$ is a nonnegative, harmonic function for the space–time process Y_n.*

PROOF: It is easily verified by direct computation that $K_{(0,\omega)}$ is harmonic. It is also possible to prove directly that $K_{(\lambda,\omega)}$ and $K_{(R,y)}$ are harmonic, but it is easier to proceed indirectly, using Props. 6–7. Let (n_j, y_j) be as in Prop. 6, with $y_j \to \omega \in \Lambda$; then

$$\begin{aligned} K_{(\lambda,\omega)}(m,x) &= \lim_{j\to\infty} K_{(n_j,y_j)}(m,x) \\ &= \lim_{j\to\infty} p^{*(n_j-m)}(x^{-1}y_j)/p^{*n_j}(y_j). \end{aligned}$$

But

$$p^{*(n_j-m)}(x^{-1}y_j) = \sum_{i=1}^{L} p_i p^{*(n_j-m-1)}(a_i x^{-1} y_j) + p_e p^{*(n_j-m-1)}(x^{-1}y_j)$$

so

$$K_{(\lambda,\omega)}(m,x) = \sum_{i=1}^{L} p_i K_{(\lambda,\omega)}(m+1, xa_i) + p_e K_{(\lambda,\omega)}(m+1,x),$$

proving that $K_{(\lambda,\omega)}$ is harmonic. A similar argument, using Prop. 7, shows that $K_{(R,y)}$ is harmonic. ///

Having identified the space–time Martin boundary, we now identify the set B_e of extreme points of the boundary (the "space of exits", in the terminology of [Dy]).

PROPOSITION 10: $B_e = \{K_{(\lambda,\omega)}: 0 \le \lambda \le R,\ \omega \in \Lambda\}$.

PROOF: First consider $K_{(R,y)}$ where $y \in \mathcal{G}$; we will show that $K_{(R,y)}$ is not an extreme point by exhibiting it as a nontrivial convex combination of nonnegative harmonic functions. By the Markov property,

$$p^{*(n-m)}(x^{-1}y) = p_e p^{*(n-1-m)}(x^{-1}y) + \sum_{i=1}^{L} p_i p^{*(n-1-m)}(x^{-1}ya_i),$$

$$p^{*n}(y) = p_e p^{*(n-1)}(y) + \sum_{i=1}^{L} p_i p^{*(n-1)}(ya_i),$$

and by Cor. 1,

$$p^{*(n-1)}(ya_i)/p^{*n}(y) \to Rb_{ya_i}/b_y,$$

$$p^{*(n-1)}(y)/p^{*n}(y) \to R,$$

hence

$$p_eR + \sum_{i=1}^{L} p_iR(b_{ya_i}/b_y) = 1.$$

Therefore, it follows from Prop. 8 and (3.2) that

$$K_{(R,y)}(m,x) = p_eRK_{(R,y)}(m,x) \\ + \sum_{i=1}^{L} p_iR(b_{ya_i}/b_y)K_{(R,ya_i)}(m,x).$$

This shows that $K_{(R,y)}$ is a convex combination of the functions $K_{(R,ya_i)}$, $i = 1, 2, \ldots, L$. Since the functions $K_{(R,y)}$, $y \in \mathcal{G}$, are distinct, it follows that $K_{(R,y)} \notin B_e$.

Thus, all elements of B_e are to be found among the functions $K_{(\lambda,\omega)}$, where $0 \leq \lambda \leq R$ and $\omega \in \Lambda$. We will show that all of these functions are extreme points. Consider first $K_{(0,\omega)}$ where $\omega \in \Lambda$; we have

$$K_{(0,\omega)}(0,e) = 1,$$

so each $K_{(0,\omega)}$ is "normalized" ([KSK], sec. 10–6). Hence, to prove that $K_{(0,\omega)} \in B_e$ it suffices to prove that $K_{(0,\omega)}$ is *minimal* harmonic, i.e., that if $h(m,x)$ is a nonnegative harmonic function satisfying $0 \leq h \leq K_{(0,\omega)}$ on Γ then there exists a constant $0 \leq c \leq 1$ such that $h = cK_{(0,\omega)}$. So suppose $0 \leq h \leq K_{(0,\omega)}$, where h is harmonic and $\omega = a_{i_1}a_{i_2}\ldots \in \Lambda$. By definition, $K_{(0,\omega)}(m,x) = 0$ unless $x = a_{i_1}a_{i_2}\ldots a_{i_m}$, in which case $K_{(0,\omega)}(m,x) = (p_{i_1}p_{i_2}\ldots p_{i_m})^{-1}$. Consequently, $h(m,x) = 0$ unless $x = a_{i_1}a_{i_2}\ldots a_{i_m}$. But h is harmonic; therefore, for each $m \geq 0$

$$h(m, a_{i_1}a_{i_2}\ldots a_{i_m}) = p_{i_{m+1}}h(m+1, a_{i_1}a_{i_2}\ldots a_{i_m}a_{i_{m+1}}).$$

It follows that h is a scalar multiple of $K_{(0,\omega)}$. This proves that for each $\omega \in \Lambda$, $K_{(0,\omega)} \in B_e$.

Finally, consider $K_{(\lambda,\omega)}$, where $0 < \lambda \leq R$ and $\omega \in \Lambda$. We will show that $K_{(\lambda,\omega)}$ is not a nontrivial convex combination of the functions $K_{(\lambda',\omega')}$, $0 \leq \lambda' \leq R$ and $\omega' \in \Lambda$. Suppose to the contrary that

$$K_{(\lambda,\omega)} = \int_{[0,R]\times\Lambda} K_{(\lambda',\omega')}d\nu(\lambda',\omega').$$

for some probability measure ν on $[0,R]\times\Lambda$. Evaluating both sides at the arguments $(0,e)$, $(1,e),\ldots$, using (3.8), (3.10), and appealing to the fact that a probability distribution on $[0,R]$ is uniquely determined by its moments, we find that

$$\nu\{(\lambda',\omega')\colon \lambda'\neq\lambda\}=0;$$

thus

$$K_{(\lambda,\omega)}=\int_\Lambda K_{(\lambda,\omega')}d\nu(\omega')$$

for some probability measure ν on Λ. It follows that

$$Q_\lambda(x,\omega)=\int_\Lambda Q_\lambda(x,\omega')d\nu(\omega')\quad \forall x\in\mathcal{G}.$$

We will now show that the only such representation of Q_λ is the trivial one, i.e., ν is the unit point mass at ω. Write $\omega=a_{i_1}a_{i_2}\ldots$; fix $m\geq 1$, and consider $x=a_{i_1}a_{i_2}\ldots a_{i_m}$. By (3.1), for any $\omega'\in\Lambda$

$$Q_\lambda(x,\omega')\leq Q_\lambda(x,\omega)$$

with equality iff $N(x,\omega')=m$ (recall that $F_i(\lambda)<1$ for each $i=1,2,\ldots,L$, by Prop. 1). Consequently

$$\nu\{\omega'\colon N(x,\omega')=m\}=1.$$

But $m\geq 1$ is arbitrary; by letting $m\to\infty$ we find that $\nu(\{\omega\})=1$.

This proves that $K_{(\lambda,\omega)}$ is not a nontrival convex combination of the functions $K_{(\lambda',\omega')}$, where $0\leq\lambda'\leq R$, $\omega'\in\Lambda$. Since these are the only candidates for extreme points, and every nonnegative harmonic function is a convex combination of extreme points ([KKS], sec. 10–5 to 10–7), it follows that $K_{(\lambda,\omega)}\in B_e$. ///

In summary, our results are as follows.

THEOREM 2: *The space–time Martin boundary is*

$$\mathcal{M}=([0,R]\times\Lambda)\cup(\{R\}\times\mathcal{G}).$$

The topology on $[0,R] \times (\Lambda \cup \mathcal{G})$ *is the product topology, with* $(\Lambda \cup \mathcal{G})$ *having the word topology (the topology of letter-by-letter convergence); the topology on* $\mathcal{M}$ *is that induced by* $[0,R] \times (\Lambda \cup \mathcal{G})$*. The space of exits for the space–time process is*

$$B_e = [0,R] \times \Lambda.$$

Thus every nonnegative, space–time harmonic function $h(m,x)$ *has a unique representation*

$$\begin{aligned} h(n,x) &= \int_{[0,R]\times\Lambda} K_{(\lambda,\omega)}(n,x)\,d\nu(\lambda,\omega) \\ &= \int_{(0,R]\times\Lambda} \lambda^n Q_\lambda(x,\omega)\,d\nu(\lambda,\omega) \\ &+ \int_\Lambda K_{(0,\omega)}(n,x)\,d\nu(0,\omega), \end{aligned}$$

where ν *is a finite, positive Borel measure on* $[0,R] \times \Lambda$*.*

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