CHARACTERIZING EXPONENTIAL DISTRIBUTIONS VIA CONDITIONAL INDEPENDENCE

by

TaChen Liang
Wayne State University

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Purdue University

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TaChen Liang
Department of Mathematics
Wayne State University
Detroit, MI 48202

Abstract

Let $Y_1, \ldots, Y_n$ be n mutually independent, nonnegative random variables such that for each $i = 1, \ldots, n, Y_i$ has an absolutely continuous distribution function $F(x;\theta_i) = F(\frac{x}{\theta_i})$, where $\theta_i > 0$, and $F(\cdot)$ has support $[0, \infty)$. We show that given $Y_i - Y_{i+1} > 0$ for all $i = 1, \ldots, n-1$, the necessary and sufficient condition for the random variables $Y_1 - Y_2, \ldots, Y_{n-1} - Y_n$ and Y_n to be conditionally mutually independent is that for each $i = 1, \ldots, n, Y_i$ has an exponential distribution.

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1. Introduction

The problem of characterizing the exponential distributions has been studied in the literature for a long time. For most recent references, for examples, see Ahsanullah (1984), Gather (1989) and Too and Lin (1989) among the many papers in this research area.

In this paper, we consider the following problem. Let $Y_1, \ldots, Y_n$ be n mutually independent, nonnegative random variables such that for each $i = 1, \ldots, n, Y_i$ has an absolutely continuous distribution function $F(x;\theta_i) = F(\frac{x}{\theta_i})$, where $\theta_i > 0$, and $F(\cdot)$ has support $[0, \infty)$. Let A denote the event that $Y_i - Y_{i+1} > 0$ for all $i = 1, \ldots, n-1$. Given A, we seek a necessary and sufficient condition for $Y_1 - Y_2, \ldots, Y_{n-1} - Y_n$ and Y_n to be conditionally mutually independent. We find that the N & S condition is: for each $i = 1, \ldots, n, Y_i$ has an exponential distribution. The main result is given in the next section.

2. The main result

Let $Y_1, \ldots, Y_n$ denote n mutually independent, nonnegative random variables such that Y_i has absolutely continuous distribution function $F_i(x)$ and density function $f_i(x)$, where $f_i(x) > 0$ for all $x > 0$. Let A be the event that $Y_i - Y_{i+1} > 0$ for all $i = 1, \ldots, n-1$. First, we have the following lemma.

Lemma 1. Conditional on the event A, $Y_1 - Y_2$ and Y_2 are independent iff Y_1 has an exponential distribution.

Proof: (Sufficiency) The conditional independence implies that for fixed $a \geq 0$ and $b \geq 0$,

$$P\{Y_1 - Y_2 > a,\ Y_2 > b|A\} = P\{Y_1 - Y_2 > a|A\}P\{Y_2 > b|A\}. \tag{1}$$

After some computations, from (1), we can obtain:

$$\begin{aligned}
&\frac{P\{Y_1 - Y_2 > 0,\ Y_2 > b,\ Y_2 - Y_3 > 0, \ldots, Y_{n-1} - Y_n > 0\}}{P\{Y_1 - Y_2 > 0,\ Y_2 - Y_3 > 0, \ldots, Y_{n-1} - Y_n > 0\}} \\
=&\frac{P\{Y_1 - Y_2 > a,\ Y_2 > b,\ Y_2 - Y_3 > 0, \ldots, Y_{n-1} - Y_n > 0\}}{P\{Y_1 - Y_2 > a,\ Y_2 - Y_3 > 0, \ldots, Y_{n-1} - Y_n > 0\}} \\
=&\frac{\int_{y=b}^{\infty} P\{Y_1 > y + a,\ Y_3 < y,\ Y_3 - Y_4 > 0, \ldots, Y_{n-1} - Y_n > 0\} f_2(y)dy}{\int_{y=0}^{\infty} P\{Y_1 > y + a,\ Y_3 < y,\ Y_3 - Y_4 > 0, \ldots, Y_{n-1} - Y_n > 0\} f_2(y)dy} \\
=&\frac{\int_{y=b}^{\infty}[1 - F_1(y+a)]G(y)f_2(y)dy}{\int_{y=0}^{\infty}[1 - F_1(y+a)]G(y)f_2(y)dy} \equiv H_b(a) \text{ (say)},
\end{aligned} \tag{2}$$

where $G(y) = P\{Y_3 < y,\ Y_3 - Y_4 > 0, \ldots, Y_{n-1} - Y_n > 0\}$, and the last equality is obtained due to the assumption that $Y_1, \ldots, Y_n$ are mutually independent. Note that for each fixed $b \geq 0$, $H_b(a)$ is a constant function of the variable a. Thus, $\frac{\partial H_b(a)}{\partial a} = 0$, which yields

$$\begin{aligned}
M_a(b) \equiv &\int_{y=b}^{\infty} f_1(y+a)G(y)f_2(y)dy \times \int_{y=0}^{\infty}[1 - F_1(y+a)]G(y)dy \\
&- \int_{y=b}^{\infty}[1 - F_1(y+a)]G(y)f_2(y)dy \times \int_{y=0}^{\infty} f_1(y+a)G(y)f_2(y)dy \\
=&0.
\end{aligned}$$

Letting $a = 0$ and differentiating $M_0(b)$ with respect to the variable b yields the following:

$$\frac{f_1(b)}{1 - F_1(b)} = \frac{\int_{y=0}^{\infty} f_1(y)G(y)f_2(y)dy}{\int_{y=0}^{\infty}[1 - F_1(y)]G(y)f_2(y)dy} \tag{3}$$

which is a positive constant since by the assumption, F_i has support $[0, \infty)$ for each $i = 1, \ldots, n$.

Since b can be any positive number, thus (3) implies that $F_1(\cdot)$ is an exponential distribution with mean $\int_{y=0}^{\infty}[1 - F_1(y)]G(y)f_2(y)dy / \int_{y=0}^{\infty} f_1(y)G(y)f_2(y)dy$.

(Necessity) Since F_1 is an exponential distribution, we can see the right–hand–side of (2) is independent of the value a and hence $H_b(a)$ is a constant function of the variable a, which implies (1). Thus, given A, $Y_1 - Y_2$ and Y_2 are conditionally independent.

Remark: In Lemma 1, we only require $F_i(\cdot)$ has support $[0, \infty)$ for each $i = 1, \ldots, n$. It is not necessary that $F_i, i = 1, \ldots, n$, belong to the same class of scale family distributions.

Theorem 1. Suppose that $F_i, i = 1, \ldots, n$, belong to the same class of scale family distributions, that is, $F_i(x) = F(x; \theta_i) = F(\frac{x}{\theta_i})$, $\theta_i > 0$, where the distribution function $F(\cdot)$ has support $[0, \infty)$. Then, conditional on the event A, a necessary and sufficient condition for $Y_1 - Y_2$, $Y_2 - Y_3, \ldots, Y_{n-1} - Y_n$ and Y_n to be mutually independent is that for each $i = 1, \ldots, n, Y_i$ has an exponential distribution.

Proof: (Sufficiency) Given the event A, the conditionally mutual independence among $Y_1 - Y_2, \ldots, Y_{n-1} - Y_n$ and Y_n implies that given A, $Y_1 - Y_2$ and Y_2 are conditionally independent. Then, by Lemma 1, Y_1 has an exponential distribution. By the assumption, Y_i has distribution F_i belonging to the same class of scale family distributions for each $i = 1, \ldots, n$, which implies that Y_i has an exponential distribution for each $i = 2, \ldots, n$ since F_1 is an exponential distribution function.

(Necessity) See Lemma 2.1 (ii) of Sackrowitz and Samuel–Cahn (1984).

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