Travelling Salesman with a Self-Similar Itinerary

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Abstract

ABSTRACT. Let $X_1, X_2, \ldots$ be i.i.d. random points in $\mathbb{R}^2$ with distribution ν, and let L_n be the length of the shortest path through $X_1, \ldots, X_n$. The exact almost sure rate of growth of L_n is obtained under the assumption that ν is self-similar in an appropriate sense. This extends a well known theorem of Beardwood, Halton, and Hammersley.

SHORT TITLE. Travelling Salesman with a Self-Similar Itinerary.

1. The Beardwood-Halton-Hammersley Theorem

Theorem 0: *If $X_1, X_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^d$ from a probability distribution with compact support and if L_n is the length of the shortest path through $X_1, X_2, \ldots, X_n$, then*

$$\frac{L_n}{n^{(d-1)/d}} \longrightarrow c_d \int f(x)^{(d-1)/d} dx \quad a.s.,$$

where $c_d > 0$ is a constant and $f(x)$ is the density of the absolutely continuous component of the distribution of X_n.

This theorem was discovered by Beardwood, Halton, and Hammersley [1]. It is of interest in the study of probabilistic algorithms for the travelling salesman problem [5], [7].

If the distribution of X_n is singular then Th. 0 implies that $L_n/n^{(d-1)/d} \longrightarrow 0$ a.s. It is natural to inquire about the rate of growth of L_n in such cases. The purpose of this note is to exhibit the exact rate of growth of L_n when the distribution of X_n and its support are "self-similar" in an appropriate sense. For simplicity, we shall consider only distributions in $\mathbb{R}^2$.

The Beardwood-Halton-Hammersley theorem is ultimately a consequence of the local self-similarity of the ambient space, because this forces a degree of hierarchical organization upon the shortest path (see Lemmas 4 and 8 below). It is easy to find singular distributions lacking self-similarity for which the growth of L_n is highly erratic. Beardwood-Halton-Hammersley-type theorems undoubtedly hold for distributions not considered in this paper, provided some approximate, local self-similarity is present (e.g., normalized Hausdorff measures on Julia sets).

2. A BHH Theorem for Self-Similar Probability Distributions

Let K be a compact subset of $\mathbb{R}^2$ satisfying $K = \bigcup_{i=1}^{m} \psi_i(K)$, where ψ_i is a contractive similarity transformation of $\mathbb{R}^2$ with similarity ratio $r_i \epsilon (0,1)$ (i.e., for any points $x, y \epsilon \mathbb{R}^2, ||\psi_i(x) - \psi_i(y)|| = r_i||x-y||$). Say that K is *strongly self-similar* if there exists a bounded, connected, open set V whose boundary ∂V is a simple, closed, rectifiable curve and such that $V \cap K \neq \varnothing, \psi_i(V) \cap \psi_j(V) = \varnothing \;\; \forall\, i \neq j$, and $\bigcup_{i=1}^{m} \psi_i(V) \subset V$. If $\psi_1(K), \ldots, \psi_m(K)$ are pairwise disjoint, say that K is *strongly disconnected.*

Examples: (1) $K =$ the closed unit square; $V =$ the open unit square; $\psi_1, \psi_2, \psi_3, \psi_4$ are the affine mappings that take V to the four nonoverlapping subsquares of side $r_1 = r_2 = r_3 = r_4 = 1/2$, each with one vertex at a vertex of V.

(2) $K = \{(x_1, x_2) : x_i \epsilon C\}$, where C is the usual Cantor set; $V =$ the open unit square; $\psi_1, \psi_2, \psi_3, \psi_4$ are the affine mappings that take V to the four nonoverlapping subsquares

of side $r_1 = r_2 = r_3 = r_4 = 1/3$, each with one vertex at a vertex of V. This set K is strongly disconnected.

(3) K = the Sierpinski gasket ([6], plate 141); V = an open equilateral triangle of side 1; ψ_1, ψ_2, ψ_3 are the affine mappings that take V to the three equilateral (sub)triangles of side $r_1 = r_2 = r_3 = 1/2$, each with a vertex at a vertex of V.

Assume henceforth that K is a strongly self-similar set and that $r_1, r_2, \ldots, r_n$ are the similarity ratios. It is known [4] that the Hausdorff dimension δ of K is the unique positive real number such that $\sum_{i=1}^{m} r_i^\delta = 1$. Note that $\delta < 1$ iff $\Sigma r_i < 1$.

Theorem 1: *Assume that $\delta < 1$. Let $X_1, X_2, \ldots$ be i.i.d. with distribution ν, where $\nu(K) = 1$, and let L_n be the length of the shortest path through $X_1, X_2, \ldots, X_n$. Then there is a constant $C < \infty$ such that*

$$\lim_{n \longrightarrow \infty} L_n = C \quad a.s. \tag{2.1}$$

The cases $\delta > 1$ and $\delta = 1$ are more interesting. Let ν be a probability measure on K; say that ν is *self-similar* if there exist positive real numbers $p_1, p_2, \ldots, p_m$ satisfying $p_1 + \ldots + p_m = 1$ and

$$\nu(\psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(K)) = p_{i_1} p_{i_2} \ldots p_{i_n}$$

for all finite sequences $i_1 i_2 \ldots i_n$ of indices from $\{1, 2, \ldots, m\}$. (If $p_i = r_i^\delta$ then ν is the normalized δ-dimensional Hausdorff measure on K.) Define the *similarity exponent* θ of ν to be the unique real number such that

$$\sum_{i=1}^{m} r_i p_i^\theta = 1.$$

Observe that $\theta > 0$ iff $\delta > 1$ and $\theta = 0$ iff $\delta = 1$. In general, $(1-\theta)^{-1}$ is the Hausdorff dimension of the set of generic points of ν.

Theorem 2: *Let $X_1, X_2, \ldots$ be i.i.d. with distribution ν, where ν is a self-similar probability measure on K with similarity exponent $\theta > 0$. Let L_n be the length of the shortest path through $X_1, \ldots, X_n$.*

(a) (Nonarithmetic case) If $\log p_1, \ldots, \log p_m$ are contained in no discrete additive subgroup of $\mathbb{R}$ then there is a constant $0 < \mu < \infty$ such that

$$\lim_{n \longrightarrow \infty} L_n / n^\theta = \mu \quad a.s. \tag{2.2}$$

(b) *(Arithmetic case) If* $\log p_1, \ldots, \log p_m$ *are contained in* $h\mathbb{Z}$ *but in no proper subgroup of* $h\mathbb{Z}$ *then there exists a continuous, positive, h-periodic function* $C(t)$ *such that*

$$\lim_{n \longrightarrow \infty} \frac{L_n}{n^\theta C(\log n)} = 1 \ a.s. \tag{2.3}$$

Notice that (K, ν) may have more than one self-similarity structure. Consider, for example $K = [0,1] \times [0,1]$ and $\nu =$ Lebesgue measure on K. One self-similarity structure is given by $V =$ open unit square; $\psi_1, \psi_2, \psi_3, \psi_4 =$ the affine mappings that take V to the four nonoverlapping subsquares $\left(\frac{i}{2}, \frac{(i+1)}{2}\right) \times \left(\frac{j}{2}, \frac{(j+1)}{2}\right), i, j = 0$ or 1; and $p_1 = p_2 = p_3 = p_4 = 1/4$. Another self-similarity structure is given by $V =$ open unit square; $\psi_1, \ldots, \psi_9 =$ the affine mappings taking V to the nine nonoverlapping subsquares $\left(\frac{i}{3}, \frac{(i+1)}{3}\right) \times \left(\frac{j}{3}, \frac{(j+1)}{3}\right), i, j = 0, 1,$ or 2; and $p_1 = p_2 = \ldots = p_9 = \frac{1}{9}$. Observe that $\theta = \frac{1}{2}$ for each of the structures, and that the closed additive group generated by $\log\left(\frac{1}{4}\right)$ and $\log\left(\frac{1}{9}\right)$ is $\mathbb{R}$. Consequently, (2.3) applied to each of the structures separately implies that (2.2) must hold, because if $C(t)$ is $\log 4$-periodic and $\log 9$-periodic it must be constant. In general, if there exist distinct self-similarity structures $(\psi_1, \ldots, \psi_m; p_1, \ldots, p_m)$ and $(\tilde{\psi}_1, \ldots, \tilde{\psi}_{\tilde{m}}; \tilde{p}_1, \ldots, \tilde{p}_{\tilde{m}})$ such that the smallest closed subgroup of $\mathbb{R}$ containing $\log p_1, \ldots, \log p_m, \log \tilde{p}_1, \ldots, \log \tilde{p}_{\tilde{m}}$ is $\mathbb{R}$, then (2.2) holds.

One might wonder if the periodic function $C(t)$ in (2.3) is ever non-constant. I am convinced that in general it is, for reasons that I will explain in sec. 9.

The case $\delta = 1$ occurs iff $\Sigma_{i=1}^m r_i = 1$. One example is $K =$ any closed line segment in $\mathbb{R}^2$. In this instance $L_n \longrightarrow$ constant a.s. as $n \longrightarrow \infty$; thus, one might expect that, in general, if $\delta = 1$ then L_n is bounded. This is false.

Theorem 3: *Let* $X_1, X_2, \ldots$ *be i.i.d. with distribution* ν, *where* ν *is a self-similar probability measure on* K, *and let* L_n *be the length of the shortest path through* $X_1, X_2, \ldots, X_n$. *Assume that* K *is strongly disconnected and strongly self-similar, with dimension* $\delta = 1$. *Then there is a constant* $0 < \mu < \infty$ *such that*

$$\lim_{n \longrightarrow \infty} L_n / \log n = \mu \ \ a.s. \tag{2.4}$$

The proof of Th. 1 will be given in sec. 4; the proofs of Th. 2–3 occupy sec. 5–8. Section 3 is a resumé of some important facts about self-similar sets and self-similar distributions.

3. The Structure of Self-Similar Sets and Distributions

The elements of a strongly self-similar set K may be represented as infinite sequences in the following way. Let $i_1 i_2 \ldots$ be an infinite sequence of indices from $\{1, 2, \ldots, m\}$; define

$$K_{i_1 i_2 \ldots i_n} = \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(K),$$
$$V_{i_1 i_2 \ldots i_n} = \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(V), \text{ and}$$
$$k_{i_1 i_2 \ldots} = \bigcap_{n=1}^{\infty} K_{i_1 i_2 \ldots i_n} .$$

Observe that $K \supset K_{i_1} \supset K_{i_1 i_2} \supset \ldots$ is a nested sequence of nonempty, compact sets, so the intersection is nonempty. Moreover,

$$\text{diameter } (K_{i_1 i_2 \ldots i_n}) = r_{i_1} r_{i_2} \ldots r_{i_n} \text{ diameter } (K),$$

so $k_{i_1 i_2 \ldots}$ is a single point. Conversely, each point $x \in K$ has the form $x = k_{i_1 i_2 \ldots}$ for some sequence of indices, because there is a nested chain $K \supset K_{i_1} \supset K_{i_1 i_2} \supset \ldots$ containing x. Notice that some points may have multiple representations.

Recall that V is a bounded open set satisfying $\bigcup_{i=1}^{m} \psi_i(V) \subset V$ and $\psi_i(V) \cap \psi_j(V) = \emptyset$ if $i \neq j$. It follows that for any sequence $i_1 i_2 \ldots, \overline{V} \supset \overline{V}_{i_1} \supset \overline{V}_{i_1 i_2} \supset \ldots$ is a nested sequence of nonempty, compact sets with diameters converging to zero, hence $\bigcap_n \overline{V}_{i_1 i_2 \ldots i_n}$ is a single point. This point must be $k_{i_1 i_2 \ldots}$, because for each $n, K_{i_1 i_2 \ldots i_n} \cap \overline{V}_{i_1 i_2 \ldots i_n} \neq \emptyset$, since $K \cap V \neq \emptyset$. It follows that each $k_{i_1 i_2 \ldots}$ is an element of $\overline{V}$; thus

$$K \subset \overline{V} \text{ and } K_{i_1 i_2 \ldots i_n} \subset \overline{V}_{i_1 i_2 \ldots i_n} \tag{3.1}$$

for any n and any sequence $i_1 i_2 \ldots$. Note that for any finite sequences $i_1 i_2 \ldots i_n$ and $i'_1 i'_2 \ldots i'_{n'}$ that do *not* agree in all of the first $(n \wedge n')$ entries,

$$V_{i_1 i_2 \ldots i_n} \cap V_{i'_1 i'_2 \ldots i'_{n'}} = \emptyset. \tag{3.2}$$

Lemma 1: *There exist a finite sequence $j_1 j_2 \ldots j_\ell$ of indices such that $K_{j_1 j_2 \ldots j_\ell} \subset V$.*

Proof: By hypothesis $K \cap V \neq \emptyset$, so there is a point $k_{j_1 j_2 \ldots} \in V$. But the sets $K_{j_1 j_2 \ldots j_n}$ shrink to $k_{j_1 j_2 \ldots}$ as $n \to \infty$ (the diameters converge to zero) and V is open; hence, for ℓ sufficiently large, $K_{j_1 j_2 \ldots j_\ell} \subset V$. □

Proposition 1: *Let $p_1, p_2, \ldots, p_m$ be positive real numbers satisfying $p_1 + \ldots + p_m = 1$. There is a unique probability measure $\nu = \nu_{p_1, p_2, \ldots, p_m}$ satisfying $\nu(K) = 1$ and*

$$\nu(K_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \ldots p_{i_n} \tag{3.3}$$

for every finite sequence $i_1 i_2 \dots i_n$ *from* $\{1, 2, \dots, m\}$. *This probability measure also satisfies*

$$\nu(K_{i_1 i_2 \dots i_n} \cap K_{i'_1 i'_2 \dots i'_{n'}}) = 0 \tag{3.4}$$

for all sequences $i_1 i_2 \dots i_n$ *and* $i'_1 i'_2 \dots i'_{n'}$ *that do not agree in each of the first* $(n \wedge n')$ *entries.*

Proof: It follows from (3.1)–(3.2) and Lemma 1 that

$$K_{i_1 i_2 \dots i_n j_1 \dots j_\ell} \cap K_{i'_1 i'_2 \dots i'_{n'}} = \varnothing \tag{3.5}$$

unless $i_q = i'_q$ for $q = 1, 2, \dots, (n \wedge n')$.

Let ν be a probability measure satisfying (3.3) for all finite sequences $i_1 i_2 \dots i_n$. Since $p_{j_1} p_{j_2} \cdots p_{j_\ell} > 0$, a simple calculation shows that for any sequence $i_1 i_2 \dots i_n$

$$\nu(K_{i_1 i_2 \dots i_n}) = \nu(\bigcup_{\mathbb{X}} K_{x_1 x_2 \dots x_q})$$

where $\mathbb{X}$ is the set of all finite sequences $x_1 x_2 \dots x_q$ such that $q \geq n + \ell$, $x_s = i_s$ for $1 \leq s \leq n$, and $x_{q-\ell+s} = j_s$ for $1 \leq s \leq \ell$. Note that if $x_1 x_2 \dots x_q \in \mathbb{X}$ then $K_{x_1 x_2 \dots x_q} \subset K_{i_1 i_2 \dots i_n}$. It now follows from (3.5) that (3.4) must hold unless $i_s = i'_s$ for $s = 1, 2, \dots, (n \wedge n')$, because for any sequence $x_1 x_2 \dots x_q \in \mathbb{X}, K_{x_1 \dots x_q} \cap K_{i'_1 \dots i'_{n'}} = \varnothing$. Consequently, ν assigns probability 1 to the set of points in K with *unique* representations $k_{i_1 i_2 \dots}$.

Let X be a random point of K with distribution ν, where ν is a probability measure satisfying (3.3). By the preceding paragraph, X has a unique representation $k_{I_1 I_2 \dots}$ where $I_1, I_2, \dots$ are random variables valued in $\{1, 2, \dots, m\}$. But (3.3) requires that $I_1, I_2, \dots$ be i.i.d. with

$$P\{I_n = i\} = p_i, \quad i = 1, 2, \dots, m. \tag{3.6}$$

This uniquely determines ν.

Finally, let $I_1, I_2, \dots$ be i.i.d. with distribution (3.6), let $X = k_{I_1 I_2 \dots}$, and let ν be the distribution of X. With probability one, the finite sequence $j_1 j_2 \dots j_\ell$ occurs somewhere in the infinite sequence $I_1 I_2 \dots$, so $k_{I_1 I_2 \dots}$ is the unique representation of X. Hence, ν satisfies (3.4). It follows that

$$\begin{aligned}\nu(K_{i_1 i_2 \dots i_n}) &= P\{I_1 = i_1, \dots, I_n = i_n\} \\ &= p_{i_1} p_{i_2} \cdots p_{i_n}\end{aligned}$$

for any finite sequence $i_1 i_2 \dots i_n$; this proves that there is a probability measure on K satisfying (3.3). □

4. Hausdorff Dimension < 1

Recall that the similarity exponent θ is the unique real number such that $\Sigma r_i p_i^\theta = 1$. If $\theta < 0$ then clearly $\Sigma r_i < 1$, so the Hausdorff dimension of K is less than 1.

Lemma 2: *If $\Sigma r_i < 1$ then there exists a rectifiable path that visits every point of K.*

Proof: Let $d =$ diameter (K) and let $x \in K$. For each finite sequence $i_1 i_2 \dots i_n$ of indices define $x(i_1 i_2 \dots i_n) = \psi_{i_1} \circ \psi_{i_2} \circ \dots \circ \psi_{i_n}(x)$. There is obviously a closed path γ_1 beginning and ending at x that visits $x(1), \dots, x(m)$ in that order, such that length $(\gamma_1) \le (m+1)d$. Define closed paths $\gamma_2, \gamma_3, \dots$, each beginning and ending at x, recursively, as follows. To obtain γ_{n+1}, follow γ_1 from x to $x(1)$; then follow $\psi_1(\gamma_n)$ from $x(1)$ back to $x(1)$; then follow γ_1 from $x(1)$ to $x(2)$; then follow $\psi_2(\gamma_n)$ from $x(2)$ back to $x(2)$; ...; finally, follow γ_1 from $x(m)$ back to x. Note that γ_n visits x and each $x(i_1 i_2 \dots i_k), k \le n$. Also

$$\text{length } (\gamma_{n+1}) = \text{ length } (\gamma_1) + \sum_{i=1}^{m} r_i \text{ length } (\gamma_n)$$

$$\Longrightarrow \text{ length } (\gamma_n) = \left(\sum_{k=0}^{n} \left(\sum_{i=1}^{m} r_i \right)^k \right) \text{ length } (\gamma_1)$$

$$\le (1 - \Sigma r_i)^{-1}(m+1)d \quad \forall\, n.$$

Assume that each γ_n is parametrized by arclength. It is easy to see that as $n \to \infty$ the paths γ_n converge uniformly to a path γ with arclength $\le (1 - \Sigma r_i)^{-1}(m+1)d$. (In fact, all we need is a subsequence γ_{n_k} converging uniformly to a path γ, and this follows from the Arzela-Ascoli theorem, since $\gamma_n, n \ge 1$, are uniformly equicontinuous.) The path γ must visit every point $x(i_1 i_2 \dots i_k)$, since γ_n does for every $n \ge k$; consequently, γ visits every point of K since $\{x(i_1 i_2 \dots i_k) : i_1 \dots i_k \text{ any finite sequence}\}$ is dense in K(because $x(i_1 i_2 \dots i_k) \in K_{i_1 i_2 \dots i_k}$). □

Proof of Th. 1: By hypothesis, the support of ν is a closed subset of K. Define $C = \inf_{\gamma \in \Gamma}$ length γ, where Γ is the set of all continuous paths which visit every point of support (ν). By Lemma 2, $C < \infty$.

Let ζ_n be a minimal path through $X_1, \dots, X_n$. Then $L_n =$ length $\zeta_n \le C$, since $X_1, X_2, \dots, X_n \epsilon$ support (ν). Assume that ζ_n is parametrized by arc length; then $\{\zeta_n\}_{n \ge 1}$ is a uniformly equicontinuous family of functions, so there exists a subsequence $n_k \uparrow \infty$ such that ζ_{n_k} converges uniformly to a continuous path ζ. With probability one, ζ visits every point of support (ν), because ζ visits $X_1, X_2, \dots$ and $\{X_1, X_2, \dots\}$ is dense in support (ν).

Hence, length $(\zeta) \geq C$ a.s. But since ζ_{n_k} converges uniformly to ζ and ζ_{n_k} is parametrized by arc length, length $(\zeta_{n_k}) \to$ length (ζ). Finally, since length (ζ_n) is nondecreasing in n, we must have $L_n \uparrow C$. □

5. The Circle Freeway Lemma

The proofs of Th. 2 and Th. 3 use a "Poissonization" argument. Let $X_1, X_2, \ldots$ be i.i.d. with distribution $\nu = \nu_{p_1,\ldots,p_m}$ (cf. Prop. 1), and let L_n be the length of the shortest path through $X_1, \ldots, X_n$. Let $N(t), t \geq 0$, be a rate 1 Poisson process independent of $X_1, X_2, \ldots$. Then $N(t)/t \to 1$ a.s. as $t \to \infty$. Since L_n is nondecreasing in n, we can easily recover the asymptotic behavior of L_n as $n \to \infty$ from the asymptotic behavior of $L_{N(t)}$ as $t \to \infty$. The advantage of introducing $N(t)$ is that if $U_1, U_2, \ldots, U_m$ are pairwise disjoint then the random sets $U_i \cap \{X_1, \ldots, X_{N(t)}\}, i = 1, 2, \ldots, m$, are independent.

For each $t > 0$ and each finite sequence $i_1 i_2 \ldots i_n$ of indices from $\{1, 2, \ldots, m\}$, define $\lambda(t; i_1 i_2 \ldots i_n)$ to be the length of the shortest path through $\{X_1, X_2, \ldots, X_{N(t)}\} \cap K_{i_1 i_2 \ldots i_n}$, and define $\lambda(t) = L_{N(t)}$.

Lemma 3: *Let $i_1^{(s)} i_2^{(s)} \ldots i_{n(s)}^{(s)}, s = 1, 2, \ldots, q$ be finite sequences of indices from $\{1, 2, \ldots, m\}$ such that no two sequences satisfy $i_j^{(s)} = i_j^{(s')}$ for $j = 1, 2, \ldots, (n(s) \wedge n(s'))$. Then for each $t > 0$ the random variables $\lambda\left(t; i_1^{(s)} i_2^{(s)} \ldots i_{n(s)}^{(s)}\right), s = 1, \ldots, q$ are independent. Moreover, for any $t > 0$ and any sequence $i_1 i_2 \ldots i_n$, the random variable $\lambda(t; i_1 i_2 \ldots i_n)$ has the same distribution as $r_{i_1} r_{i_2} \ldots r_{i_n} \lambda(p_{i_1} p_{i_2} \ldots p_{i_n} t)$.*

Proof: This is a routine consequence of (3.4) and the self-similarity of K and ν. □

Lemma 4: *There exists a constant $C < \infty$ such that for every $t > 0$*

$$\sum_{i=1}^{m} \lambda(t; i) - C \leq \lambda(t) \leq \sum_{i=1}^{m} \lambda(t; i) + C. \tag{5.1}$$

Proof: Construct a path through $X_1, \ldots, X_{N(t)}$ as follows. First construct the shortest path through $\{X_1, \ldots, X_{N(t)}\} \cap K_i$ for each $i = 1, 2, \ldots, m$, then join these m paths together in sequence. The length of this path is $\leq \Sigma_{i=1}^{m} \lambda(t; i) + C$ for some $C < \infty$ independent of t (by the compactness of $K_1, \ldots, K_m$). This proves the upper bound.

Next, consider the shortest path γ through $X_1, X_2, \ldots, X_{N(t)}$. The initial and terminal points of γ lie in $V_i, V_{i'}$, say; extend γ from its endpoints to a path $\overline{\gamma}$ whose initial and terminal points lie on $\partial V_i, \partial V_{i'}$ for some i, i'. This may be done in such a way that length $(\overline{\gamma}) \leq$ length $(\gamma) + 2 \max_{1 \leq i \leq m}$ diameter (V_i). Now for any $i \in \{1, 2, \ldots, m\}$, the intersection of $\overline{\gamma}$ with $\overline{V}_i$ consists of a finite collection of paths $\gamma_j, j = 1, 2, \ldots, q$ in $\overline{V}_i$, each having its endpoints on ∂V_i. Since ∂V_i is a simple, closed, rectifiable curve, Lemma 5 below implies that $\gamma_1, \gamma_2, \ldots, \gamma_q$ may be joined together to form a single path $\gamma^{(i)}$ satisfying length $(\gamma^{(i)}) \le \sum_{j=1}^{q}$ length $(\gamma_j) + 2$ length (∂V_i). This path $\gamma^{(i)}$ goes through each point in $\{X_1, X_2, \ldots, X_{N(t)}\} \cap K_i$, because $K_i \subset \overline{V}_i$ and the original path $\overline{\gamma}$ goes through $X_1, \ldots, X_{N(t)}$. Thus

$$\sum_{i=1}^{m} \lambda(t;i) \le \lambda(t) + 2 \max_{1 \le i \le m} \text{ diam } (V_i) + 2 \sum_{i=1}^{m} \text{ length } (\partial V_i). \qquad \square$$

Corollary 1: *If $C < \infty$ is as in Lemma 4 then for every $t < \infty$ and every finite sequence $i_1 i_2 \ldots i_n$ of indices,*

$$\begin{aligned}(5.2) \qquad & \sum_{i=1}^{m} \lambda(t; i_1 i_2 \ldots i_n i) - C r_{i_1} r_{i_2} \ldots r_{i_n} \\ & \le \lambda(t; i_1 i_2 \ldots i_n) \\ & \le \sum_{i=1}^{m} \lambda(t; i_1 i_2 \ldots i_n i) + C r_{i_1} r_{i_2} \ldots r_{i_n}\end{aligned}$$

Proof: The self-similarity of K and ν imply that the joint distribution of $\lambda(t; i_1 i_2 \ldots i_n)$, $\lambda(t; i_1 i_2 \ldots i_n 1), \ldots, \lambda(t; i_1 i_2 \ldots i_n m)$ is the same as that of $\left(\prod_{j=1}^{n} r_{i_j}\right) \lambda(t)$, $\left(\prod_{j=1}^{n} r_{i_j}\right) \lambda(t;1), \ldots, \left(\prod_{j=1}^{n} r_{i_j}\right) \lambda(t;m)$. Thus, (5.2) follows from (5.1). $\square$

Lemma 5 *(Circle Freeway Lemma): Let $\gamma_1, \gamma_2, \ldots, \gamma_q$ be piecewise smooth paths in a domain U whose boundary ∂U is a simple, closed, rectifiable curve. Assume that for each i the endpoints of γ_i are on ∂U. Then there is a path γ in $(\partial U) \cup \left(\bigcup_{i=1}^{q} \gamma_i\right)$ that goes through every point of $\bigcup_{i=1}^{q} \gamma_i$ such that*

$$(5.3) \qquad \text{length } (\gamma) \le \sum_{i=1}^{q} \text{ length } (\gamma_i) + 2 \text{ length } (\partial U).$$

Proof: The $2q$ endpoints of $\gamma_1, \ldots, \gamma_q$ all lie on ∂U; label these points $Q_1, Q_2, \ldots, Q_{2q}$ so that they appear in clockwise order around ∂U. The paths γ_i, γ_j may meet inside U at "traffic lights" and may coincide on road segments between traffic lights. Define a graph G whose vertices are the traffic lights and the points $Q_1, \ldots, Q_{2q}$ and with edges as follows. For vertices V and V' there is one edge between V and V' for *each* path γ_i that goes through V and V' without going through any other vertex in between. Also, there is *one* edge between Q_{2i-1} and Q_{2i} and *two* edges between Q_{2i} and Q_{2i+1} for each $i = 1, 2, \ldots, q$ (here $Q_{2q+1} = Q_1$) *in addition to* any edges between Q_i, Q_j already in place because of paths γ_s that connect Q_i, Q_j without going through any vertex in between. The graph G is connected and each vertex has even degree. Therefore, by Euler's "Konigsburg bridges" theorem ([2], Ch. 1, Th. 10) there is an Eulerian circuit. The Eulerian circuit determines a path γ with the desired property.

(Note: It is possible that some of the points Q_i may coincide. If this is the case, they should be listed according to multiplicity, and in the construction of the graph G each should be counted as a separate vertex. Thus, some of the edges in G may correspond to paths of length zero in $(\partial U) \cup \left(\bigcup_{i=1}^{q} \gamma_i\right)$. This does not affect the validity of (5.3).) □

6. Expectation of $\lambda(t)$ when $\delta > 1$

Recall that the similarity exponent θ of $\nu = \nu_{p_1,\ldots,p_m}$ is the unique real number such that $\sum_{i=1}^{m} r_i p_i^\theta = 1$. Observe that $\theta > 0$ iff $\Sigma r_i > 1$ iff $\delta > 1$.

Lemma 6: *If $\theta > 0$ then $E\lambda(t) \uparrow \infty$ as $t \to \infty$.*

Proof: $E\lambda(t)$ is clearly monotone in t. Let $\mathbf{i}^{(s)} = i_1^{(s)} i_2^{(s)} \ldots i_n^{(s)}, s = 1, 2, \ldots, m^n$ be all the finite sequences of indices of length n from $\{1, 2, \ldots, m\}$. Then

$$\lim_{t\to\infty} P\{\{X_1, X_2, \ldots, X_{N(t)}\} \cap K_{\mathbf{i}^{(s)}} \neq \varnothing \ \ \forall\, s = 1, 2, \ldots, m^n\} = 1;$$

consequently, it suffices to prove that the length of the shortest path through $K_{\mathbf{i}^{(1)}}, \ldots, K_{\mathbf{i}^{(m^n)}}$ converges to ∞ as $n \to \infty$.

Any path through all the sets $K_{\mathbf{i}^{(s)}}, s = 1, 2, \ldots, m^n$ must pass through each set of the subcollection $K_{\mathbf{i}^{(s)}}, s = 1, 2, \ldots, m^{n-\ell}$ in which $\mathbf{i}^{(s)}$ has the form $\mathbf{i}^{(s)} = i_1 i_2 \ldots i_{n-\ell} j_1 j_2 \ldots j_\ell$, where $i_1, i_2, \ldots, i_{n-\ell}$ are arbitrary and $j_1 j_2 \ldots j_\ell$ is the sequence constructed in Lemma 1. By (3.1)–(3.2) and Lemma 1,

$$\text{distance}\,(K_{\mathbf{i}^{(s)}}, K_{\mathbf{i}^{(s_*)}}) \geq r_{i_1} r_{i_2} \ldots r_{i_{n-\ell}} d,$$

where $d =$ distance $(K_{j_1 \ldots j_\ell}, \partial V)$ and $\mathbf{i}^{(s)} = i_1 i_2 \ldots i_{n-\ell} j_1 \ldots j_\ell, \mathbf{i}^{(s_*)} = i'_1 i'_2 \ldots i'_{n-\ell} j_1 \ldots j_\ell$ are distinct sequences. Now any path that goes through all the sets $K_{\mathbf{i}^{(s)}}$, where $\mathbf{i}^{(s)} = i_1 \ldots i_{n-\ell} j_1 \ldots j_\ell$, must exit each on its way to another, with at most one exception. Therefore, its length must be at least

$$\sum_{i_1 i_2 \ldots i_{n-\ell}} r_{i_1} r_{i_2} \ldots r_{i_{n-\ell}} d - d = \left(\sum_{i=1}^{m} r_i\right)^{n-\ell} d - d.$$

Since $\Sigma r_i > 1$, this becomes large as $n \to \infty$. □

Proposition 2: *Assume that $\theta > 0$.*

(a) (Nonarithmetic case) If $\log p_1, \log p_2, \ldots, \log p_m$ *are not contained in any discrete additive subgroup of* $\mathbb{R}$ *then there is a constant* $\mu > 0$ *such that*

$$\lim_{t \longrightarrow \infty} E\lambda(t)/t^\theta = \mu. \tag{6.1}$$

(b) (Arithmetic case) If $\log p_1, \log p_2, \ldots, \log p_m$ *are contained in* $h\mathbb{Z}$ *but not in a proper subgroup of* $h\mathbb{Z}$ *then there is a continuous monotone function* $\mu(\beta), 0 \leq \beta \leq h$, *such that* $\mu(\beta) > 0$ *and*

$$\lim_{n \longrightarrow \infty} E\lambda(\exp\{nh + \beta\})/e^{nh\theta} = \mu(\beta) \tag{6.2}$$

uniformly for $0 \leq \beta \leq h$.

Proof: Consider first the nonarithmetic case. Let C be the constant in the inequalities (5.3)–(5.4). Fix t_* (large). Then Lemma 4 and Corollary 1 imply that

$$\begin{aligned} &\Sigma_t \lambda(t; i_1 i_2 \ldots i_n) - \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \\ \leq &\lambda(t) \\ &\Sigma_t \lambda(t; i_1 i_2 \ldots i_n) + \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \end{aligned}$$

where the sum Σ_t ranges over all finite sequences $i_1 i_2 \ldots i_n, n \geq 1$, satisfying

$$-\sum_{j=1}^{n-1} \log p_{i_j} < \log(t/t_*) \leq -\sum_{j=1}^{n} \log p_{i_j}$$

and the sum Σ_t^* ranges over all finite sequences $i_1 i_2 \ldots i_n, n \geq 0$($n = 0$ is the empty sequence), satisfying

$$-\sum_{j=1}^{n} \log p_{i_j} < \log(t/t_*).$$

It follows from Lemma 3 that

$$\begin{aligned} &\Sigma_t r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(p_{i_1} \ldots p_{i_n} t) - \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \\ \leq &E\lambda(t) \\ \leq &\Sigma_t r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(p_{i_1} \ldots p_{i_n} t) + \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n}. \end{aligned}$$

Define $G(t)$ and $H(t)$ by

$$\begin{aligned} G(t) &= \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \text{ and} \\ H(t) &= \Sigma_t r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(p_{i_1} p_{i_2} \ldots p_{i_n} t) \end{aligned}$$

For each of these there is a functional equation, which may be obtained by conditioning on the first coordinate i_1:

$$G(t) = \sum_{i=1}^{m} r_i G(p_i t) + C1\{t > t_*\}$$

and

$$H(t) = \sum_{i=1}^{m} r_i H(p_i t) + R(t),$$

where

$$R(t) = \sum_{i=1}^{m} r_i E\lambda(p_i t) 1\{t_* \geq p_i t > p_i t_*\}.$$

Each of these may be rewritten as a renewal equation, using the fact that $\Sigma r_i p_i^\theta = 1$:

$$e^{-\theta s} G(e^s) = \sum_{i=1}^{m} (r_i p_i^\theta)(p_i^{-\theta} G(p_i e^s)) + Ce^{-\theta s} 1\{s > \log t_*\}$$

and

$$e^{-\theta s} H(e^s) = e^{-\theta s} R(e^s) + \sum_{i=1}^{m} (r_i p_i^\theta)\left(p_i^{-\theta} e^{-\theta s} H(p_i e^s)\right).$$

Therefore, the renewal theorem for $\mathbb{R}$ implies that

$$\lim_{t \longrightarrow \infty} G(t)/t^\theta = C \int_{\log t_*}^{\infty} e^{-\theta s} ds/\gamma = C_1(t_*)$$

$$\lim_{t \longrightarrow \infty} H(t)/t^\theta = \int e^{-\theta s} R(e^s) ds/\gamma = C_2(t_*)$$

where $\gamma = \sum_{i=1}^{m} (r_i p_i^\theta)(-\log p_i) > 0$.

Now observe that for any $\epsilon > 0$, if t_* is chosen sufficiently large then $C_1(t_*) < \epsilon C_2(t_*)$. This is because $E\lambda(t) \uparrow \infty$ (Lemma 6) and

$$\frac{C_1(t_*)}{C_2(t_*)} = \frac{C t_*^{-\theta}}{\theta \sum_{i=1}^{m} r_i \int_{\log p_i t_*}^{\log t_*} e^{-\theta s} E\lambda(p_i e^s) ds}.$$

We have already shown that

$$-G(t) \leq E\lambda(t) - H(t) \leq G(t)$$

Since $\epsilon > 0$ is arbitrary, (6.1) follows.

The arithmetic case is similar – it uses the renewal theorem for $\mathbb{Z}$. The details are omitted. □

7. Expectation of $\lambda(t)$ when $\delta = 1$ and K is Strongly Disconnected

The set K is strongly disconnected if $K_1, K_2, \ldots, K_m$ are pairwise disjoint. When this is the case, the open set V may be chosen so that the compact sets $\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_m$ are pairwise disjoint. Recall that $\delta = 1$ iff $\sum_{i=1}^{m} r_i = 1$. *Assume throughout this section that* $\delta = 1$ *and that* K *is strongly disconnected.*

Lemma 7: *For* $n \geq 1$, *let* β_n *be the length of the shortest path that visits each of the* m^n *sets* $\overline{V}_{i_1 i_2 \ldots i_n}$. *Then* $\lim_{n \longrightarrow \infty} \beta_n = \infty$.

Proof: Let γ_n be the shortest path, and let $\overline{\gamma}_n$ be the closed curve obtained by connecting the endpoints of γ_n. Then length $(\overline{\gamma}_n) \leq$ length $(\gamma_n) +$ diameter (V), so to show that γ_n is long it suffices to show that $\overline{\gamma}_n$ is long.

Define $d = \min_{i \neq j}$ distance $(\overline{V}_i, \overline{V}_j) > 0$. If $i_1 i_2 \ldots i_n$ and $i'_1 i'_2 \ldots i'_n$ are any two distinct sequences of indices from $\{1, 2, \ldots, m\}$ then

$$\text{(7.1)} \qquad \text{distance } (\overline{V}_{i_1 i_2 \ldots i_n}, \overline{V}_{i'_1 i'_2 \ldots i'_n}) \geq r_{i_1} r_{i_2} \ldots r_{i_k} d,$$

where $k = \max \{q : i_j = i'_j \ \ \forall\, 1 \leq j \leq q\}$, by self-similarity and the nesting property $\overline{V} \supset \overline{V}_{i_1} \supset \overline{V}_{i_1 i_2} \supset \ldots$. Let $\mathbf{i}^{(1)}, \mathbf{i}^{(2)}, \ldots, \mathbf{i}^{(m^n)}$ be the distinct sequences of length n. Each $\overline{V}_{\mathbf{i}^{(s)}}$ has a segment of $\overline{\gamma}_n$ going out; hence (7.1) implies that

$$\text{(7.2)} \qquad \text{length } (\overline{\gamma}_n) \geq \sum_{i_1, \ldots, i_n = 1}^{m} r_{i_1} r_{i_2} \ldots r_{i_{n-1}} d = \left(\sum_{i=1}^{m} r_i \right)^{n-1} md = md.$$

But this is a rather crude estimate, since some $\overline{V}_{\mathbf{i}^{(s)}}$ may have an outgoing segment connecting it to some $\overline{V}_{\mathbf{i}^{(s')}}$ where $\mathbf{i}^{(s')}$ differs from $\mathbf{i}^{(s)}$ in some coordinate before the nth. Thus, some of the terms $r_{i_1} r_{i_2} \ldots r_{i_{n-1}} d$ in the above sum may be upgraded to $r_{i_1} r_{i_2} \ldots r_{i_k} d$ for some $k < n - 1$. In fact, for *each* sequence $i_1 i_2 \ldots i_k, k < n$, there is at least one sequence $i_{k+1} i_{k+2} \ldots i_n$ such that $\overline{V}_{i_1 i_2 \ldots i_n}$ has an outgoing segment to some $\overline{V}_{i'_1 i'_2 \ldots i'_n}$ where $i_1 i_2 \ldots i_k \neq i'_1 i'_2 \ldots i'_k$. For any such sequence $i_1 i_2 \ldots i_n$ the term $r_{i_1} r_{i_2} \ldots r_{i_{n-1}} d$ in the estimate (7.2) may be upgraded to $r_{i_1} r_{i_2} \ldots r_{i_{k-1}} d$. This upgrading may be done one step at a time, first from $n - 1$ to $n - 2$, then $n - 2$ to $n - 3$, etc. At each step the size of the deleted term is no more than $\overline{r}$ times the upgraded term, where $\overline{r} = \max (r_1, r_2, \ldots, r_m) < 1$. Thus (7.2) may be improved to

$$\begin{aligned} \text{length } (\overline{\gamma}_n) &\geq \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k = 1}^{m} r_{i_1} r_{i_2} \ldots r_{i_{k-1}} d(1 - \overline{r}) \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{m} r_i \right)^{k-1} md(1 - \overline{r}) = nmd(1 - \overline{r}). \end{aligned}$$

It follows that length $(\overline{\gamma}_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. □

Lemma 8: *Let $p_n(t) = P\{\{X_1, X_2, \ldots, X_{N(t)}\} \cap \overline{V}_{i_1 i_2 \ldots i_n} = \emptyset$ for some sequence $i_1 i_2 \ldots i_n\}$. Then for any $t > 0$ and $n = 1, 2, \ldots$,*

$$\begin{aligned}(7.3)\qquad &\sum_{i_1 i_2 \ldots i_n} r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(tp_{i_1} \ldots p_{i_n}) + \beta_n(1 - p_n(t)) - 2 \text{ length } (\partial V)\\ &\leq E\lambda(t)\\ &\leq \sum_{i_1 i_2 \ldots i_n} r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(tp_{i_1} \ldots p_{i_n}) + \beta_n + 2 \text{ diam } (V)\end{aligned}$$

Proof: This is a refinement of Lemma 4 and may be proved in a similar manner. The difference here is that the sets $\overline{V}_{\mathbf{i}^{(1)}}, \ldots, \overline{V}_{\mathbf{i}^{(m^n)}}$, where $\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(m^n)}$ are the distinct sequences of length n, are pairwise disjoint and therefore are separated by positive distances.

Construct the shortest path through $\{X_1, \ldots, X_{N(t)}\} \cap \overline{V}_{\mathbf{i}^{(s)}}$ for each sequence $\mathbf{i}^{(s)}$ of length n, then join these together. This may be done in such a way that the length of the resulting path is $\leq \Sigma_{i_1 i_2 \ldots i_n}\{\lambda(t; i_1 \ldots i_n) + 2 \text{ diam } (\overline{V}_{i_1 i_2 \ldots i_n})\} + \beta_n$. By Lemma 3, $E\lambda(t; i_1 i_2 \ldots i_n) = r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(tp_{i_1} \ldots p_{i_n})$. Moreover, $\Sigma_{i_1 i_2 \ldots i_n} \text{ diam } (\overline{V}_{i_1 i_2 \ldots i_n}) = \Sigma_{i_1 \ldots i_n} r_{i_1} \ldots r_{i_n} \text{ diam } (V) = \text{ diam } (V)$, since $\Sigma r_i = 1$. This proves the upper bound.

The proof of the lower bound is virtually the same as the proof of the lower bound in Lemma 4. □

Proposition 3: *There exists a constant $0 < C < \infty$ depending only on K (not on $p_1, p_2, \ldots, p_m$) such that*

$$(7.4)\qquad \lim_{t \longrightarrow \infty} \frac{E\lambda(t)}{\log t} = C / \sum_{i=1}^{m} r_i \log p_i^{-1}$$

Proof: Choose $\epsilon > 0$ small. By Lemma 7 there is an $n \geq 1$ such that 2 length $(\partial V) + 2 \text{ diam } (V) < \epsilon\beta_n$. Fix t_* so large that $p_n(t) < \epsilon$ for all $t \geq \underline{p} t_*$, where $\underline{p} = \min(p_1, \ldots, p_m)^n > 0$. By Lemma 8, if $t \geq t_*$ then

$$\begin{aligned}&\Sigma_t E\lambda(tp_{i_1} \ldots p_{i_{nk}}) r_{i_1} \ldots r_{i_{nk}} + (1 - 2\epsilon)\beta_n \Sigma_t^* r_{i_1} \ldots r_{i_{nk}}\\ &\leq E\lambda(t)\\ &\leq \Sigma_t E\lambda(tp_{i_1} \ldots p_{i_{nk}}) r_{i_1} \ldots r_{i_{nk}} + (1 + \epsilon)\beta_n \Sigma_t^* r_{i_1} \ldots r_{i_{nk}}\end{aligned}$$

where the sum Σ_t ranges over all sequences $i_1 i_2 \ldots i_{nk}$, $k \geq 1$, satisfying

$$-\sum_{j=1}^{nk-1} \log p_{i_j} < \log(t/t_*) \leq -\sum_{j=1}^{nk} \log p_{i_j}$$

and the sum Σ_t^* ranges over all sequences $i_1 i_2 \ldots i_{nk}$, $k \geq 1$, satisfying

$$-\sum_{j=1}^{nk} \log p_{i_j} < \log(t/t_*).$$

Define $G(t)$ and $H(t)$ by

$$G(t) = \Sigma_t^* r_{i_1} r_{i_2} \dots r_{i_{nk}} \text{ and}$$
$$H(t) = \Sigma_t r_{i_1} r_{i_2} \dots r_{i_{nk}} E\lambda(p_{i_1} \dots p_{i_{nk}} t).$$

We have shown that

$$H(t) + (1-2\epsilon)\beta_n G(t) \leq E\lambda(t) \leq H(t) + (1+\epsilon)G(t).$$

The function $H(t)$ satisfies a functional equation equivalent to a renewal equation; so the renewal theorem implies that $H(t) = 0(1)$ as $t \to \infty$. On the other hand, the "elementary renewal theorem" ([3], Th. 5.52) implies that

$$\lim_{t \longrightarrow \infty} G(t)/\log t = 1/\sum_{i_1 \dots i_n} r_{i_1} \dots r_{i_n} \log(p_{i_1} \dots p_{i_n})^{-1}$$
$$= 1/n \sum_i r_i \log p_i^{-1}.$$

Therefore

$$\frac{(1-2\epsilon)\beta_n}{n\Sigma r_i \log p_i^{-1}} \leq \liminf_{t \longrightarrow \infty} \frac{E\lambda(t)}{\log t} \leq \limsup_{t \longrightarrow \infty} \frac{E\lambda(t)}{\log t} \leq \frac{(1+\epsilon)\beta_n}{n\Sigma r_i \log p_i^{-1}}.$$

Since $\epsilon > 0$ is arbitrary, (7.4) follows, with $C = \lim_{n \to \infty} (\beta_n/n)$. □

8. Almost Sure Convergence

Theorems 2–3 follow from Props. 2–3 and Lemma 4 by routine arguments. The key is the following.

Lemma 9: *If* $0 < \delta < 2$ *then var* $(\lambda(t)) = 0(1)$ *as* $t \to \infty$. *If* $\delta = 2$ *then var* $(\lambda(t)) = 0(\log t)$ *as* $t \to \infty$.

Note: In fact, Steele [7] has shown that var $(\lambda(t))$ is $0(1)$ even when $\delta = 2$. This is not needed for the proofs of Th. 2–3, however.

Proof: Lemmas 3–4 imply that

(8.1) $$\text{var}\,(\lambda(t)) \leq \sum_{i=1}^m r_i^2 \text{var}\,(\lambda(tp_i)) + C$$

for a suitable constant $C < \infty$. If $0 < \delta < 2$ then $\Sigma_{i=1}^m r_i^2 < 1$; consequently, iteration of (8.1) gives

$$\text{var}\,(\lambda(t)) \leq C \sum_{k=0}^{\infty} \left(\sum_{i=1}^m r_i^2 \right)^k < \infty.$$

If $\delta = 2$ then $\sum_{i=1}^{m} r_i^2 = 1$. Let $\overline{p} = \max(p_1, \ldots, p_m) < 1$, and let k be the smallest integer larger than $(\log t)/(\log \overline{p}^{-1})$. Iterating (8.1) k times gives

$$\begin{aligned}
\text{var }(\lambda(t)) &\leq \sum_{i_1,\ldots,i_k=1}^{m} r_{i_1}^2 r_{i_2}^2 \ldots r_{i_k}^2 \text{ var } \lambda(tp_{i_1} \ldots p_{i_k}) \\
&+ \sum_{j=0}^{k-1} C \sum_{i_1,i_2,\ldots,i_j=1}^{m} r_{i_1}^2 r_{i_2}^2 \ldots r_{i_j}^2 \\
&\leq \left(\sum_{i=1}^{m} r_i^2 \right)^k E\lambda(1)^2 + C \sum_{j=0}^{k-1} \left(\sum_{i=1}^{m} r_i^2 \right)^j \\
&= E\lambda(1)^2 + Ck;
\end{aligned}$$

thus var $(\lambda(t)) = 0(\log t)$ as $t \to \infty$. □

Proof of Th. 2: Consider first the nonarithmetic case. By Lemma 9, var $(\lambda(t)) = 0(\log t)$, and by Prop. 2, $E\lambda(t) \sim \mu t^\theta$ as $t \to \infty$. Chebychev's inequality implies that for any $\alpha > 0, \epsilon > 0$,

$$P\{|\lambda((1+\alpha)^n)/(1+\alpha)^{n\theta} - \mu| > \epsilon\} \leq (\text{constant})\epsilon^2 n/(1+\alpha)^{2n\theta};$$

consequently, the Borel-Cantelli Lemma implies that $\lambda((1+\alpha)^n)/(1+\alpha)^{n\theta} \to \mu$ a.s. Since $\alpha > 0$ is arbitrary and $\lambda(t)$ is nondecreasing in t it follows that $\lambda(t)/t^\theta \to \mu$ a.s. as $t \to \infty$. Finally, recall that $\lambda(t) = L_{N(t)}$ where $N(t)$ is a rate 1 Poisson process; since $N(t)/t \to 1$ a.s., we must have

$$L_n/n^\theta \longrightarrow \mu \quad \text{a.s.} \qquad \text{as } n \longrightarrow \infty.$$

A similar argument applies in the arithmetic case. □

Proof of Th. 3: By Lemma 8, var $(\lambda(t)) = 0(1)$, and by Prop. 3, $E\lambda(t) \sim \mu \log t$. Consequently, Chebychev's inequality implies that for any $\epsilon > 0$

$$P\{|\lambda(e^n)/n - \mu| > \epsilon\} \leq (\text{constant})\epsilon^2/n^2$$

from which it follows, by the Borel-Cantelli Lemma, that $\lambda(e^n)/n \to \mu$ a.s. Since $\lambda(t)$ is monotone in $t, \lambda(t)/\log t \to \mu$ a.s. as $t \to \infty$. Since $\lambda(t) = L_{N(t)}$ where $N(t)$ is a standard Poisson process, it follows that $L_n/\log n \to \mu$ a.s. as $n \to \infty$. □

9. Periodicity

The purpose of this section is to give a heuristic argument explaining why the function $C(t)$ occurring in (2.3) may, in general, be non-constant. For simplicity we shall restrict our attention to the special case in which $K = C \times C$, where C is the usual Cantor set, and $\nu = \mu_C \times \mu_C$, where μ_C is the Cantor measure. The self-similarity structure is as follows: V = open unit square; $\psi_1, \psi_2, \psi_3, \psi_4$ are the affine mappings taking V onto the four disjoint squares $\left(\frac{i}{3}, \frac{i+1}{3}\right) \times \left(\frac{j}{3}, \frac{j+1}{3}\right), i, j \in \{0, 2\}; r_1 = r_2 = r_3 = r_4 = 1/3$; and $p_1 = p_2 = p_3 = p_4 = 1/4$.

The arguments of secs. 5–8 suggest that a nearly minimal path through $X_1, X_2, \ldots, X_n$ may be constructed as follows. Fix a large (an integer), and let $k = [a(\log n)/(\log 4)]$. For each of the 4^k sets $K_{i_1 i_2 \ldots i_k}$, assemble the minimal path through $\{X_1, \ldots, X_n\} \cap K_{i_1 i_2 \ldots i_k}$, then connect these 4^k paths to form a single path. The connection of the 4^k subpaths may be done by joining the sets $K_{i_1 i_2 \ldots i_k}$ in lexicographic order, e.g., if $k = 2, 11 \rightarrow 12 \rightarrow 13 \rightarrow \ldots \rightarrow 43 \rightarrow 44$. It doesn't matter if the lexicographic ordering is suboptimal, because most of the length of the complete path comes from the 4^k subpaths, not from the connections (provided the constant a is large).

NOTE: This is where the cases $\delta = 1$ and $\delta > 1$ differ. When $\delta = 1$ the hierarchical connections make up most of the total length of the path. This is why the periodicity phenomenon does not occur in Theorem 3.

The lengths of the 4^k paths through the sets $\{X_1, \ldots, X_n\} \cap K_{\mathbf{i}(s)}, s = 1, \ldots, 4^k$, are nearly i.i.d. random variables whose sum is the primary contribution to L_n. These lengths, after rescaling by a factor of 3^k, are determined by the relative configuration of the points $\{X_1, \ldots, X_n\}$ in the "squares" $K_{\mathbf{i}(s)}$. As n goes from $4^{k/a}$ to $4^{(k+1)/a}$ the distribution of the configuration of points in a square changes (but notice that this distribution is the same at $4^{k/a}$ and $4^{(k+1)/a}$). When n is somewhat greater than $4^{k/a}$ the probability of a configuration like that shown in Fig. 1 may be high, whereas when n is slightly less than $4^{(k+1)/a}$ the probability of a configuration like that shown in Fig. 2 may be high. Thus, the distribution of the length of the shortest path through $\{X_1, \ldots, X_n\} \cap K_{\mathbf{i}(s)}$ multiplied by 3^k should be genuinely periodic in $\log n / \log 4$, and therefore EL_n / n^θ (where $\theta = 1 - \log 3 / \log 4$) should also be genuinely periodic in $\log n / \log 4$.

Figure 1 here

Figure 2 here

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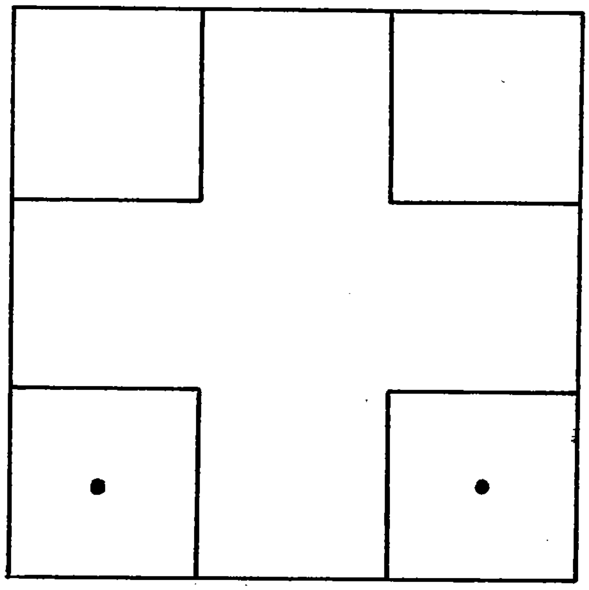

Figure 1

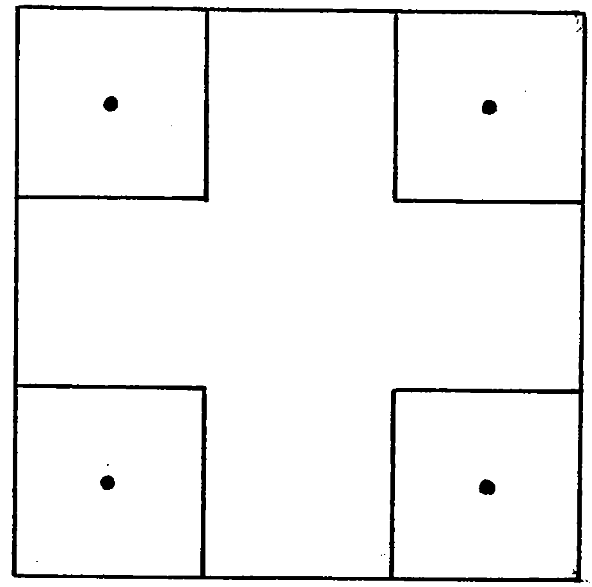

Figure 2