Intrinsic Ultracontractivity and the Dirichlet Laplacian

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ABSTRACT

We construct a class of domains for which necessary and sufficient conditions can be found for the semigroup associated with the Dirichlet Laplacian to be intrinsically ultracontractive. Some of the domains for which intrinsic ultracontractivity hold have infinite area. Our methods are probabilistic.

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1. Introduction Let D be a domain in $\mathbf{R}^n, n \geq 1$, and let $p_t^D(x,y) = p_t(x,y), t > 0, x, y \in D$, be the heat kernel for D, that is, the fundamental solution of $\frac{1}{2}\Delta - \frac{\partial}{\partial t} = 0$ with Dirichlet boundary values. Following the fundamental paper [DS] of Davies and Simon we define the semigroup associated with $p_t(x,y)$ to be intrinsically ultracontractive if there is a positive eigenfunction for the Dirichlet Laplacian in D and there are positive constant c_t and C_t such that

$$(1.1) \qquad c_t\phi(x)\phi(y) < p_t(x,y) < C_t\phi(x)\phi(y), t > 0, x, y \in D.$$

An eigenfunction free characterization of intrinsic ultracontractivity is the existence, for each $t > 0$, of a positive constant K_t such that

$$(1.2) \qquad p_t(x,y)/\int p_t(x,y)dy \leq K_t p_t(z,y)/\int p_t(z,y)dy, x, y, z \in D, t > 0.$$

Of course it is immediate that (1.1) implies (1.2), and it is easy to show, using essentially Perron-Frobenius theory, that (1.2) implies the existence of a positive eigenfunction ϕ, and that (1.1) holds. This is done at the beginning of Section 2.

For brevity, if the semigroup connected to $p_t(x,y)$ is intrinsically ultracontractive, we will call the domain *iuc*. For a positive sequence $\mathbf{a} = \{a_i, i \geq 1\}$ of real numbers we define the domain $\Omega_{\mathbf{a}}$ as follows. Put $\lambda_n = \sum_{i=1}^{n} 2^{-i-1}, n \geq 1$. Let $f_{\mathbf{a}}(x) = -a_n, \lambda_n < x < \lambda_{n+1}, f_{\mathbf{a}}(x) = 0, x \notin \bigcup_{n=1}^{\infty}(\lambda_n, \lambda_{n+1})$. Then $\Omega_{\mathbf{a}} = \{(x,y) : 0 < x < 1, f(x) < y < 1\}$. We prove the following theorem.

Theorem 1. *$\Omega_{\mathbf{a}}$ is* iuc *if and only if* $\lim_{n\to\infty} a_n 2^{-n} = 0$.

This theorem provides examples of *iuc* domains of infinite area, answering a question on p. 375 of [DS]. In addition it provides, in an admittedly specialized setting, a description of the "edge" of *iuc*. It was in fact such an edge type theorem for a very different class of domains which lead to this infinite area question of [DS]. A conjecture of a necessary and sufficient condition, with a sketch of the proof of necessity, for a planar domain "above the graph of a function" to be *iuc* is given at the end of Section 3.

There are both analytic and probabilistic motivations for the study of intrinsic ultracontractivity. We refer the reader to [DS] for the analytic implications connected with the name itself. In addition, [DS] says "Intrinsic ultracontractivity especially interesting since it implies ϕ_n/ϕ_0 is bounded for any L^2 eigenfunction ϕ_n, and so these results are a contribution to the large literature on the decay of eigenfunctions."

The density of standard n dimensional Brownian motion started at x and killed when it leaves D, conditioned on this killing taking place after time t, is $p_t(x,y)/\int_D p_t(x,y)dy$, and so by virtue of (1.2) we see that intrinsic ultracontractivity is a very strong mixing condition for this motion. A more subtle probabilistic connection occurs in the study of the lifetimes of h-processes in D, a subject which has recently been studied by both analysts and probabilists. If D is *iuc* each h process in D not only has finite lifetime but also this lifetime satisfies the stronger condition of having exponential tails, the exponential constant being the eigenvalue corresponding to ϕ. Deblassie ([De 1], [De 2]) was the first to make the connection between *iuc* type estimates on the heat kernel and tails of the lifetime of h processes. See also [KP]. We remark that intrinsic ultracontractivity is strictly stronger than either of these lifetime conditions. See [BD]. Recently, J. Xu [X] has constructed a domain of infinite area in which each h process has finite expected lifetime. His domain is in fact not *iuc*.

We also prove the following result. Our proof is an adaption of part of the argument used to prove *iuc* in Theorem 1.

Theorem 2. *Let f be a real valued uppersemicontinuous function on $(0,1)$ such that $-M < f < 0$ for a positive constant M. The domain $D_f = \{(x,y) : f(x) < y < 1\}$ is iuc.*

Davies and Simon in [DS] show that bounded domains with boundaries satisfying certain inner and outer cone conditions (including Lipschitz domains) are *iuc*, but their results are not applicable due to the lack of regularity of the bottom boundaries of our domains. Also the proof of [KP] that Lipschitz domains are *iuc* does not extend to prove Theorem 2. We also prove an analog of Theorem 2 in higher dimensions. See Theorem 4.3. We remark that R. Bass and K. Burdzy have independently, a few months before this work, proved that the expected lifetimes of h processes in the domains treated in Theorem 4.3, as well as in so called Holder domains, are finite.

This paper is organized as follows. In the next section we set up the general framework. Some of this section easily generalizes to more general semigroups than the one considered here, but our results are stated only for the heat kernel. Theorem 1 is proved in Section 3. This paper was inspired by [DS], but may be read independently of it. Our methods are probabilistic.

2. Notation and Preliminaries. Points in $\mathbf{R}^n$ are written $x = (x_1, \ldots, x_n)$, and in two dimensions we sometimes use complex notation, writing $x = (Re x,\ Im x)$. If $g(t), t \geq 0$, is an $\mathbf{R}^n$ valued function and if $A \subset \mathbf{R}^n$, we put $\tau_A = \tau_A(g) = \inf\{t > 0 : g(t) \in \partial A\}$. Usually τ_A is a first exit time, sometimes a first hitting time. We use D to stand for a generic domain and we designate by $X_t^D = X_t = (X_t^1, \ldots, X_t^n)$, standard n dimensional Brownian motion *killed at* τ_D. Thus $p_t(x,y)$ is the transition density for X_t. We use $P_x, E_x (P_\nu, E_\nu)$ to denote probability and expectation associated with X started at $x \in \mathbf{R}^n$, (with initial distribution ν). We let $p_t(x,y;F)$, where F is some event, denote the density of $X_t; F$ under P_x, i.e., if $A \subset \mathbf{R}^n$ is Borel, $\int_A p_t(x,y;F)dy = P_x(\{X_t \in A\} \cap F)$. We also put

$p_t(x, A) = P_x(X_t \in A)$, if $A \subset \mathbf{R}^n$ is Borel. The operators $T_t, t \geq 0$, are the semigroup associated with $p_t(x,y)$, so that $T_t f(x) = \int_D p_t(x,y) f(y) dy, f \in L^1(D)$.

Constants c, C, etc change and may depend on the domain and any fixed points x_0 or fixed compact sets, but unless otherwise specified do not depend on the variable points x, y, z, w. Constants which depend on t are written c_t, C_t. We here note that all such constants in this paper may be and are assumed to be bounded above and bounded away from 0 on compact t subintervals of $(0, \infty)$. This always follows easily from the arguments which demonstrate the existence of such constants, and will not be mentioned further. What happens outsde D does not concern us, so $g > 0$ will stand for $g(x) > 0, x \in D$, for example. If g is a positive integrable function on D we write $\hat{g} = g / \int_D g$, and call $\hat{g}$ the normalization of g. The complement of a set A is denoted A^c, and $\theta(t)$ is the usual shift operator (see [Du]).

Lemma 2.1. *If (1.2) holds, there is* $0 < \phi \in L^1(D)$ *and* $\lambda > 0$ *such that* $\int \phi = 1$ *and*

$$T_t \phi = e^{-\lambda t} \phi, t > 0.$$

Proof. Consider T_t as a map of $L^1(D)$ to $L^1(D)$. Let x_0 be a distinguished point of D, and put $h_t(y) = p_t(x_0, y) / \int p_t(x_0, y) dy$. Then (1.2) implies (as in, e.g., the proof of Lemma 3.3 of [Da])

$$K_t^{-1} h_t(y) < \widehat{T_t f} < K_t h_t(y), f > 0, f \in L^1(D). \tag{2.2}$$

In addition (2.2) and the observation that $T_s = T_\epsilon \circ T_{s-\epsilon}$, gives easily that $\{\widehat{T_t f} : f > 0, \int f = 1\}$ is equicontinuous on compact subsets of D, and, together with (2.2), this shows that T_t is a compact operator. It is immediate that T_t satisfies all the other conditions of Jentzch's Theorem (see [S], Theorem 6.6, page 337), and so by this theorem we know there is $\Psi_t > 0$ such that $T_t\Psi_t = c_t\Psi_t$. Furthermore Ψ_t is unique up to normalization. For Jentzch's Theorem, as stated in [S], says that there is a unique positive eigenfunction corresponding to the largest eigenvalue of T_t. Now (1.2) guarantees

$$c_{t,\nu,\eta} < P_\nu(\tau_D > s)/P_\eta(\tau_D > s) < C_{t,\nu,\eta}, s \geq t,$$

where ν and η are any probability distributions on D, and especially, any two positive integrable eigenfunctions must decay at the same rate and thus have the same eigenvalues.

Since $\Psi_{2^{-n}}$ is an eigenfunction for $T_{2^{-k}}$ for any $k < n$, it follows that $\hat{\Psi}_{2^{-n}} = \hat{\Psi}_{2^{-m}}, 0 < m < n$, and we denote both here and later, these common normalizations by ϕ. It is routine to show ϕ is an eigenfunction for T_r for each dyadic rational r, and continuity properties of our semigroup imply ϕ is an eigenfunction of T_t for each t, yielding (1.1). □

Proposition 2.2. *If (1.2) holds there is a positive eigenfunction ϕ such that (1.1) holds. Furthermore, ϕ is the unique integrable positive eigenfunction.*

Proof. The only statement above not verified in the proof of the last lemma is that (1.1) holds. Using (2.2) both with $f = \phi$ and for a sequence of functions approaching point mass at x in an appropriate sense, we have $K_t^{-2} \leq (p_t(x,y)/\int p_t(x,y)dy)/\phi(y) \leq K_t^2, x, y \in \Omega$, implying $p_t(x,y)/p_t(x,y_0) \sim \phi(y)/\phi(y_0)$, where the $\sim$ indicates the ratios of the two sides are bounded above and below by positive constants depending only on t. By symmetry $p_t(z_0,y)/p_t(z,y) \sim \phi(z_0)/\phi(z)$. Now let x_0 be a distinguished point of D. We have $p_t(x,y)/p_t(x_0,x_0) = (p_t(x_0,y)/p_t(x_0,x_0))(p_t(x,y)/p_t(x_0,y)) \sim (\phi(y)/\phi(x_0))(\phi(x)/\phi(x_0))$, yielding $p_t(x,y)/\phi(x)\phi(y) \sim p_t(x_0,x_0)/\phi(x_0)\phi(x_0)$. Since this last quantity is a constant, (1.1) is proved. □

An alternative approach to the preceeding proposition and lemma, which uses heavier machinery, is to use the L^1 compactness together with Theorem 1.6.4 of [Dae] and then to work with all the eigenfunctions. This was pointed out to us by R. Bañuelos.

Lemma 2.3. A) *If there is a compact $F \subset D$ such that*

(2.3) $$p_t(y, F) \geq c_t p_t(y, D), y \in D, t > 0,$$

then

(2.4) $$p_t(y, z) \leq c_t p_t(y, w), y \in D, z, w \in F.$$

B) *If (2.3) holds and if furthermore*

(2.5) *for each $z \in D$, there is $w = w(z) \in F$ such that $p_t(w, y) \geq c_t p_t(z, y), y \in D$,*

then (1.2) holds.

Proof. A) The positivity and continuity of $p_t(y, z)$ imply

(2.6) $$\inf_{y,z \in F} p_t(y, z) = c_t > 0.$$

Using respectively the boundedness of $p_{t/2}(u, v)$ in u, v, for t fixed, (2.3), and (2.6), together with the aforementioned boundedness, we have

$$\begin{aligned} p_t(y, z) &= \int p_{t/2}(y, w) p_{t/2}(w, z) dw \\ &< c_t p_{t/2}(y, D) \\ &< c_t p_{t/2}(y, F) \\ &< c_t p_t(y, w), y \in D, z, w \in F. \end{aligned}$$

B) Next we show that (2.3) and (2.5) imply (1.2). Let x_0 be a distinguished point of F. Now

$$\begin{aligned} p_t(x, y) &= \int_D p_{t/2}(x, z) p_{t/2}(z, y) dz \\ &= \int_F p_{t/2}(x, z) p_{t/2}(z, y) dz + \int_{F^c} p_{t/2}(x, z) p_{t/2}(z, y) dz \\ &= I + II, \end{aligned}$$

say. We have

$$I \geq c_t \int_F p_{t/2}(x,z)p_{t/2}(x_0,y)dz = c_t p_{t/2}(x,F)p_{t/2}(x_0,y) \geq c_t p_{t/2}(x,D)p_{t/2}(x_0,y),$$

using (2.4) and (2.3).

Furthermore, using (2.5) and (2.4), the latter holding by part A) of this lemma, we have

$$\begin{aligned} I + II &\leq c_t \int_D p_{t/2}(x,z)p_{t/2}(w(z),y)dz \\ &\leq c_t \int_D p_{t/2}(x,z)p_{t/2}(x_0,y)dz \\ &= c_t p_{t/2}(x,D)p_{t/2}(x_0,y). \end{aligned}$$

Thus $\gamma_x c_t h(y) < p_t(x,y) < \gamma_x C_t h(y)$, where $\gamma_x = p_{t/2}(x,D)$ and where $h(y) = p_{t/2}(x_0,y)$ does not depend on x. This easily yields (1.2). □

Lemma 2.4. *There is a constant k depending on s, t, and the distance of x to ∂D such that*

$$p_t(x,y) > kp_s(x,y), s < t, y \in D. \tag{2.7}$$

Proof. Let B be the closed ball of radius half the distance from x to ∂D. For $y \in B$, (2.7) holds since $p_r(x,y)$ is bounded above and below for $y \in B$. Let $T = \inf\{t > 0 : X_t \in \partial B\}$, and let f be the density of T. The density of T is completely known (see [CT]). We use only the facts that f is continuous, positive, and bounded above on $(0,\infty)$, and we also use that given T, X_T under P_x has the uniform distribution on ∂B, which we denote η. If $y \notin B$,

$$\begin{aligned} p_t(x,y) &= \int_0^t \left(\int_{\partial B} p_{t-a}(z,y)d\eta(z)\right) f(a)da \\ &> \int_{t-s}^t \left(\int_{\partial B} p_{t-a}(z,y)d\eta(z)\right) f(a)da \end{aligned}$$

$$> k \int_0^s \left(\int_{\partial B} p_{s-w}(z,y) d\eta(z) \right) f(w) dw$$
$$= kp_s(x,y). \qquad \square$$

Lemma 2.5. *Let Δ be a Borel subset of D, and put $\tau_\Delta = T$. Let $\nu_{x,t}(T) = \nu_{x,t}$ be the distribution of $(T, X_T); T \le t$ under P_x. If*

$$\nu_{z,t} \le c_t \nu_{w,t}, \text{ then } p_t(z,\cdot; T \le t) \le c_t p_t(w,\cdot; T \le t).$$

Proof. This lemma is a simple consequence of the strong Markov property, which gives

$$p_t(z,y; T \le t) = \int_{[0,t]\times\Delta} p_{t-s}(w,y) d\nu_{z,t}(s,w). \qquad \square$$

We remark that variants of Lemmas 2.4 and 2.5 have been used to compare densities for many years.

Next we turn to Girsanov's theorem. A special case of this theorem is a central tool for proving that (2.3) and (2.5) hold for the domains we study later. Let μ be a Wiener measure on $C[0,t]$, that is, μ is the distribution of a standard one-dimensional Brownian motion on $[0,t]$, with perhaps a random initial position. Let $f(s)$ be a continuously differentiable function on $[0,t], f(0) = 0$. Let ν be the measure induced on $C[0,t]$ by the map $g \to g - f$, that is, if A is a Borel subset of $C[0,t], \nu(A - f) = \mu(A)$. Then Girsanov's theorem (see [Du]) says

$$\frac{d\nu}{d\mu} = \exp\left[\int_0^t f'(s) dB_s + \tfrac{1}{2} \int_0^t f'(s)^2 ds \right],$$

where $B_s(h) = h(s), 0 \le s \le t, h \in C[0,t]$, is the Brownian motion associated with μ. Especially if F is a Borel set of $C[0,t]$ such that there is a constant M for which $|g(t) - g(0)| < M, g \in F$, and if k is a constant, we get with $f(s) = ks$ above,

$$\frac{d\nu}{d\mu} = \exp\left[\int_0^t k dB_s + \tfrac{1}{2} tk^2 \right] = \exp\left[k(B_t - B_0) + \tfrac{1}{2} tk^2 \right]$$
$$\in \left(\exp\left[-Mk + \tfrac{1}{2} tk^2 \right], \exp\left[Mk + \tfrac{1}{2} tk^2 \right] \right) \text{ on } F,$$

so that

$$\mu(F+ks)=\nu(F)>\mu(F)\exp(-Mk+\tfrac{1}{2}tk^2). \tag{2.8}$$

More generally, if η is the distribution (on the Borels of $C_2[0,t]$, the continuous functions from $[0,t]$ to $\mathbf{R}^2$) of a standard 2-dimensional Brownian motion and if H is any Borel subset of $C_2[0,t]$, such that $g=(g_1,g_2)\in H$ implies $|g_2(0)-g_2(t)|<M$, then if $\gamma_{k,t}$ is the function $(0,ks), 0\le s\le t$, we have

$$\eta(H+\gamma_{k,t})>\eta(H)\exp(-Mk+\tfrac{1}{2}tk^2). \tag{2.9}$$

This follows from (2.8) and the independence of the components of 2-dimensional Brownian motion. Now let $r>0$ and let $R_{(r)}=R$ be the rectangle $(0,r)\times(0,1)$. Let $p_t(x,y)=p_t^R(x,y)$.

Lemma 2.6. *There exists a constant c, such that for all numbers $r>1$,*

i) $p_{r^2}(x,(a,a+1)\times(0,1))>cp_{2r^2}(x,R), 0<a<r-1, x\in R,$

and

ii) $p_{r^2}(y,z)>cp_{2r^2}(x,z), x\in R, y\in(1,r-1)\times(\frac{1}{4},\frac{3}{4}).$

Proof. Let $p_t^1(u,v)$ be the transition density of killed one-dimensional Brownian motion in $(0,1)$, let $\varphi^1(u)=(\pi/2)\sin\pi u$ be the corresponding eigenfunction, and denote its eigenvalue by $-\theta$. Let $p_t^r(u,v)$ be the transition density of killed one-dimensional Brownian motion in $(0,r)$ and $\varphi^r(u)=(\pi/2r)\sin(\pi u/r)$ be the corresponding eigenfunction, which has eigenvalue $-\theta/r^2$. Now $p_t(x,y)=p_t^r(Rex,\ Rey)p_t^1(Imx,\ Imy)$, and $\varphi^R(u)=\varphi^r(Reu)\varphi^1$ (Imu). Furthermore, since $(0,1)$ is *iuc*,

$$c_t\varphi^1(u)\varphi^1(v)<p_t^1(u,v)<C_t\varphi^1(u)\varphi^1(v), u,v\in R, t>0,$$

and convoluting this inequality for $t = 1$ with p_{t-1} yields, with $c_2 = c_1 \exp(\theta)$ and $C_2 = C_1 \exp(\theta)$, c_1 and C_1 as just above,

$$c_2 \exp(-\theta t)\phi^1(v)\varphi^1(v) < p_t^1(u,v) < C_2 \exp(-\theta t)\varphi^1(v)\varphi^1(v), t \geq 1. \tag{2.10}$$

Scaling gives $p^r_{r^2t}(ar, br) = p^1_t(a,b)/r$ and $\varphi^r(rx) = \varphi^1(x)/r$, and so (2.10) implies

$$\begin{aligned} &c_2 \exp(-(\theta/r^2)r^2 t)\varphi^r(u)\varphi^r(v) < p^r_{r^2t}(u,v) \\ &\qquad < C_2 \exp(-(\theta/r^2)r^2 t)\varphi^r(u)\varphi^r(v), t \geq r^2. \end{aligned} \tag{2.11}$$

Both assertions of Lemma 2.4 follow easily from (2.10), and (2.11). □

3. Proof of Theorem 1.

Before proving Theorem 1, we remark that the product of an *iuc* domain in m dimensions with an *iuc* domain in k dimensions is *iuc* in $m + k$ dimensions, so that Theorem 1 easily yields examples of *iuc* domains of infinite volume in all dimensions.

First we prove the "if" part of Theorem 1. Let $\mathbf{a}$ be a positive sequence, considered fixed, such that $lim\ 2^{-n}a_n = 0$ and shorten $\Omega_{\mathbf{a}}$ to Ω. Put $R_n = (\lambda_n, \lambda_{n+1}) \times (-a_n, 0], n > 0, \lambda_k$ as in the description of Ω in Section 1. Let $S = (0,1) \times (0,1)$, so that $\Omega = S \cup \bigcup_{n=1}^{\infty} R_n$. Let $D_j = (\lambda_n, \lambda_{n+1}) \times \{0\}$ be the "door" between R_j and S, put $\Delta = \bigcup_{i=1}^{\infty} D_i$ and $\Lambda_j = S \cup \bigcup_{k=1}^{j} R_j$. The set $A = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$ will serve as the compact set F of Lemma 2.3. In this section $p_t(x,y) = p_t^{\Omega}(x,y)$ and $X_t, t \geq 0$, is two dimensional Brownian motion killed when it leaves Ω. The next proposition establishes (2.3) for $D = \Omega$.

Proposition 3.1. *The following inequalities hold*

$$p_t(x,A) \geq c_t p_t(x,\Omega),\ Im x \in [-\tfrac{1}{4}, \tfrac{1}{4}], \tag{3.1}$$

$$p_t(x,A) \geq c_{t,j} p_t(x,\Omega), x \in \Lambda_j,\ \text{and} \tag{3.2}$$

$$p_t(x,A) \geq c_t p_t(x,\Omega), x \in \bigcup_{j=1}^{\infty} R_j. \tag{3.3}$$

Proof of (3.1). Let $L_1 = (0, \frac{1}{4}) \times \{\frac{3}{4}\}, L_2 = (\frac{3}{4}, 1) \times \{\frac{3}{4}\}$, so that $L_1 \cup L_2 = L$ together with the top line of A give a line across S. We first show

$$(3.4) \qquad P_x(\tau_L < t) \leq P(\tau_A < t), \ Im x \in [-\tfrac{1}{4}, \tfrac{1}{4}], t > 0.$$

It suffices to prove

$$(3.5) \qquad P_x(\tau_L < t) \leq P_x(\tau_A < t), Im x = \tfrac{1}{2}, t > 0,$$

for the conditions on x in (3.4) imply that under P_x, X_t must hit $\{Im\ x = \frac{1}{2}\}$ before it hits L, and an application of the strong Markov property at the time $\tau_{\{Im\ x = \frac{1}{2}\}}$, and (3.5), yields (3.4).

To prove (3.5), assume *WLOG* $Re x < \frac{1}{4}$, let $\mathcal{J}$ be the box $(0, \frac{1}{4}) \times (\frac{1}{4}, \frac{3}{4})$, and let $\Gamma = (0, \frac{1}{4}] \times \{\frac{3}{4}\}, \Psi = (0, \frac{1}{4}] \times \{\frac{1}{4}\}, Q = \{\frac{1}{4}\} \times [\frac{1}{4}, \frac{1}{2}], V = \{\frac{1}{4}\} \times (\frac{1}{2}, \frac{3}{4}]$ be the top, bottom, bottom of right side, top of right side, of $\mathcal{J}$. Shorten $\tau_{\mathcal{J}}$ to τ. Note $Q \cup V \in A$, and $\tau \leq \tau_L$. To prove (3.5) we will show

$$(3.5)' \qquad P_x(\tau < t, X_\tau \in \Gamma \cup \Psi) \leq P_x(\tau < t, X_\tau \in Q \cup V), Im\ x = \tfrac{1}{2}, t > 0,$$

which in turn follows from

$$(3.5)'' \qquad P_x(\tau < t, X_\tau \in \Gamma) \leq P_x(\tau < t, X_\tau \in V), \text{ and}$$

$$(3.5)''' \qquad P_x(\tau < t, X_\tau \in \Psi) \leq P_x(\tau < t, X_\tau \in Q).$$

These both follow from the same argument, so we prove $(3.5)''$. Let W be the straight line segment connecting $(0, \frac{1}{2})$ and $(\frac{1}{4}, \frac{3}{4})$. Note V reflected about W is Γ. Let $\eta_1 = \inf\{t > 0 : X_t \in W\}, \eta_2 = \inf\{t > \eta_1 : Im\ X_t = \frac{1}{2}\}, \eta_{2i+1} = \inf\{t > \eta_{2i} : X_t \in W\}, \eta_{2i+2} = \inf\{t > \eta_{2i+1} : Im\ X_t = \frac{1}{2}\}, i \geq 1$. Now $P_x(\tau < t, X_\tau \in \Gamma) = \sum_i P_x(\eta_{2i-1} < \tau < \eta_{2i}, X_\tau \in$

$\Gamma, \tau < t) = \sum_i P_x(\eta_{2i-1} < \tau < \eta_{2i}, X_\tau \in V, \tau < t) < P_x(\tau < t, X_\tau \in V)$, if $Im\ x = \frac{1}{2}$ and $0 < Re\ x < \frac{1}{4}$, the second equality using the strong Markov property and the fact that Brownian motion, reflected about W, is still Brownian motion. This gives (3.5)$''$ and thus (3.4).

Having established (3.4), we observe that if $z \in A$ then at least one "quadrant" of the oriented square of side length $\frac{1}{2}$ and center z lies in A, so that if c_t is the probability that two dimensional Brownian motion started at z does not leave a square of side length $\frac{1}{2}$ and center z by time t, we have c_t is decreasing in t and $p_t(z, A) \geq c_t/4$, which together with the strong Markov property yields

(3.6) $$p_s(z, A) \geq (c_t/4) P_z(\tau_A \leq s), 0 \leq s \leq t.$$

Let $B = L_1 \cup L_2 \cup A$. In view of (3.6) and (3.4), to prove (3.1) it suffices to prove

(3.7) $$P_x(\tau_B < t) \geq c_t p_t(x, \Omega),\ \ Imx \in [-\tfrac{1}{4}, \tfrac{1}{4}].$$

Write $p_t(x, \Omega) = \alpha + \beta$, where

$$\alpha = \alpha_t = P_x(Im\ X_t \leq -\tfrac{5}{4}, t < \tau_\Omega),\ \text{and}$$

$$\beta = \beta_t = P_x(Im\ X_t > -\tfrac{5}{4}, t < \tau_\Omega).$$

Now

(3.8) $$P_x(\tau_B < t) \geq \alpha,\ \ Imx \in [-\tfrac{1}{4}, \tfrac{1}{4}].$$

This follows from the fact that Brownian motion started at x reflected about the horizontal line H through x after the *last* time before t that this motion lies in H is still Brownian motion (the complete symmetry about H of the definition of two dimensional Brownian motion started at x guarantees this invariance), and the fact that any path in the set of which α is the probability, so reflected, hits B before it exits Ω.

We also have

$$P_x(\tau_B < t) \geq C_t\beta, Im\ x \in [-\tfrac{1}{4}, \tfrac{1}{4}]. \tag{3.9}$$

This follows from (2.9) with $M = 1\frac{1}{2}, H$ the set of all functions f in $C_2[0,t]$ such that $f(0) = x, f(s) \in \Omega, 0 \leq s \leq t, Im\ f(t) > -\frac{5}{4}$, and $k = (\frac{9}{4})/t$, and the fact that any function in this H under the transformation of (2.9) is a function which hits B before it leaves Ω. Together (3.9) and (3.8) prove (3.7) which yields (3.1).

Proof of (3.2). For any fixed j, Λ_j is *iuc*. This follows from the results of [DS] or [KP], but the reader unfamiliar with these papers should just assume, for now, that Λ_j is *iuc*, since Theorem 2 implies this result. Write $T = T_{\Lambda_j}, J = \{T > \frac{t}{2}\}$. We have $P_x(X_t \in A, J) \geq c_{t,j} P_x J, x \in \Lambda_j$, which follows from (1.1) with $D = \Lambda_j$. This implies

$$P_x(\tau_A < t|J) \geq c_{t,j} P_x(\tau_\Omega > t|J). \tag{3.10}$$

To complete the proof of (3.2) it suffices to show

$$P_x(\tau_A < t|J^c, X_T \in \Delta) \geq c_t P_x(\tau_\Omega < t|J^c, X_T \in \Delta). \tag{3.11}$$

Recall $\Delta = \bigcup_{i=1}^{\infty} D_i$. Note $\{X_T \in \Delta\} = \{\tau_\Omega > T\}$. Since each $x \in \Delta$ satisfies $-\frac{1}{4} \leq Im\ x \leq \frac{1}{4}$, the strong Markov property and (3.1) give that if $c_t = \inf\{c_s(3.1) : t/2 \leq s \leq t\})$, then $P_x(\tau_A < t|X_T) \geq c_t P_x(\tau_\Omega < t|X_T)$ on $J^c \cap \{X_T \in \Delta\}$, which gives (3.11).

Proof of (3.3). In view of (3.2) it suffices to prove (3.3) only for $x \in \bigcup_{j=j_0}^{\infty} R_j$ for any fixed $j_0 = j_0(t)$. Let $R'_n = (\lambda_n, \lambda_{n+1}) \times (-a_n, 1)$ be R_n together with the extension of R_n across S. Put $\tau_n = \tau_{R'_n}$. By Lemma 2.7 i) and scaling, we see that there is a sequence $\delta_n \to 0$ such that

$$P_x(X_{\delta_n} \in A, \delta_n < \tau_n) > c P_x(2\delta_n < \tau_n), x \in R'_n. \tag{3.12}$$

In fact, A may be replaced in (3.12) by a tiny subsquare of R'_n contained in A. Pick $j_0(t) = j_0$ so large that $\delta_n < t/4, n \geq j_0$. If $n \geq j_0$, and if $x \in R_n$, (3.12) and (3.6) guarantee

(3.13) $$P_x(\tau_A < t) \geq c_t P(\tau_\Omega < t, \tau_n > 2\delta_n), n \geq j_0.$$

To complete the proof of (3.3) we show

(3.14) $$P_x(\tau_A < t | \tau_n \leq 2\delta_n, X_{\tau_n} \in \Omega) \geq c_t P(\tau_\Omega < t | \tau_n \leq 2\delta_n, X_{\tau_n} \in \Omega), n \geq j_0.$$

Since $\tau_n < t/2$ if $n \geq j_0$, and $X_{\tau_n} \in S$ on $\{\tau_n \leq 2\delta_n, X_{\tau_n} \in \Omega\}$, (3.14) follows from the same argument used to prove (3.11), where the constants from (3.2) with $j = 1$ are used in place of those of (3.1). □

Let $\hat{x}$ be the point in A closest to x except if $Im x > \frac{3}{4}$, in which case $\hat{x}$ is the closest point on $[\frac{1}{4}, \frac{3}{4}] \times \{\frac{5}{8}\}$ to x. The following, gives (2.5) for $D = \Omega, A = F$.

Proposition 3.2. *We have*

(3.15) $$p_t(\hat{x}, y) \geq c_t p_t(x, y), x, y \in \Omega.$$

Proof. First we prove (3.15) when $Im\ x \geq \frac{1}{4}, x \notin A$. Let Ψ be the (straight) line through $\hat{x}$ perpendicular to and bisecting the line segment connecting $\hat{x}$ and x, and let x^* be the reflection of x about Ψ. Note $x^* \in A$. We will show

(3.16) $$p_t(x^*, y) \geq c_t p_t(x, y), y \in \Omega,$$

which in view of (2.4) implies (3.15). Let V be the part of S which lies on the x side of Ψ, and let U be the reflection of V about Ψ. Note $U \subset \Omega$. Now

$$p_t(x^*, z) = p_t(x^*, z; t \leq \tau_\Psi) + p_t(x^*, z; t > \tau_\Psi).$$

By symmetry, and the (unstated) equality case of Lemma 2.5,

$$p_t(x^*,z;t\geq\tau_\Psi)=p_t(x,z;t\geq\tau_\Psi,\tau_\Psi=\tau_U)\leq p_t(x,z),$$

while

$$p_t(x^*,y;t<\tau_\Psi)\leq p_t^S(x^*,y)\leq c_tp_t^S(x,y)\leq c_tp_t(x,y),$$

the middle inequality following from (1.1), since S is *iuc*. This completes the proof of (3.16) and thus the $Im\ x>\frac{1}{4}$ case of (3.15).

Next we prove (3.15) in the case $Im\ x\leq-\frac{1}{4}$. Let $x\in R_j$. Note $Im\hat{x}=\frac{1}{4}$. We have

$$p_t(x,y)=p_t^{R_j}(x,y)+p_t(x,y;t>\tau_{R_j}). \tag{3.17}$$

Note $\tau_{R_j}=\tau_{D_j}$ if $X_0=x$ and $\tau_\Omega>\tau_{R_j}$. To handle the second term in (3.17) we use Lemma 2.5. Now $\nu_{x,s}(\tau_{D_j})$ has a density h on $(0,\infty)\times(\lambda_j,\lambda_{j+1})$, where $h(s,a)=f(s)g_s(a)$, where $f(s)$ is the density of $\tau_{(-a_j,0)};Z_{\tau_{(-a_j,0)}}=0$ under P^*_{Imx}, and g_s is the density of $Z_s;\tau_{(\lambda_j,\lambda_{j+1})}>s$ under P^*_{Rex}. Here P^* stands for probability associated with standard *one* dimensional Brownian motion. Similarly, $\nu_{\hat{x},s}(\tau_{D_j})$ has density on $(0,\infty)\times(\lambda_j,\lambda_{j+1})$ given by $\hat{f}(s)\hat{g}_s(a)$, where $\hat{f}_s$ is the density of $\tau_{(0,1)};Z_{\tau_{(0,1)}}=0$ under $P^*_{\frac{1}{4}}$ and $\hat{g}_s$ is the density of $Z_s;\tau_{(0,1)}>s$ given $Z_0=Rex(=Re\hat{x})$. Since $(0,1)\supset(\lambda_j,\lambda_{j+1}),\hat{g}_s>g_s$ for each s. To complete the proof that $\nu_{x,s}(\tau_{D_j})\leq c_t\nu_{\hat{x},s}(\tau_{D_j})$ it suffices to show $\hat{f}(s)\geq c_tf(s),0\leq s\leq t$. Shrinking $(0,1)$ and expanding and reflecting $(-a_n,0)$, we see that it suffices to show that if γ is the density of $\tau_{(0,\frac{1}{2})};X_{\tau_{(0,\frac{1}{2})}}=0$ under $P^*_{\frac{1}{4}}$ and if η is the density of $\tau_{(0,\infty)}$ under $P^*_r,r>\frac{1}{4}$ then $\gamma(s)\geq c_t\eta(s),0\leq s\leq t$. Clearly γ is half the density of $\tau_{(0,\frac{1}{2})}$. Both γ and η are easily found, and it is easy to check this result. Thus we have, by Lemma 2.5,

$$p_t(x,y;t>\tau_{R_j})\leq c_tp_t(\hat{x},y;t>\tau_{D_j})<c_tp_t(\hat{x},y). \tag{3.18}$$

To finish the $Im\ x \le -\frac{1}{4}$ case we prove there is c_t not depending on j such that

$$p_t^{R_j}(x,y) < c_t p_t(\hat{x},y), y \in R_j. \tag{3.19}$$

Let R'_j be as in the proof of (3.3). Let $p_t^{R_j} = p_t^j, p_t^{R'_j} = p_t^J$. Then

$$p_t^j(x,y) < p_t^J(x,y) < c_{t,j} p_t^J(\hat{x},y) < c_{t,j} p_t(\hat{x},y), \tag{3.20}$$

the middle inequality since R'_j is *iuc*. Now Lemma 2.7 ii) gives $\gamma_n \to 0$ such that if $Im\ x^* = \frac{1}{4}$ and x^* is directly above the center of D_j, there is c_t not depending on j such that

$$p_{\gamma_j}^J(x^*,y) > c_t p_{2\gamma_j}^J(x,y), y \in R'_j. \tag{3.21}$$

Pick j_0 so large that $\gamma_j < \frac{t}{4}, j \ge j_0$. Then

$$\begin{aligned} p_t^j(x,y) < p_t^J(x,y) &= \int p_{2\gamma_j}^J(x,z) p_{t-2\gamma_j}^J(z,y)dz \\ &< c_t \int p_{\gamma_j}^J(x^*,z) p_{t-2\gamma_j}^J(z,y)dz \\ &= c_t p_{t-\gamma_j}^J(x^*,y) \\ &< c_t p_{t-\gamma_j}(x^*,y) \\ &< c_t p_t(x^*,y) < c_t p_t(\hat{x},y), j \ge j_0, \end{aligned} \tag{3.22}$$

the next to last inequality by Lemma 2.4 and the last by (2.4) with $D = \Omega, F = A$, which holds since we have already proved (2.3) in this case. Together (3.22) and (3.20) give (3.19), with $c_t(3.19) = \max(c_t(3.22), c_{t,j}(3.20), 1 \le j < j_0)$.

To finish the proof of Proposition 3.2 we establish (3.15) in the case $-\frac{1}{4} < Im\ x < \frac{1}{4}$. Let $G = \{x \in \Omega : -\frac{1}{4} < Im\ x < \frac{1}{4}\}$, and shorten τ_G to τ. We first show

$$p_{t/2}(x,y;\tau > t/2) \le c_t p_{t/2}(\hat{x},y), x \in G. \tag{3.23}$$

We need only establish this for $y \in G$. Let $B(y,\epsilon) = B$ be a disc of radius ε around y, which is contained in G. Let $\tilde{x}$ satisfy $Im\ \tilde{x} = \frac{1}{4}, Re\ \tilde{x} = Re\ x$. Let $\theta = \frac{1}{4} - Im\ x = \tilde{x} - x$. We will use (2.9) with $k = -2\theta/t, r = t/2, \gamma = \gamma_{-2\theta/t,t/2}$, and $H = \Gamma = \{g \in C_2[0,r] : g(0) = x, \tau(g) > t/2$, and $g(t/2) \in B\}$. Then if σ_x denotes the distribution of standard (unkilled) two dimensional Brownian motion started at x, (2.12) gives

$$\sigma_x(\Gamma + \gamma) \geq c_t\sigma_x(\Gamma) = c_t p_{t/2}(x, B; \tau > t/2) \tag{3.24}$$

so that

$$\sigma_x(\Gamma + \gamma) = \sigma_{\tilde{x}}\left((\tilde{x} - x) + \Gamma + \gamma\right) \geq c_t\sigma_x(\Gamma). \tag{3.25}$$

Now if $g \in \Gamma$ then $(\tilde{x} - x) + g + \gamma = f$ satisfies $f(0) = \tilde{x}, \tau_\Omega(f) > t/2, f(t/2) \in B$. Thus

$$p_{t/2}(\tilde{x}, B) \geq \sigma_x\left((\tilde{x} - x) + \Gamma + \gamma\right) \geq c_t p_{t/2}(x, B; \tau > t/2),$$

and dividing this inequality by $\pi\varepsilon^2$ and letting $\varepsilon \to 0$ gives the first inequality of $p_{t/2}(x, y; \tau > t/2) \leq c_t p_{t/2}(\tilde{x}, y) \leq c_t p_{t/2}(\hat{x}, y)$, the second following, since $\hat{\tilde{x}} = \hat{x}$ and $Im\ \tilde{x} = \frac{1}{4}$, from (3.15) in the case $Im\ x = \frac{1}{4}$. This proves (3.23).

Now let $\xi = t - (\tau \wedge t/2)$. Note $t/2 \leq \xi \leq t$. We have

$$\begin{aligned}
p_t(x,y) &= E_x p_\xi(X_{\tau\wedge t/2}, y) \\
&= E_x p_{t-\tau}(X_\tau, y) I(\tau \leq t/2) + E_x p_{t/2}(X_{t/2}, y) I(\tau > t/2) \\
&\leq c_t E_x p_{t-\tau}(\widehat{X_\tau}, y) I(\tau \leq t/2) + \int_G p_{t/2}(x, z; \tau > t/2) p_{t/2}(z, y)\,dz \\
&\leq c_t E_x p_{t-\tau}(\hat{x}, y) I(\tau \leq t/2) + c_t \int_\Omega p_{t/2}(\hat{x}, z) p_{t/2}(z, y)\,dz \\
&\leq c_t p_t(\hat{x}, y) + c_t p_t(\hat{x}, y) = c_t p_t(\hat{x}, y).
\end{aligned}$$

The first inequality holds since on $\{\tau > \tau_\Omega, \tau < t/2\}, Im\ X_\tau = \pm\frac{1}{4}$, so we may use the results from the earlier part of this proof. The second follows from (2.4) and (3.23), and the third from Lemma 2.4.

This ends the proof of the proposition and thus of the "if" part of Theorem 1. □

Now the proof of the "only if" part of Theorem 1 is given. If D is *iuc* then the exponential decay of ϕ implies that $\int_t^{2t} p_s^D(x,y)ds \geq c_t \int_{2t}^{\infty} p_s^D(x,y)ds, x, y \in D$, or, to put it another way, if τ_{D_y} is the lifetime of the h process in the domain $D - \{y\} = D_y$ with corresponding harmonic function $G^D(\cdot, y), G$ the Green function, then

$$(4.24) \qquad P_x^{D_y}(t < \tau_{D_y} < 2t) \geq c_t P_x^{D_y}(\tau_{D_y} > 2t).$$

See [D] and [BD] for more detail on these matters.

Now let **a** be a sequence such that a subsequence of $a_n 2^{-n}$ (for notational purposes assume this subsequence is in fact the sequence $a_n 2^{-n}$) satisfies $\lim a_n 2^{-n} \in (0, \infty]$. Let x_n be any point in the bottom half of R_n, and let $\Gamma = \Omega_{\mathbf{a}} - \{(\frac{1}{2}, \frac{1}{2})\}$. For the remainder of the paper, P_x, E_x denote probability and expectation with respect to the h process associated with $G^{\Gamma}((\frac{1}{2}, \frac{1}{2}), \cdot)$ in the domain Γ, and τ is the lifetime of this process. Let $T_n = \int_0^{\tau} I(X_s \in R_n)ds$. The methods of [X] easily give $\underline{\lim} E_{x_n} T_n / a_n 2^{-n} > 0$. The arguments of Section 5 of [Da], without change, give lim $\mathrm{Var}_{x_n} T_n / (a_n 2^{-n})^2 = 0$. Thus there is a constant K such that $\lim P_{x_n}(T_n > K) = 1$, and thus $\lim P_{x_n}(\tau > K) = 1$, which contradicts (4.24) with $y = (\frac{1}{2}, \frac{1}{2}), D = \Omega_{\mathbf{a}}, t = K/2$. □

If f is an uppersemicontinuous function on $(0,1)$ such that $-\infty \leq f \leq 0$, we let D_f be the domain $\{(x,y) : 0 < x < 1, f(x) < y < 1\}$. For $x \in D_f$ define the domain D_f^x as a union of horizontal line segments as follows. Let $\alpha = \alpha(x) = \inf\{r : (Rex, r) \in D_f\}$, and for each $s \in (\alpha, 1)$ let $H_s^x = (a(s), b(s)) \times \{x\}$, where $(a(s), b(s))$ is the largest open interval such that $(a(s)b(s)) \times \{s\}$ contains (Rex, s) and is contained in D_f, and put $G_x = \cup\{H_s^x : \alpha < s < 1\}$. Let for $\delta < 1, G_x^{\delta} = G^x \cap \{z : Imx < \delta\}$. The proof at the end of the last section can be adapted to prove that if $\lim_{\delta \to -\infty} (\sup_{x \in D_f} \text{area } (G_x^{\delta})) > 0$ then D_f is not *iuc*. We conjecture that if this limit is 0, then D_f is *iuc*.

4. Domains above the graph of a function. First we prove Theorem 2. The proof involves modifications of part of the proof of Theorem 1, which are now sketched. We replace A by $B = [\frac{1}{8}, \frac{7}{8}] \times [\frac{1}{8}, \frac{7}{8}]$. The role of Δ is played by $(0,1) \times \{0\}$. The following holds (now $p_t = p_t^{D_f}$).

Proposition 4.1. *Let $x \in D_f$. Then*

$$p_t(x, B) \geq c_{t,M} p_t(x, D_f), \quad Imx \leq \tfrac{1}{4}. \tag{4.1}$$

$$p_t(x, B) \geq c_{t,M} p_t(x, D_f), x \in S. \tag{4.2}$$

The proofs of (4.1) and (4.2) exactly parallel the proofs of (3.1) and (3.2) respectively, and we observe that all we have to use in the proof of (4.2) is that S is *iuc*, which follows from the fact that (0, 1) is *iuc* in one dimension. More extensive alteration is required of Proposition 3.2. Let $\alpha = \frac{7}{8}, \beta = \frac{15}{16}$. Let Γ be $(0,1) \times \{\alpha\}, \Theta = (0,1) \times \{1\}$. If $Imx \geq \alpha, x \in D_f$, let $x^* \in A \subset B$ be the same x^* as in the previous section. If $Imx < \alpha$, let $x^+ = (Rex, \beta)$, if $Imx \geq \alpha$, let $x^+ = x$, and put $x' = (x^+)^*$.

Proposition 4.2. *Let $x, y \in D_f$. Then*

$$p_t(x, y) \leq c_t p_t(x', y), \quad Imy < \alpha. \tag{4.3}$$

$$p_t(x, y) \leq c_t p_t(x', y), \quad Imy \geq \alpha. \tag{4.4}$$

Proof. We first note that $p_t(w, y) \leq c_t p_t(w^*, y), Imw > \frac{7}{8}, y \in D_f$, by the same argument that proved (3.16), so it suffices to prove the variants of (4.3) and (4.4) in which x^+ replaces x', denoted $(4.3)^+$ and $(4.4)^+$.

Of course $(4.4)^+$ is immediate if $x^+ = x$. Otherwise we have $Imx^+ = \beta$ and $(4.4)^+$

follows from

$$p_t(x^+,y) > p_t^S(x^+,y), \quad Imy \geq \alpha,$$

$$p_t(x,y) < p_t^H(x,y), Imy \geq \alpha, \quad H = (0,1) \times (-\infty, 1), \text{ and}$$

$$p_t^S(x^+,y) \geq c_t p_t^H(x,y), \quad Imy \geq \alpha.$$

To prove the last of these, let $g_s(a,\cdot), h_s(a,\cdot)$ be the densities of $Z_s; s < \tau_{(0,1)}, Z_s; s < \tau_{(-\infty,1)}$ where $Z_t, t \geq 0$ is one dimensional Brownian motion started at a. Then we have

$$p_t^S(x^+,y) = g_t(Rex^+, Rey)g_t(\beta, Imy),$$

while, since $Rex^+ = Rex$,

$$p_t^H(x,y) = g_t(Rex^+, Rey)h_t(Imx, Imy),$$

and it is a straightforward calculation involving one dimensional Brownian motion that

$$h_t(u,v) \leq c_t g_t(\beta, v) \text{ if } v \in [\alpha, 1) \text{ and } u < \alpha.$$

To prove $(4.3)^+$, let $\Psi = D_f \cup (0,1) \times [1,\infty)$. An argument exactly like the one which gave (3.23) gives

$$p_t^\Psi(x,y) \leq c_{t,M} p_t^\Psi(x^+,y), Imy < \alpha,$$

so that, since $p_t^{D_f} \leq p_t^\Psi$, to complete the proof of $(4.3)^+$ it suffices to prove

$$p_t^\Psi(x,y) \leq c_t p_t^{D_f}(x,y), Imy < \alpha, Imx = \beta. \tag{4.5}$$

Let $T_1 = \tau_\Gamma, T_{2i} = \tau_\Theta \circ \theta(T_{2i-1}) + \theta(T_{2i-1}), T_{2i+1} = \tau_\Gamma \circ \theta(T_{2i}) + \theta(T_{2i}), i \geq 1$. Then if $Imx = \beta, Imy < \alpha$, which we assume throughout the rest of this proof, we have

$$p_t^{D_f}(x,y) = Ep_{t-T_1}(X_{T_1}, y)I(T_1 < t) = \int_{(0,t)\times\Gamma} p_{t-s}(v,y)d\nu_{x,t}(T_1), \tag{4.6}$$

in the notation of Lemma 2.5, while

$$(4.7) \qquad \begin{aligned} p_t^{\Psi}(x,y) &= E\sum_{i=0}^{\infty} p^{\Psi}_{t-T_{2i+1}}(X_{T_{2i+1}},y)I(T_{2i+1}<t<T_{2i+2}, t<\tau_\Psi) \\ &= E\sum_{i=0}^{\infty} p_{t-T_{2i+1}}(X_{T_{2i+1}},y)I(T_{2i+1}<t, T_{2i+1}<\tau_\Psi) \\ &= \int_{(0,t)\times\Gamma} p_{t-s}(u,y)d\eta(s,u), \end{aligned}$$

where the second equality follows since $\tau_\Psi \wedge \tau_\Theta = \tau_{D_f}$, and where $\eta = \sum_{i=0}^{\infty}\gamma_i, \gamma_i$ the distribution of $(T_{2i+1}, X_{T_{2i+1}}); T_{2i+1} < \tau_\Psi$ on $[0,t)\times\Gamma$. Now let λ_i be the distribution of $(T_{2i+1}, X_{T_{2i+1}}); T_{2i+1} < \tau_{(0,1)\times(-\infty,\infty)}$. Clearly $\lambda_i \geq \gamma_i$. We will show

$$(4.8) \qquad \sum_{i=0}^{\infty} \lambda_i < c_t \nu_{x,t}(T_1),$$

which, together with (4.6) and (4.7), yields (4.5). Let $B_s, s\geq 0$, be standard one dimensional Brownian motion started at α. Let $\tau_1 = \inf\{t>0 : |B_t - \frac{7}{8}| = \frac{1}{16}\}, \gamma = \inf\{t > 0 : B_t = \frac{7}{8}\}, \tau_{2i+1} = \tau_{2i} + \tau_1\circ\theta_{\tau_{2i}}, \tau_{2i} = \tau_{2i-1} + \gamma\circ\theta_{\tau_{2i-1}}, i>0$. Let f_i be the density of τ_{2i+1}. It is easy to compute f_1, and writing $\tau_{2i+1} = (\tau_{2i+1} - \tau_1) + \tau_1$ we note that $\tau_{2i+1} - \tau_1$ is independent of τ_1 and it is readily shown that $\sum_{i=1}^{\infty} P(\tau_{2i+1} - \tau_1 < t) < \infty$ for each t from which, together with the fact that $f_1(u) \leq c_t f_1(v)$ whenever $0 \leq u < v \leq t$, we get $\sum_{i=1}^{\infty} f_i(s) < c_t f_1(s), 0 \leq s \leq t$. Since at a point (s,v) of $(0,t)\times\Gamma$ the density of $\Sigma\lambda_i$ equals $\left(\sum_{i=1}^{\infty} f_i(s)\right) g_s(Rex, Rev)$ and the density of $\nu_{x,t}(T_1)$ equals $(\frac{1}{2})f_1(s)g_s(Rex, Rev)$, this gives (4.8) □

Finally we state and prove an extension of Theorem 2 to higher dimensions. We denote points in $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n)$, and let $|x|_{n-1}$ stand for $\left(\sum_{i=1}^{n-1} x_i^2\right)^{\frac{1}{2}}$. We prove

Theorem 4.3. *Let $f(x_1,\ldots,x_{n-1})$ be uppersemicontinuous on $\{|x|_{n-1} < 1\}$ and satisfy $-M < f \leq 0$ for a positive constant M. Then $D_f = \{(x_1,\ldots,x_n) : |x|_{n-1} < 1, f(x_1,\ldots,x_{n-1}) < x_n < 1\}$ is iuc.*

Proof. We shorten D_f to D, let $X_t = (X_t^1, \ldots, X_t^n), t \geq 0$, be standard n-dimensional Brownian motion, and put $|X_t|_{n-1} = \left(\sum_{i=1}^{n-1} (X_t^i)^2\right)^{\frac{1}{2}}$. With one exception, to be mentioned shortly, our adaption of the last proof is routine. The compact set B of the last proof is replaced by $B_n = \{x : \delta_n < x_n < 1 - \delta_n, |x|_{n-1} < 1 - \delta_n\}$, where δ_n is chosen to have properties i) and ii) below, and α, β in that proof are replaced by $1 - (\delta_n/2)$ and $1 - (\delta_n/4)$ respectively. We replace Γ, Θ with $\Gamma_n = D \cap \{x_n = \alpha_n\}$ and $\Theta_n = \{x_n = 1, |x|_{n-1} < 1\}$. If $x \in D \cap \{x_n > \alpha_n\} = Q_n$, let $\hat{x}$ be the closest point in B_n to x and let x^* be the reflection of x about the hyperplane F through $\hat{x}$ which is perpindicular to the line through x and $\hat{x}$. Let U_x be the reflection about F of $D \cap H$, where H is all those points on that side of F which contains x. We choose δ_n so small that

$$i) \qquad U_x \subset \{x : 0 < x_n < 1, |x|_{n-1} < 1\}, x \in Q_n.$$

The exception mentioned earlier is the analog of (3.5). What we must show is

$$(4.9) \qquad P_x(\tau_{L_n} < t) \leq P_x(\tau_{A_n} < t), x_n = \tfrac{1}{2}, t > 0,$$

where $L_n = \{x_n = 1 - \delta_n, 1 - \delta_n < |x|_{n-1} < 1\}$. To prove (4.9) we show, mimicing the proof of (3.5), that for suitably small δ_n, if we let Γ_n, Ψ_n, and $Q_n \cup V_n$ represent the top, bottom, and union of the top and bottom halves of the inner boundary of $\mathcal{J}' = \{x : \delta_n < x_n < 1 - \delta_n, 1 - \delta_n < |x|_{n-1} < 1\}$, then

$$ii) \qquad P_x(\tau_{\mathcal{J}'} < t, \tau_{\mathcal{J}'} \in \Gamma_n \cup \Psi_n) \leq P_x(\tau_{\mathcal{J}'} < t, \tau_{\mathcal{J}'} \in Q_n \cup V_n), x_n = \tfrac{1}{2}.$$

Now we know from the proof of (3.5) that if (Z_t, W_t) are independent one dimensional Brownian motions started at a point $(a, \frac{1}{2})$, $0 < a < \frac{1}{4}$, and if $\mathcal{J}$ is as in the proof of (3.5), then the probability (Z, W) exits from $\mathcal{J}$ at $\Gamma \cup \Psi$ by time t exceeds the probability it exits from J at $Q \cup V$ by time t. Scaling gives that if $0 < \theta < \frac{1}{2}$ and if Z'_t is a motion satisfying $Z'_{\lambda t}, t \geq 0$, is Brownian motion, where $\lambda = \sqrt{(\frac{1}{2} - \theta)/\theta}$, if W'_t is an independent Brownian motion, if $\mathcal{J}' = (0, \theta) \times (\theta, 1 - \theta)$, and if $(Z'_0, W'_0) = (u, \frac{1}{2}), 0 < u < \theta$, then the probability (Z', W') exits $\mathcal{J}'$ at the union of the top and bottom by time t exceeds the probability t exits $\mathcal{J}'$ at the right side by time t. This must also hold if instead of $Z'_{\lambda t}$ being a Brownian motion, we have $Z'_{\eta(t)}, t \geq 0$, is Brownian motion, where $\eta(t)$ is increasing, continuous, and $\eta(t) \leq \lambda t, t \geq 0$, since speeding up the horizontal motion can only increase the probability of first exit from the right hand side by time t, and only decrease the probability of first exit from the top and bottom by time t.

Now return to n dimensions, and put $Z''_t = \psi(|X_t|_{n-1}) - \psi(1), t > 0$, where $\psi(s) = 1 + \log s, s > 0, n = 2, \psi(s) = s^{-n+2}, s > 0, n > 2$. Let $T = \inf\{t : |X_t|_{n-1} = 1\}$. Then Z''_t is a local martingale, since ψ is harmonic away from 0, and the standard time change formula (see Section 2.8 of [M]) and Ito's lemma give that there is a constant $c_n > 0$ and a function $\eta(t)$ such that $\eta(t) \leq c_n t, t \leq T$, and $Z''_{\eta(t)}, t \geq 0$, is Brownian motion. Put $W''_t = X^n_t$. By the last paragraph there is a $\theta'' = \theta''_n$ such that if $Z''_0 \in (0, \theta''), W''_0 = \frac{1}{2}$, and $\mathcal{J}'' = (0, \theta'') \times (\theta'', 1 - \theta'')$, then the probability that (Z'', W'') exits $\mathcal{J}''$ at the union of the top and bottom by time t exceeds the probability it exits the right side by time t. It is now straightforward to show that for small enough δ_n we have ii). □

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