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ABSTRACT

We consider the problem of sequentially estimating one parameter in a two- parameter exponential family of distributions. Known members of this family are the normal, gamma and the inverse Gaussian distribution. The asymptotic normality of the stopping variable is established. We also provide approximations to its mean and to the regret associated with it. The known results for the normal distribution follow as a particular case.

1. Introduction

Let

$$(1.1) \qquad f(x;\theta) = a(x)exp\{\theta_1 U_1(x) + \theta_2 U_2(x) + c(\theta)\}, \quad \theta = (\theta_1, \theta_2),$$

be a density function (w.r.t. Lebesgue measure on $\mathbf{R}$), which characterizes a *regular* two-parameter exponential family of distributions, (see Brown (1986)), i.e.; the natural parameter space Θ is defined by;

$$\Theta = \{\theta \in \mathbf{R}^2 \; ; e^{-c(\theta)} = \int a(x)exp\{\theta_1 U_1(x) + \theta_2 U_2(x)\}\, dx \; < \infty\},$$

so that $\Theta \equiv int\Theta \neq \emptyset$. It is well known that for any $\theta \in \Theta$ the r.v. $\mathbf{U} = (U_1, U_2)$ has moments of all orders. In particular, we denote;

$$(1.2) \qquad E_\theta(\mathbf{U}) = (\mu_1, \mu_2), \qquad \mu_i = -\partial c(\theta)/\partial\theta_i, \qquad i = 1,2$$

and

$$V_\theta(\mathbf{U}) = (\,\sigma_{ij}\,), \qquad \sigma_{ij} = -\partial^2 c(\theta)/\partial\theta_i\partial\theta_j \qquad i,j = 1,2,$$

where $V_\theta(\mathbf{U})$ is the corresponding (positive definite) variance-covariance matrix.

Let $X_1, \ldots, X_n$, $n > 1$, be n independent identically distributed *r.v.'s* having a common density of the form (1.1). We set $T_{i:n} = \sum_{j=1}^n U_i(X_j)$ and denote by $\bar{T}_{i:n}$, $i = 1,2$ the usual average. The joint distribution of $\mathbf{T} = (T_{1:n}, T_{2:n})$ is a member of the two-parameter exponential family, and

$$(1.3) \qquad E_\theta(\mathbf{T}) = (n\mu_1, n\mu_2), \qquad V_\theta(\mathbf{T}) = (n\sigma_{ij}) \quad i,j = 1,2\;.$$

In the present paper we consider a subfamily of (1.1), as was first introduced by Bar-Lev and Reiser (1982) (henceforth referred to as BLR), in the context of construction of UMPU tests. This subfamily is characterized by the following two assumptions:

ASSUMPTION A.1. *The parameter θ_2 can be represented as; $\theta_2 = -\theta_1\psi'(\mu_2)$, where $\psi'(\mu_2) = d\psi(\mu_2)/d\mu_2$, for some function ψ.*

ASSUMPTION A.2. *$U_2(x) = h(x)$, where $h(x)$ is a $1-1$ function on the support of (1.1).*

By using the mixed parametrization $(\theta_1, \theta_2) \rightarrow (\theta_1, \mu_2)$, which is a homeomorphism with $(\theta_1, \mu_2) \in \Theta_1 \times \mathcal{N}_2$, (varying independently, respectively), it can be shown that under Assumptions A.1- A.2 the following relations hold, (see BLR):

a) The variance of U_2 is given by:

$$\sigma_{22}(\theta) \equiv \frac{\partial \mu_2}{\partial \theta_2} = \frac{-1}{\theta_1 \psi''(\mu_2)}, (> 0), \tag{1.4}$$

b) The functions $c(\theta)$ and $\mu_1(\theta)$ when expressed by θ_1 and μ_2, have the following form:

$$\begin{cases} c(\theta_1, \mu_2) = \theta_1[\mu_2\psi'(\mu_2) - \psi(\mu_2)] - G(\theta_1) \\ \\ \mu_1 = \psi(\mu_2) + G'(\theta_1) \end{cases} \tag{1.5}$$

where $G(\theta_1)$ is an infinitely differentiable function on Θ_1 for which $G''(\theta_1) > 0$, for all $\theta_1 \in \Theta_1$. Here G' and G'' denote the first and second derivatives of G, respectively. One of the immediate consequences of (1.4) (see BLR), is that either $\Theta_1 \subset \mathbf{R}^-$ or that $\Theta_1 \subset \mathbf{R}^+$.

Suppose now, that on the basis of n independent observations $x_1, \ldots, x_n$ from (1.1), we wish to estimate $\mu_2 \equiv E_\theta(U_2)$ in the presence of the nuisance parameter θ_1. Let $\hat{\theta}_1$ and $\hat{\mu}_2$ denote the maximum likelihood estimators of θ_1 and μ_2, respectively. It is easy to show that under the above assumptions, $\hat{\mu}_2 = \bar{T}_{2:n}$ and that $\hat{\theta}_1$ satisfies the equation:

$$nG'(\hat{\theta}_1) = T_{1:n} - n\psi(\bar{T}_{2:n}) \equiv Z_n . \tag{1.6}$$

It has been shown by BLR that the distribution of the statistic Z_n is a member of the one parameter exponential family of distributions having a cumulant generating function $H_n(\theta_1) = nG(\theta_1) - G(n\theta_1)$, $\theta_1 \in \Theta_1$, $n \geq 2$.

Suppose that the loss incurred by using $\bar{T}_{2:n}$ as an estimate for μ_2 is:

$$L_\rho(\bar{T}_{2:n}) = \rho|\psi''(\mu_2)|(\bar{T}_{2:n} - \mu_2)^2 + n ,$$

where $\rho > 0$. The factor $\rho|\psi''(\mu_2)|$ represents the importance of the estimation error relative to the cost of one observation.

¿From (1.3) and (1.4) it follows that for a fixed $\theta_1 \in \Theta_1$ the corresponding risk is:

$$R_\rho(n) = E_\theta[L_\rho(\bar{T}_{2:n})] = \frac{\rho}{n|\theta_1|} + n .$$

The optimal sample size which minimizes the risk is obtain by choosing an integer adjacent to $n_0 = (\rho/|\theta_1|)^{\frac{1}{2}}$, at which $R_\rho(n_0) = 2n_0$. However, since θ_1 is an unknown nuisance parameter, this cannot be implemented. So one may use a random sample size $\widetilde{N}_\rho$, based on the following stopping rule:

$$\widetilde{N}_\rho = \inf\{n \geq m_0 ; |\hat{\theta}_1| > \rho/n^2\} \tag{1.7}$$

for some initial sample size m_0 , $(m_0 \geq 2)$.

Since the function $G'(\theta_1)$ is strictly increasing on Θ_1, it follows from (1.6) that the stopping rule (1.7) has the following forms:

(*i*) If $\Theta_1 \subset \mathbf{R}^-$ then; $\quad \widetilde{N}_\rho = \inf\{n \geq m_0 \,; Z_n < nG'(\frac{-\rho}{n^2})\}$,

(*ii*) If $\Theta_1 \subset \mathbf{R}^+$ then; $\quad \widetilde{N}_\rho = \inf\{n \geq m_0 \,; Z_n > nG'(\frac{\rho}{n^2})\}$.

In a recent paper (Bose and Boukai (1990), henceforth referred to as BB), we have investigated the first order properties of the sequential estimation procedure based on $\widetilde{N}_\rho$. Based on certain properties of Z_n, it was shown that the two cases (i)-(ii) are symmetrical, thus there is no loss of generality to assume that $\Theta_1 \subset \mathbf{R}^-$. It was also shown that $\widetilde{N}_\rho/n_0 \to 1$, w.p. 1 (as $\rho \to \infty$) and that under rather general conditions similar to Assumptions A.3 and A.4 below;

$$\lim_{\rho\to\infty} \frac{R_\rho(\widetilde{N}_\rho)}{R_\rho(n_0)} = 1 \,, \tag{1.8}$$

where $R_\rho(\widetilde{N}_\rho)$ denotes the risk associated with the stopping variable $\widetilde{N}_\rho$. Thus the suggested sequential estimation procedure is *risk efficient.* A crucial key in proving (1.8) was a new independence result presented in BB (1990). We restate it here and for details and proof, we refer the reader to BB.

<u>THEOREM 1</u>. *Under the above assumptions, for all $n \geq 2$ and $\theta \in \Theta$, the random variables $(Z_2, \ldots, Z_n)$ are jointly independent of $T_{2:n}$.*

<u>Remark 1.</u>: Clearly the event $\{\widetilde{N}_\rho = n\}$ is determined only by $(Z_{m_0}, \ldots, Z_n)$, and therefore by Theorem 1 above is independent of $\bar{T}_{2:n}$.

Let us consider now a simple modification of the stopping rule $\widetilde{N}_\rho$ in (*i*) above. This modification is intended to reduce bias incurred by underestimating n_0 using $\widetilde{N}_\rho$. Consider the stopping variable:

$$N_\rho = \inf\{n \geq m_0 \,; Z_n a_n < nG'(\frac{-\rho}{n^2})\} \,, \tag{1.9}$$

where we suppose that $a_n > 1$, $n \geq 1$, are of the form $a_n = 1+a_0/n+\delta_n$, with $\delta_n = o(1/n)$ as $n \to \infty$. Notice that since $a_n > 1$, the event $\{\widetilde{N}_\rho > k\}$ implies $\{N_\rho > k\}$, $k \geq m_0$. It can be easily verified that the first order result stated in (1.8) and the premise of Remark

1 remain valid with the modified stopping variable N_ρ. In this paper we present some asymptotic properties of the stopping variable N_ρ (as $\rho \to \infty$) and of the risk associated with it. In particular we establish the asymptotic normality of (appropriately normalized) N_ρ and obtain approximation to $E_\theta(N_\rho)$. Denoting by $\mathcal{R}(\rho, \theta_1) = R_\rho(N_\rho) - R_\rho(n_0)$, the so called regret, (i.e.; the additional risk incurred by the sequential estimation procedure based on N_ρ instead of n_0) we also show that

$$\lim_{\rho \to \infty} \mathcal{R}(\rho, \theta_1) = [4\theta_1^2 G''(\theta_1)]^{-1} ,$$

which is positive for all $\theta_1 \in \Theta_1$. These types of results are usually referred to as second order approximations.

Sequential estimation procedures similar to the one discussed here have been considered by several researchers. For the estimation of the mean of a normal population, Robbins (1959) suggested a stopping rule N, based on the successive estimates of the population variance. So that the estimate $\bar{x}_n$ of the mean and the event $\{N = n\}$ are independent for every n. This property was heavily exploited by most researcher who worked on the normal problem. Starr (1966) showed that for the normal case the first order property (1.8) holds iff $m_0 \geq 3$. Woodroofe and Starr (1969) found that the regret is bounded. Woodroofe (1977) has used second order approximations to study the regret and proved that $\mathcal{R}(\rho) \to \frac{1}{2}$, as $\rho \to \infty$ if $m_0 \geq 6$.

Extensions of this procedure to nonnormal cases are also discussed in the literature. Starr and Woodroofe (1972) deal with the negative exponential distribution and proved results on the regret. A survey of results concerning sequential estimation procedures for the negative exponential distribution, with and without a truncation parameter, can be found in Mukhopadhyay (1988). Aras (1987, 1989) provided first and second order results for the case of censored data from negative exponential distribution. Ghosh and Mukhpadhyay (1979) and Chow and Yu (1981), with a "distribution free" approach, allowed the initial sample size m_0 to be a function of ρ and to $\to \infty$ as $\rho \to \infty$. They show the risk efficiency of the estimation procedure when the distributions are unspecified and as $m_0 \to \infty$.

It should be pointed out that the normal distribution and the ("nonregular") truncated negative exponential distribution were the only cases known to have the independence property mentioned above. In this paper we utilize the result of Theorem 1 to obtain second order approximations for the regret and the mean of our stopping time in the general case discussed. These approximations are based on results from nonlinear renewal theory (developed mainly by Lai and Siegmund (1977, 1979) and Woodroofe (1977)), detialed in Woodroofe (1982).

2. Main results

Let $\theta_1 \in \Theta_1$ be fixed and Z_n be as defined in (1.6). We will assume without loss of generality that $\Theta_1 \subset \mathbf{R}^-$, so that N_ρ is as defined in (1.9) above, $G'(\theta_1) > 0$ on Θ_1 and $0 < Z_{n-1} < Z_n$ *a.s.*, (see Lemma 2.2 in BB).

As was shown by Starr (1966) and by Woodroofe (1977, 1982), the initial sample size m_0 plays a crucial role in an attempt to analyze the *risk* as well as the *regret* associated with N_ρ. It was also shown, (see Woodroofe (1977), pp. 987), that the left tail behavior of the underlying c.d.f. is also crucial in the risk's assessments. For the general case discussed here, we impose the following two natural conditions on the model at hand. The first condition is imposed on the function G'. Notice that G' determines both; the boundary for the stopping rule N_ρ, as well as the moments of Z_n. The second condition is imposed to ensure an appropriate initial sample size m_0.

<u>ASSUMPTION A.3</u>. *For some* $\gamma > 1/2$, $\sup_{x \geq 4|\theta_1|} x^\gamma G'(-x) \leq M < \infty$.

<u>ASSUMPTION A.4</u>. . *The initial sample size* m_0 *is such that* $\forall\, \theta_1 \in \Theta_1$, $E_{\theta_1}(Z_{m_0}^{-\beta}) < \infty$ *for some* $\beta > \frac{3}{(2\gamma-1)}$.

In the following two theorems we present the main results of this paper. Their proofs are based on several lemmas presented in Section 3. These theorems pertain to the asymptotic mean of N_ρ and its regret, as the cost ρ of the sampling error relative to the cost of one observation tends to infinity. It is understood that the two statements $\{\rho \to \infty\}$ and $\{n_0 \to \infty\}$ are equivalent. In the following we set $\tau^2(\theta_1) = -2\theta_1 G''(\theta_1)$, (>0).

<u>LEMMA 1</u>. *As* $n_0 \to \infty$

(2.1)
$$N_\rho^* \equiv \frac{(N_\rho - n_0)}{\sqrt{n_0}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \frac{G''(\theta_1)}{\tau^2(\theta_1)}) \ .$$

<u>Proof:</u> The proof of the Lemma is deferred to Section 3.

<u>THEOREM 2</u>. *Suppose that* G' *and* m_0 *satisfy conditions A.3 and A.4. Then*

$$E(N_\rho) = n_0 + b_0 - \frac{1}{\tau^2(\theta_1)}[\frac{G''(\theta_1)}{2} - \theta_1 G'''(\theta_1) - \tau(\theta_1)G'(\theta_1)a_0] + o(1) \ ,$$

where b_0 *is*

$$b_0 = \frac{1}{2} + \frac{G''(\theta_1)}{2\tau^2(\theta_1)} - \sum_{k=1}^{\infty} \frac{1}{k} E(\widetilde{S}_k I[\widetilde{S}_k < 0]) \ ,$$

and $\widetilde{S}_k$, $k \geq 1$, *are partial sums of i.i.d. r.vs. defined in (3.6) below.*

Proof: The assertion follows immediately from Lemmas 2 and 4 below and Theorem 4.5 in Woodroofe (1982). ■

THEOREM 3. *If Assumptions A.3, and A.4 with* $\beta > 5/(2\gamma - 1)$ *hold, then*

$$\lim_{\rho \to \infty} \mathcal{R}(\rho, \theta_1) = \frac{G''(\theta_1)}{\tau^2(\theta_1)}$$

Proof: Let $R_\rho(N_\rho)$ denote the risk associated with the stopping time N_ρ. Then

$$R_\rho(N_\rho) = \rho|\psi''(\mu_2)|E\big((\bar{T}_{2:N_\rho} - \mu_2)^2 + N_\rho\big) .$$

By using the relation $n_0^2 = \rho/|\theta_1|$, and (1.4) along with the independence result stated in Theorem 1, we obtain that $R_\rho(N_\rho) = E\big(\frac{n_0^2}{N_\rho^*} + N_\rho\big)$. Accordingly the regret $\mathcal{R}$ may be written as:

$$\mathcal{R}(\rho, \theta_1) = E\big(\frac{n_0^2}{N_\rho} + N_\rho\big) - 2n_0 = n_0 E\big[u(\frac{N_\rho}{n_0}) - u(1)\big] ,$$

where $u(x) = x + 1/x$. By using a second order Taylor series expansion of $u(x)$ about the point 1, we get that,

$$u(\frac{N_\rho}{n_0}) - u(1) = (\frac{N_\rho}{n_0} - 1)^2(\frac{1}{b})^3$$

where b is some intermediate point with $|b - 1| \leq |N_\rho/n_0 - 1|$, $b \to 1$ a.s. and $b \geq \frac{1}{2}$ on the set $\{N_\rho > n_0/2\}$. Accordingly

$$E\big[n_0(u(\frac{N_\rho}{n_0}) - u(1))I[N_\rho > n_0/2]\big] = E\big[N_\rho^{*2}(\frac{1}{b})^3 I[N_\rho > n_0/2]\big] \to \frac{G''(\theta_1)}{\tau^2(\theta_1)}$$

as $n_0 \to \infty$, by Lemma 1, Lemma 6 and Lemma 7. On the other hand, since on the set $\{N_\rho \leq n_0/2\}$ with $n_0 > 1$, $0 \leq (u(\frac{N_\rho}{n_0}) - u(1)) \leq cn_0$, for some constant $c > 0$, it follows that

$$E\big[n_0(u(\frac{N_\rho}{n_0}) - u(1))I[N_\rho \leq n_0/2]\big] \leq cn_0^2 P(N_\rho \leq n_0/2) .$$

By Lemma 3 and Remark 4, the right side of the above inequality tends to 0 as $n_0 \to \infty$, provided that $\beta > \frac{5}{(2\gamma-1)}$. This completes the proof. ■

Remark 2:

a) Known members of the exponential family discussed here are the (two-parameter) Normal, Gamma and Inverse Gaussian distributions (with θ being an interior point), (see

Examples 5.1-5.3 in BLR and the discussion in BB). In all of these cases, explicit expressions for the statistic Z_n and the function G are readily available. It can be shown that in these three cases, Assumption A.3 holds with $\gamma = 1$ and Assumption A.4 is satisfied with $m_0 > 1 + 2\beta$. So that the second order result of Theorem 3 requires an initial sample size $m_0 \geq 12$. For specific cases the initial sample size can be reduced. For instance, Woodroofe (1977) has shown that in the normal case $m_0 \geq 6$ is suffices.

b) It is interesting to observe that for the normal case and for the inverse Gaussian case $G''(\theta_1)/\tau^2(\theta_1)$ is independent of θ_1 and is equal to $\frac{1}{2}$. However, for the gamma case the limiting regret depends on θ_1.

3. Auxiliary results and proofs

Let Z_n be as defined in (1.6), $Z_n = T_{1:n} - n\psi(\bar{T}_{2:n})$, $n \geq 2$. In the following lemma we state some of the large sample properties of Z_n.

<u>LEMMA 2</u>. *For each $\theta_1 \in \Theta_1$, as $n \to \infty$,*

(i) $\bar{Z}_n \equiv Z_n/n \xrightarrow{a.s.} G'(\theta_1)$,

(ii) $\sqrt{n}(\bar{Z}_n - G'(\theta_1)) \xrightarrow{\mathcal{D}} N(0,\ G''(\theta_1))$

<u>Proof</u>: (i) follows by the strong law of large numbers and (1.5). To prove (ii) we expand $\psi(\bar{T}_{2:n})$ about μ_2, to get

$$\psi(\bar{T}_{2:n}) = \psi(\mu_2) + (\bar{T}_{2:n} - \mu_2)\psi'(\mu_2) + \xi_n/n \ ,$$

where $\xi_n = n(\bar{T}_{2:n} - \mu_2)^2\psi''(\mu_n)/2$, and μ_n is some intermediate point satisfying $|\mu_n - \mu_2| \leq |\bar{T}_{2:n} - \mu_2|$. Accordingly, Z_n can be rewritten in the form:

$$Z_n = \sum_{j=1}^{n} Y_j - \xi_n \ , \tag{3.1}$$

where we have put

$$Y_j = U_1(X_j) - \psi'(\mu_2)U_2(X_j) + (\psi'(\mu_2)\mu_2 - \psi(\mu_2)), \quad j = 1, \ldots, n \ . \tag{3.2}$$

Clearly, $Y_1, \ldots, Y_n$ are *i.i.d. r.v.'s*. Using (1.2)-(1.5), it follows that $E(Y_1) = G'(\theta_1)$ and $Var(Y_1) = G''(\theta_1)$. Next, observe that $\psi''(\mu_n) \longrightarrow \psi''(\mu_2)$ w.p. 1 as $n \longrightarrow \infty$ and that $\sqrt{n}(\bar{T}_{2:n} - \mu_2)^2 \xrightarrow{\mathcal{P}} 0$, so that $\xi_n/\sqrt{n} \xrightarrow{\mathcal{P}} 0$, as $n \longrightarrow \infty$. The proof of (ii) now follows from Slutsky's Theorem and the C.L.T.. ∎

Remark 3:

a) It can be easily shown that the marginal distribution of the *r.vs.* Y_j, $j = 1 \ldots n$, depends only on the parameter θ_1. In fact, under Assumption A.1, one can use (1.1) and (1.5) to show that the moments generating function $M_{Y_1}(t)$, of Y_1 is given by:

$$M_{Y_1}(t) = \exp\{G(t+\theta_1) - G(\theta_1)\}, \qquad t+\theta_1 \in \Theta_1 .$$

b) It can be verified that the sequence ξ_n satisfies conditions 4.1 and 4.2 of Woodroofe (1982), (see Example 4.1(ii) there) and thus is said to be *slowly changing.*

LEMMA 3. *Suppose that $G'(\cdot)$ and m_0 satisfy Assumptions A.3 and A.4. Then as $\rho \to \infty$*

a)
$$n_0 P(N_\rho \le n_0/2) \to 0$$

and

b)
$$E((\frac{n_0}{N_\rho})^2 I[N_\rho \le n_0/2]) \to 0 .$$

Proof: The proof is carried along the same lines as the proof of Lemma B in BB and is therefore omitted. ∎

Remark 4: It can be easily verified, by using the same arguments as in Lemma B of BB, that for $k \ge 1$,

$$n_0^k P(N_\rho \le n_0/2) \to 0 \quad \text{as } \rho \to \infty.$$

provided that the condition on β in A.4 is replaced with $\beta > \frac{(1+2k)}{(2\gamma-1)}$.

Since $G'(\cdot)$ is monotone increasing on Θ_1, we can rewrite (1.9) as:

$$N_\rho = \inf\{n \ge m_0 \,; n(-g(\bar{Z}_n a_n))^{\frac{1}{2}} > \rho^{\frac{1}{2}} \}, \tag{3.3}$$

with $g(u) = G'^{-1}(u)$. By using the relation $g(G'(\theta_1)) = \theta_1$ and a Taylor series expansion about $G'(\theta_1)$ we get:

$$n(-g(\bar{Z}_n a_n))^{\frac{1}{2}} = n(-\theta_1)^{\frac{1}{2}} - \frac{n(\bar{Z}_n a_n - G'(\theta_1))}{2(-\theta_1)^{\frac{1}{2}} G''(\theta_1)} + \frac{n(\bar{Z}_n a_n - G'(\theta_1))^2}{2} Q(\gamma_n), \tag{3.4}$$

where $Q(\gamma_n) = \frac{d^2[(-g(\theta)^{\frac{1}{2}}]}{d\theta^2}\big|_{\theta=\gamma_n}$ and γ_n is some intermediate point satisfying $|\gamma_n - G'(\theta_1)| \le |Z_n a_n - G'(\theta_1)|$. Using this and expression (3.1) for Z_n in (3.4), we immediately obtain

$$\frac{n(-g(\bar{Z}_n a_n))^{\frac{1}{2}}}{(-\theta_1)^{\frac{1}{2}}} \equiv \tilde{Z}_n = \tilde{S}_n + \tilde{\xi}_n, \tag{3.5}$$

where with $\xi_n = n(\bar{T}_{2:n} - \mu_2)^2\psi''(\mu_n)/2$ as in (3.1) and Y_i as in (3.2) and with $\tau(\theta_1) = -2\theta_1 G''(\theta_1)$, (> 0):

$$(3.6) \qquad \widetilde{S}_n = \sum_{i=1}^{n} \widetilde{Y}_i \ , \qquad \widetilde{Y}_i = 1 - \frac{(Y_i - G'(\theta_1))}{\tau(\theta_1)} \quad i \geq 1 \ ,$$

$$\widetilde{\xi}_n = \frac{\xi_n}{\tau(\theta_1)} - \frac{\bar{Z}_n(a_0 + n\delta_n)}{\tau(\theta_1)} + \frac{n(\bar{Z}_n a_n - G'(\theta_1))^2}{2(-\theta_1)^{\frac{1}{2}}} Q(\gamma_n) \ .$$

So that by (3.3), (3.5), (and since $n_0(-\theta_1)^{\frac{1}{2}} = \rho^{\frac{1}{2}}$);

$$(3.7) \qquad N_\rho = \inf\{n \geq m_0 \ ; \widetilde{Z}_n > n_0\} \ .$$

Clearly $\widetilde{S}_n$, $n \geq 1$, are partial sums of *i.i.d. r.vs.* with $E(\widetilde{Y}_i) = 1$ and $V(\widetilde{Y}_i) = G''(\theta_1)/\tau^2(\theta_1)$, $i \geq 1$. Also, by following Example 4.1 (ii) and Lemma 1.4 in Woodroofe (1982), it is easily seen that $\widetilde{\xi}_n$, $n \geq 1$ are slowly changing. Further, in light of Lemma 2 and the independence of $T_{2:n}$ and Z_n, one can easily observe that as $n \to \infty$: $\widetilde{\xi}_n \xrightarrow{\mathcal{D}} V$, where;

$$(3.8) \qquad V = \frac{1}{\tau^2(\theta_1)} \left[G''(\theta_1)V_1 - \frac{G''(\theta_1)}{2}V_2 - \theta_1 G'''(\theta_1)V_2 - \tau(\theta_1)G'(\theta_1)a_0\right] \ ,$$

with V_1 and V_2 being two *i.i.d.* $\chi^2_{(1)}$ random variables. In particular it follows that $\widetilde{\xi}_n/\sqrt{n} \xrightarrow{\mathcal{P}} 0$, as $n \to \infty$.

Proof of Lemma 1 Since (3.5) and (3.7) hold, and $\widetilde{\xi}_n/\sqrt{n} \xrightarrow{\mathcal{P}} 0$ and $\widetilde{\xi}_n$ are slowly changing, the result is an immediate consequence of Lemma 4.2 in Woodroofe (1982). ■

Now, let $\epsilon > 0$ and $A_n = \{|\bar{T}_{1:n} - \mu_1| < \epsilon$ and $|\bar{T}_{2:n} - \mu_2| < \epsilon\}$, and set $V_n = \widetilde{\xi}_n I[A_n]$. In the following lemma we show that conditions 4.10-4.15 in Woodroofe (1982) are satisfied by V_n and the sets A_n, $n \geq 1$. These conditions together with the result of Lemma 3 (a), are required to establish Theorem 2.

LEMMA 4. *Let A_n and V_n be as above. Then:*

$$(1) \qquad \sum_{n=1}^{\infty} P(\bigcup_{k \geq n} A'_k) < \infty \ .$$

(2) $$\sum_{n=1}^{\infty} P(V_n < -n\delta) < \infty, \quad \textit{for some } \delta,\ 0 < \delta < 1 .$$

(3) $$\max_{0 \le k \le n} |V_{n+k}|,\ n \ge 1, \quad \textit{is uniformly integrable} .$$

(4) $$V_n \xrightarrow{\mathcal{D}} V \quad \text{as } n \to \infty .$$

Proof: Since $I[A_n] \to 1$ w.p.1. and $\tilde{\xi}_n \xrightarrow{\mathcal{D}} V$, (4) follows trivially. Since ψ is twice differentiable we have that for a fixed μ_2

$$\psi(x) - \psi(\mu_2) = (x - \mu_2)\psi'(x^*)$$

for some intermediate point x^* between x and μ_2, and on $|x - \mu_2| < \epsilon$, $|\psi'(x^*)| < c(\epsilon)$ for some constant $c(\epsilon)$. Also on A_n the functions ψ and Q are bounded. Thus

$$|V_n| \le c_1 n(\bar{T}_{1:n} - \mu_1)^2 + c_2 n(\bar{T}_{2:n} - \mu_2)^2 + c_3,$$

for some constants c_i, $i = 1,..,3$ depending only on ϵ, ψ and Q. Now (2) and (3) follow as in Example 4.3 of Woodroofe (1982) and relation (2.14) there. To prove (1) note that

$$\sum_{n=1}^{\infty} P(\bigcup_{k \ge n} A'_k) \le \sum_{n=1}^{\infty} P(\max_{k \ge n} |\bar{T}_{1:k} - \mu_1| > \frac{\epsilon}{2}) + \sum_{n=1}^{\infty} P(\max_{k \ge n} |\bar{T}_{2:k} - \mu_2| > \frac{\epsilon}{2}),$$

and each of these last two sums is finite by applying the reverse submartingale inequality to the sequence $\{|\bar{T}_{i:n} - \mu_i|\ ; n \ge 1\}$, $i = 1, 2$. This completes the proof of the Lemma. ■

LEMMA 5. *Let $\epsilon > 1$. Then for all $n > n_0\epsilon$,*

$$P(N_\rho > n) \le e^{-(n-n_0)c},$$

for some constant $c > 0$ depending on ϵ and G''.

Proof: Since $P(N_\rho > n) \le P(Z_n a_n > nG'(-\rho/n^2))$, and $a_n = 1 + 1/n + \delta_n$, the proof follows exactly as that of Lemma A in BB. ■

LEMMA 6. *Let N^*_ρ be as defined in (2.1). If Assumptions A.3 and A.4 hold, then as $n_0 \to \infty$,*

$$E\big(N^{*2}_\rho I[N_\rho \le n_0/2]\big) + E\big(N^{*2}_\rho I[N_\rho \ge 2n_0]\big) \to 0.$$

Proof: Clearly, on the set $\{N_\rho \le n_0/2\}$, $N_\rho^{*2} \le cn_o$, a.s. for some constant c. Therefore,

$$E(N_\rho^{*2} I[N_\rho \le n_0/2]) \le cn_0 P(N_\rho \le n_0/2) \to 0,$$

as $n_0 \to \infty$, by Lemma 3. Also, since $N_\rho^{*2} \le (N_\rho{}^2/n_0 + n_0)$, we have;

$$E(N_\rho^{*2} I[N_\rho \ge 2n_0]) \le \frac{1}{n_0} E(N_\rho{}^2 I[N_\rho \ge 2n_0]) + n_0 P(N_\rho \ge 2n_0).$$

By using Lemma 5, it follows that

$$E(N_\rho^{*2} I[N_\rho \ge 2n_0]) \;\le\; \frac{2}{n_0} \sum_{k=2n_0}^{\infty} ke^{-(k-2n_0)c} + n_0 e^{-n_0 c} \;,$$

which approaches 0 as $n_0 \to \infty$.

LEMMA 7. *If Assumptions A.3 and A.4 hold, then* $N_\rho^{*2} I[n_0/2 < N_\rho \le 2n_0]$, $n_0 \ge 1$, *are uniformly integrable and*

$$\lim_{n_0 \to \infty} E(N_\rho^{*2}) = \frac{G''(\theta_1)}{\tau^2(\theta_1)} \equiv [4\theta_1^2 G''(\theta_1)]^{-1} \;.$$

Proof: The second assertion is an immediate consequence of the first assertion and Lemma 1 and Lemma 6. As for the first assertion, it suffices to show that there exists a function $A(x)$ such that $xA(x)$ is integrable (w.r. to Lebesgue measure on $(0,\infty)$), and

$$P\big(n_0/2 < N_\rho \le 2n_0 \;, |N_\rho^*| > x\big) \le A(x) \;,$$

for all x and n_0, (sufficiently large).
If $x \ge \sqrt{n_0}/2$, then clearly $P\big(n_0/2 < N_\rho \le 2n_0 \;, N_\rho^* < -x\big) = 0$. If $0 \le x < \sqrt{n}/2$, then $N_\rho > n_0/2$ and $N_\rho^* < -x$ imply that $n_0/2 < N_\rho \le n_0 - \sqrt{n_0}x$.
Define $I_{n_0:x} = \{k :\; n_0/2 < k \le n_0 - \sqrt{n_0}x\}$. Thus,

$$(3.9) \qquad P\big(N_\rho > \frac{n_0}{2},\; N_\rho^* < -x\big) \le P\big(Z_k < kG'(\tfrac{-\rho}{k^2}),\ \text{for some } k \in I_{n_0:x}\ \big),$$

(since $a_k > 1$). By using expression (1.5) for $G'(\theta_1)$ and (1.6) for Z_k, and since G'' is positive and continuous, it follows that the event $Z_k < kG'(\frac{-\rho}{k^2})$ implies that

$$T_{1:k} - k\mu_1 - k(\psi(\bar{T}_{2:k}) - \psi(\mu_2)) \le k\big(G'(\frac{-\rho}{k^2}) - G'(\theta_1)\big) \le -|\theta_1| c\sqrt{n_0}x \;,$$

for sufficiently large x and n_0. Thus the right side of (3.9) is bounded by

$$\begin{aligned} & P\big[T_{1:k} - k\mu_1 \le -\sqrt{n_0}xc_1, \text{ for some } k \in I_{n_0:x}\big] \\ & + P\big[-k(\psi(\bar{T}_{2:k}) - \psi(\mu_2)) \le -\sqrt{n_0}xc_2, \text{ for some } k \in I_{n_0:x}\big], \\ & \equiv I_1 + I_2 , \end{aligned}$$

for some constants c_1 and c_2. Now,

$$I_1 \le P\big[\max_{n_0/2<k\le n_0} |T_{1:k} - k\mu_1| > \sqrt{n_0}xc_1\ \big] \le Cx^{-4} ,$$

for some constant C, by the submartingale inequality. As for the second term I_2, note that since ψ is twice differentiable, for any $\delta > 0$ there exsits a constant $\beta(\delta)$ such that $|x - \mu_2| < \delta$ implies $|\psi(x) - \psi(\mu_2)| < \beta(\delta)|x - \mu_2|$. Let

$$A = \{|\bar{T}_{2:k} - \mu_2| \le \beta\frac{\sqrt{n_0}x}{k}, \text{ for all } k \in I_{n_0:x}\ \}.$$

Notice that $\sqrt{n_0}x/k \le 1$ for $k \in I_{n_0:x}$ and by choosing β small, on the set A, for all $k \in I_{n_0:x}$

$$|\psi(\bar{T}_{2:k}) - \psi(\mu_2)| \le \beta(1)\beta\frac{\sqrt{n_0}x}{k} \le \frac{\sqrt{n_0}xc_2}{k} ,$$

Thus again;

$$I_2 \le P\big[\max_{n_0/2<k\le n_0} |T_{2:k} - k\mu_2| > \sqrt{n_0}x\beta\ \big] \le Cx^{-4} ,$$

for some constant C, by using the submartingale inequality. The same bound can be obtain for $P\big(n_0/2 < N_\rho \le 2n_0\ , N_\rho^* > x\big)$, $x > 0$, by similar arguments. This completes the proof of the Lemma. ■

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