ON THE COVERAGE PROBABILITY OF SOME CONFIDENCE SETS FOR A MULTIVARIATE NORMAL MEAN *

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Technical Report #88-62

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December 1988

* AMS 1980 subject classifications: Primary 62E20, Secondary 62H10. Key Words and Phrases: Confidence sets, prior distribution, average coverage probability.

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ON THE COVERAGE PROBABILITY OF SOME CONFIDENCE SETS FOR A MULTIVARIATE NORMAL MEAN

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In 1981, Stein introduced a method of obtaining asymptotic approximations to the average coverage probability of confidence sets based on prior distributions. We shall adapt his method to the following problem. Let $X \sim N_p(\xi, \Sigma/n)$ and $S \sim W_p(\Sigma, n-1)$ where X and S are independent. Furthermore let H_i, $i = 1, \ldots, p$, denote the normalized eigenvectors of S. We are interested in getting approximations to the coverage probability of confidence sets for ξ of the form $\Omega = \{\xi : H_i'(\xi - X) \leq c(X, S)\}$ where $c(X, S)$ is a suitably chosen function. For each α strictly between 0 and 1, we shall show that $c(X, S)$ can be chosen such that the average coverage probability of Ω under an almost arbitrary twice (thrice) continuously differentiable prior density for (ξ, Σ) is asymptotically α with error of the order of n^{-1} $(n^{-3/2})$ respectively.

1 Introduction

In 1981, Stein introduced a simple but ingenious method of obtaining asymptotic approximations to the average coverage probability of confidence sets based on prior distributions. His method uses almost exclusively Taylor expansions. In this paper we shall adapt this method to the following problem.

Let $Y_1, \ldots, Y_n$ be independent $p \times 1$ multivariate normal random vectors where $Y_i \sim N_p(\xi, \Sigma)$, $1 \leq i \leq n$, $p < n$ and Σ nonsingular. Then X, S are sufficient statistics for ξ, Σ where

$$X = (\sum_i Y_i)/n, \qquad S = \sum_i (Y_i - X)(Y_i - X)'.$$

Since S is positive definite w.p.1, there exists an unique orthonormal matrix $H = (h_{ij})$ such that $HSH' = L = \text{diag}(l_1, \ldots, l_p)$ with $l_1 \geq \cdots \geq l_p$ and $h_{i1} \geq 0$ for $1 \leq i \leq p$. For simplicity of notation, we write for $i = 1, \ldots, p$,

$$H_i = (h_{i1}, \ldots, h_{ip})'.$$

We are interested in obtaining approximations to the coverage probability of confidence sets for ξ of the form

$$\{\xi : H_i'(\xi - X) \leq c(X,S)\}, \tag{1}$$

where $c(X,S)$ is a suitably chosen function of X and S. The geometrical interpretation of (1) would be a confidence interval for the projection of $\xi - X$ onto the ith eigenvector of S. Since S/n approximates Σ, an obvious application of this is in the area of linear discriminant analysis where in many cases it would be advantageous to know (or at least to have some information on) the direction of the mean vector relative to the eigenvectors of the covariance matrix. Furthermore, bounded confidence sets for ξ can be obtained from the Bonferroni inequality by considering confidence sets given in (1) for each $i = 1, \ldots, p$.

Also we observe that there is a need for higher order asymptotics in multiparameter problems since crude first-order asymptotics often fails.

Motivated by the above statements, the following principal results are obtained. Let S_p denote the set of $p \times p$ positive definite matrices, R^p denote the p-dimensional Euclidean space and $\pi_*(\xi, \Sigma)$ be an almost arbitrary twice continuously differentiable prior density for (ξ, Σ) with respect to $d\xi\, d\Sigma$. For $0 < \alpha < 1$, $c(X,S)$ can be chosen independently of π_* such that

$$\int_{R^p} \int_{S_p} P_{\xi,\Sigma}\{H_i(\xi - X) \leq c(X,S)\}\pi_*(\xi,\Sigma)\, d\Sigma d\xi \quad = \quad \alpha + O(n^{-1/2}). \tag{2}$$

This gives the average coverage probability of the confidence set defined by (1) with respect to $\pi_*(\xi, \Sigma)$. Furthermore second-order approximations to (2) are also determined. An important corollary of this is that $c(X,S)$ can also be chosen independently of π_* in such a way that

$$\int_{R^p} \int_{S_p} P_{\xi,\Sigma}\{H_i(\xi - X) \leq c(X,S)\}\pi_*(\xi,\Sigma)\, d\Sigma d\xi = \alpha + O(n^{-1}).$$

Finally third-order approximations to (2) are obtained for an almost arbitrary thrice continuously differentiable prior density π_*. An interesting corollary is that $c(X,S)$ can be selected independently of π_* such that

$$\int_{R^p} \int_{S_p} P_{\xi,\Sigma}\{H_i(\xi - X) \leq c(X,S)\}\pi_*(\xi,\Sigma)\, d\Sigma d\xi = \alpha + O(n^{-3/2}).$$

Woodroofe (1986), (1987) gives the following argument for the consideration of average coverage probabilities: Average coverage probabilities are much simpler and give a better picture of the confidence level near a given parameter point than does the value at the parameter point. Futhermore in repeated applications, parameters may also vary.

We should also mention that Welch and Peers (1963), Peers (1965) and Welch (1965) have worked on related problems using a different method. However it appears that their technique is not applicable to the problem at hand.

Finally we wish to remark that in this paper, the level of rigor is set somewhat at the level of Stein (1985) and Welch and Peers (1963).

2 Preliminaries

We shall use the following notation throughout. If a matrix A has entries a_{ij}, we shall denote A by (a_{ij}). Given a $r \times s$ matrix A, its $s \times r$ transpose is indicated by A'. $\mid A \mid$, A^{-1} denote the determinant, inverse of the square matrix A respectively. The trace of A is indicated by tr$A$. If the $p \times p$ matrix A is diagonal and has entries a_{ij}, we shall write $A = \text{diag}(a_{11}, \ldots, a_{pp})$. For any positive definite matrix $A = (a_{ij})$, we write $dA = \Pi_{i \leq j}\, da_{ij}$. Finally the expected value of a random vector X is denoted by EX and $I\{\Omega\}$ denotes the indicator function of the event Ω.

Let X and S be defined as in the previous section. Then X is a $p \times 1$ multivariate normal random vector and S is a $p \times p$ Wishart matrix independent of X, where $X \sim N_p(\xi, \Sigma/n)$, $S \sim W_p(\Sigma, n-1)$. For convenience, we write

$$\begin{aligned} S &= (s_{ij}), \qquad S^{-1} = (s^{ij}), \\ \Gamma &= (\tau_{ij}), \qquad \Gamma^{-1} = \Sigma = (\tau^{ij}). \end{aligned}$$

Given (ξ, Γ), we write $p_{\xi,\Gamma}(X, S)$ to be the density of (X, S) with respect to $dXdS$ and

(3) $$M_{\xi,\Gamma}(X, S) = \log p_{\xi,\Gamma}(X, S).$$

Finally, in this paper, the symbol C denotes a generic constant which does not depend on ξ, Γ.

Lemma 1 *With the above notation, we have the following:*

$$\begin{aligned} M_{\xi,\Gamma}(X, S) &= C - \frac{n}{2}(\xi - X)'\Gamma(\xi - X) - \frac{1}{2}tr(\Gamma S) \\ &\quad + \frac{n}{2}log \mid \Gamma \mid, \end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \xi_i} M_{\xi,\Gamma}(X,S) &= -n[\Gamma(\xi - X)]_i, \\
\frac{\partial}{\partial \tau_{ij}} M_{\xi,\Gamma}(X,S) &= \frac{n}{2}\{2[\Gamma^{-1} - (\xi - X)(\xi - X)' - \frac{S}{n}] \\
&\quad -diag[\Gamma^{-1} - (\xi - X)(\xi - X)' - \frac{S}{n}]\}_{ij}, \\
\frac{\partial^2}{\partial \xi_i \partial \xi_j} M_{\xi,\Gamma}(X,S) &= -n\tau_{ij}, \\
\frac{\partial^2}{\partial \xi_i \partial \tau_{jk}} M_{\xi,\Gamma}(X,S) &= -n(\xi_k - x_k)\delta_{ij} - n(\xi_j - x_j)\delta_{ik} \\
&\quad +n(\xi_i - x_i)\delta_{ik}\delta_{ij}, \\
\frac{\partial^2}{\partial \tau_{ij} \partial \tau_{kl}} M_{\xi,\Gamma}(X,S) &= -\frac{n}{4}(2 - \delta_{ij})(2 - \delta_{kl})(\tau^{ik}\tau^{jl} + \tau^{il}\tau^{jk}), \\
\frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k} M_{\xi,\Gamma}(X,S) &= 0, \\
\frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \tau_{kl}} M_{\xi,\Gamma}(X,S) &= -n(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ik}\delta_{il}\delta_{jk}), \\
\frac{\partial^3}{\partial \xi_m \partial \tau_{ij} \partial \tau_{kl}} M_{\xi,\Gamma}(X,S) &= 0, \\
\frac{\partial^3}{\partial \tau_{ab} \partial \tau_{ij} \partial \tau_{kl}} M_{\xi,\Gamma}(X,S) &= \frac{n}{8}(2 - \delta_{ab})(2 - \delta_{ij})(2 - \delta_{kl})(\tau^{ai}\tau^{bk}\tau^{jl} \\
&\quad +\tau^{bi}\tau^{ak}\tau^{jl} + \tau^{al}\tau^{bj}\tau^{ik} + \tau^{bl}\tau^{aj}\tau^{ik} + \tau^{ai}\tau^{bl}\tau^{jk} \\
&\quad +\tau^{bi}\tau^{al}\tau^{jk} + \tau^{ak}\tau^{bj}\tau^{il} + \tau^{bk}\tau^{aj}\tau^{il}),
\end{aligned}$$

where δ_{ij} *denotes the Kronecker delta and* $X = (x_1, \ldots, x_p)'$.

PROOF. This follows from a straightforward calculation using the fact that $X \sim N_p(\xi, \Sigma/n)$ and $S \sim W_p(\Sigma, n-1)$. □

Next for $1 \leq i \leq p$, we consider sets of the form

$$\{\xi : H_i'(\xi - X) \leq c(X,S)\}.$$

For definiteness, we shall assume that $i = 1$ hereafter. We wish to remark that similar calculations and results hold for $i = 2, \ldots, p$. We write

$$\begin{aligned}
\eta'(nS^{-1}, n) &= (\sqrt{l_1}/n, 0, \ldots, 0)H \\
&= (\sqrt{l_1}/n)H_1',
\end{aligned}$$

where l_1, H_1, H are defined in section 1. Also we observe that

$$\begin{aligned}
& \{\xi : H_1'(\xi - X) \le c(X,S)\} \\
= & \{\xi : (\sqrt{l_1}/n, 0, \ldots, 0) H n^2 S^{-1}(\xi - X) \le nc(X,S)/\sqrt{l_1}\} \\
= & \{\xi : \eta'(nS^{-1}, n) n^2 S^{-1}(\xi - X) \le nc(X,S)/\sqrt{l_1}\} \qquad (4)
\end{aligned}$$

and

$$\eta'(nS^{-1}, n) n^2 S^{-1} \eta(nS^{-1}, n) = 1. \qquad (5)$$

3 On Posterior Probabilities

For any prior probability density function $\pi(\xi, \Gamma)$ for (ξ, Γ) with respect to $d\xi d\Gamma$, let P_π be the joint distribution of (ξ, Γ, X, S) where for a fixed set of parameters (ξ, Γ), $X \sim N_p(\xi, \Sigma/n)$ and $S \sim W_p(\Sigma, n-1)$. We write the conditional probability given (X, S) as $P_\pi^{X,S}$.

Now let $\pi_0(\xi, \Gamma)$ be a nonvanishing continuously differentiable (generalized) prior density for (ξ, Γ) with respect to $d\xi d\Gamma$ such that its posterior distribution exists. For $0 < \alpha < 1$, determine $c(X, S)$ such that

$$P_{\pi_0}^{X,S}(H_1'(\xi - X) \le c(X,S)) = \alpha. \qquad (6)$$

It follows now from (4) that (6) can be rewritten as

$$P_{\pi_0}^{X,S}(\eta'(nS^{-1}, n) n^2 S^{-1}(\xi - X) \le c_\alpha(X,S)) = \alpha, \qquad (7)$$

with $c_\alpha(X,S) = nc(X,S)/\sqrt{l_1}$. Writing

$$\Omega_\alpha(X,S) = \{\xi : \eta'(nS^{-1}, n) n^2 S^{-1}(\xi - X) \le c_\alpha(X,S)\}, \qquad (8)$$

we have

$$\alpha = N/D, \qquad (9)$$

where

$$\begin{aligned}
N &= \int_{\Omega_\alpha} \int_{S_p} \exp(M_{\xi,\Gamma}) \pi_0(\xi, \Gamma)\, d\Gamma d\xi, \qquad (10) \\
D &= \int_{R^p} \int_{S_p} \exp(M_{\xi,\Gamma}) \pi_0(\xi, \Gamma)\, d\Gamma d\xi. \qquad (11)
\end{aligned}$$

Next let $\pi_*(\xi, \Gamma)$ be an arbitrary continuously differentible prior density for (ξ, Γ) with respect to $d\xi d\Gamma$. Then

$$P_{\pi_*}^{X,S}(\xi \in \Omega_\alpha) = N_*/D_*, \qquad (12)$$

where

$$N_* = \int_{\Omega_\alpha}\int_{S_p} \exp(M_{\xi,\Gamma})\pi_*(\xi,\Gamma)\, d\Gamma d\xi, \tag{13}$$

$$D_* = \int_{R^p}\int_{S_p} \exp(M_{\xi,\Gamma})\pi_*(\xi,\Gamma)\, d\Gamma d\xi. \tag{14}$$

For simplicity of notation, we write

$$\begin{aligned}
\rho(\xi,\Gamma) &= \pi_*(\xi,\Gamma)/\pi_0(\xi,\Gamma),\\
A &= \frac{1}{2}\sum_{i,j}(\xi_i - x_i)[\frac{\partial^2}{\partial\xi_i\partial\xi_j}M_{X,nS^{-1}}](\xi_j - x_j),\\
B &= \frac{1}{2}\sum_{i\le j,k\le l}(\tau_{ij} - ns^{ij})[\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}}M_{X,nS^{-1}}](\tau_{kl} - ns^{kl}),\\
F &= \sum_i[\frac{\partial}{\partial\xi_i}\rho(X,nS^{-1})](\xi_i - x_i) + \sum_{j\le k}[\frac{\partial}{\partial\tau_{jk}}\rho(X,nS^{-1})](\tau_{jk} - ns^{jk}),
\end{aligned}$$

where $\partial^2 M_{X,nS^{-1}}/\partial\xi_i\partial\xi_j = \partial^2 M_{\xi,\Gamma}/\partial\xi_i\partial\xi_j\,|_{X,nS^{-1}}$, etc.

First we shall prove four rather technical lemmas which will greatly simplify the algebraic manipulations that follow.

Lemma 2 *With N and N_* defined as in (10) and (13), we have*

$$\begin{aligned}
N_* &= \rho(X,nS^{-1})N\\
&\quad +\pi_0(X,nS^{-1})exp(M_{X,nS^{-1}})\int_{\Omega_\alpha}\int_{S_p} exp(A+B)F\, d\Gamma d\xi + R_1,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \int_{\Omega_\alpha}\int_{S_p} exp(M_{\xi,\Gamma})\pi_0(\xi,\Gamma)[\rho(\xi,\Gamma) - \rho(X,nS^{-1})]\, d\Gamma d\xi\\
&\quad -exp(M_{X,nS^{-1}})\pi_0(X,nS^{-1})\int_{\Omega_\alpha}\int_{S_p} exp(A+B)F\, d\Gamma d\xi.
\end{aligned}$$

PROOF. We observe that

$$\begin{aligned}
N_* &= \int_{\Omega_\alpha}\int_{S_p} \exp(M_{\xi,\Gamma})\pi_0(\xi,\Gamma)\rho(\xi,\Gamma)\, d\Gamma d\xi\\
&= \int_{\Omega_\alpha}\int_{S_p} \exp(M_{\xi,\Gamma})\pi_0(\xi,\Gamma)\{\rho(X,nS^{-1})\\
&\quad +[\rho(\xi,\Gamma) - \rho(X,nS^{-1})]\}\, d\Gamma d\xi\\
&= \rho(X,nS^{-1})N\\
&\quad +\pi_0(X,nS^{-1})\exp(M_{X,nS^{-1}})\int_{\Omega_\alpha}\int_{S_p} \exp(A+B)F\, d\Gamma d\xi + R_1.
\end{aligned}$$

The third equality makes use of the definition of N. □

Lemma 3 *With D and D_* defined as in (11) and (14), we have*

$$D_* = \rho(X, nS^{-1})D + R_2,$$

where

$$R_2 = \int_{R^p}\int_{S_p} exp(M_{\xi,\Gamma})\pi_0(\xi,\Gamma)[\rho(\xi,\Gamma) - \rho(X, nS^{-1})] \; d\Gamma d\xi.$$

PROOF. The proof is similar to that of Lemma 2 and hence is omitted. □

Lemma 4 *With D defined as in (11), we have*

$$\begin{aligned} & D \; exp(-M_{X,nS^{-1}})/\pi_0(X, nS^{-1}) \\ = \; & (2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} exp(B) \; d\Gamma + R_3, \end{aligned}$$

where

$$\begin{aligned} R_3 = & \int_{R^p}\int_{S_p} exp(M_{\xi,\Gamma} - M_{X,nS^{-1}}) \frac{\pi_0(\xi,\Gamma) - \pi_0(X, nS^{-1})}{\pi_0(X, nS^{-1})} \; d\Gamma d\xi \\ & + \int_{R^p}\int_{S_p} [exp(M_{\xi,\Gamma} - M_{X,nS^{-1}}) - exp(A + B)] \; d\Gamma d\xi. \end{aligned}$$

PROOF. We observe that

$$\begin{aligned} & D \exp(-M_{X,nS^{-1}})/\pi_0(X, nS^{-1}) \\ = \; & \int_{R^p}\int_{S_p} \exp(A + B) \; d\Gamma d\xi + R_3 \\ = \; & (2\pi)^{p/2} \mid (-\frac{\partial^2}{\partial\xi_i\partial\xi_j} M_{X,nS^{-1}}) \mid^{-1/2} \int_{S_p} \exp(B) \; d\Gamma + R_3 \\ = \; & (2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} \exp(B) \; d\Gamma + R_3. \end{aligned}$$

We note that $\mid (-\partial^2 M_{X,nS^{-1}}/\partial\xi_i\partial\xi_j) \mid$ denotes the determinant of the matrix whose ijth element is $-\partial^2 M_{X,nS^{-1}}/\partial\xi_i\partial\xi_j$. Also the last equality follows from Lemma 1. □

Lemma 5 *If $\rho(X, nS^{-1}) > 0$, we have*

$$\begin{aligned}
&\frac{\pi_0(X, nS^{-1}) \int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1}) e^{-M_{X,nS^{-1}}} D} \\
= \; & -(2\pi)^{-1/2} exp(-c_\alpha(X,S)^2/2) \eta'(nS^{-1}, n) \nabla log \; \rho(X, nS^{-1}) \\
& + \frac{\int_{\Omega_\alpha} \int_{S_p} e^{A+B} \sum_{i \leq j} [\partial log \rho(X, nS^{-1})/\partial \tau_{ij}](\tau_{ij} - ns^{ij}) \; d\Gamma d\xi}{(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B d\Gamma} \\
& - \frac{(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} R_3 \int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1}) \int_{S_p} e^B \; d\Gamma [(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B \; d\Gamma + R_3]},
\end{aligned}$$

where $\nabla = (\partial/\partial\xi_1, \ldots, \partial/\partial\xi_p)'$ and $\eta(nS^{-1}, n)$, $c_\alpha(X, S)$ are defined as in (7).

PROOF. We observe from Lemma 4 that

$$\begin{aligned}
&\frac{\pi_0(X, nS^{-1}) \int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1}) e^{-M_{X,nS^{-1}}} D} \\
= \; & \frac{\int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1})[(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B \; d\Gamma + R_3]} \\
= \; & \frac{\int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1})(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B \; d\Gamma} \\
& - \frac{(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} R_3 \int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1}) \int_{S_p} e^B \; d\Gamma [(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B \; d\Gamma + R_3]} \\
= \; & (2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} \int_{\Omega_\alpha} e^A [\nabla \log \rho(X, nS^{-1})]'(\xi - X) \; d\xi \\
& + \frac{\int_{\Omega_\alpha} \int_{S_p} e^{A+B} \sum_{i \leq j} [\partial \log \rho(X, nS^{-1})/\partial \tau_{ij}](\tau_{ij} - ns^{ij}) \; d\Gamma d\xi}{(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B d\Gamma} \\
& - \frac{(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} R_3 \int_{\Omega_\alpha} \int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1}) \int_{S_p} e^B \; d\Gamma [(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B \; d\Gamma + R_3]}.
\end{aligned} \tag{15}$$

Furthermore we observe from Lemma 1 that the first term of the right hand side of the last equation can be written as

$$(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} \int_{\Omega_\alpha} e^A [\nabla \log \rho(X, nS^{-1})]'(\xi - X) \; d\xi$$

$$
\begin{aligned}
&= (2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} \\
(16) \qquad &\times \int_{\Omega_\alpha} e^{-(\xi-X)'n^2S^{-1}(\xi-X)/2}[\nabla \log \rho(X, nS^{-1})]'(\xi - X)\, d\xi.
\end{aligned}
$$

Hence from the above equation, we have $\xi \sim N_p(X, S/n^2)$. With that in mind we define

$$
\begin{aligned}
Y &= \eta'(nS^{-1}, n)n^2S^{-1}(\xi - X), \\
Z &= [\nabla \log \rho(X, nS^{-1})]'(\xi - X).
\end{aligned}
$$

Then

$$EY = EZ = 0$$

and it follows from (5) that

$$
\begin{aligned}
&E\begin{pmatrix} Y \\ Z \end{pmatrix}\begin{pmatrix} Y & Z \end{pmatrix} \\
&= \begin{pmatrix} 1 & \eta'\nabla \log \rho(X, nS^{-1}) \\ \eta'\nabla \log \rho(X, nS^{-1}) & [\nabla \log \rho(X, nS^{-1})]'S[\nabla \log \rho(X, nS^{-1})]/n^2 \end{pmatrix}.
\end{aligned}
$$

Consequently we have

$$
\begin{aligned}
&\frac{\mid S/n^2 \mid^{-1/2}}{(2\pi)^{p/2}} \int_{\Omega_\alpha} e^{-(\xi-X)'n^2S^{-1}(\xi-X)/2}[\nabla \log \rho(X, nS^{-1})]'(\xi - X)\, d\xi \\
&= EZI\{Y \le c_\alpha(X, S)\} \\
&= E(E^Y Z)I\{Y \le c_\alpha(X, S)\} \\
&= E\eta'(nS^{-1}, n)[\nabla \log \rho(X, nS^{-1})]YI\{Y \le c_\alpha(X, S)\} \\
&= -(2\pi)^{-1/2}\eta'(nS^{-1}, n)[\nabla \log \rho(X, nS^{-1})]\exp(-c_\alpha(X, S)^2/2).
\end{aligned}
$$

(17)

Here $E^Y Z$ denotes the conditional expectation of Z given Y and the first equality uses (8). Now the result easily follows from (15), (16) and (17). □

With these four lemmas in hand, we shall compute the posterior probability of Ω_α under the prior $\pi_*(\xi, \Gamma)$.

Proposition 1 *If $\rho(X, nS^{-1}) > 0$, the posterior probability of Ω_α under the prior $\pi_*(\xi, \Gamma)$ is given by*

$$
\begin{aligned}
&P_{\pi_*}^{X,S}(\xi \in \Omega_\alpha) \\
&= \alpha - (2\pi)^{-1/2} exp(-c_\alpha(X, S)^2/2)\eta'(nS^{-1}, n)\nabla log\ \rho(X, nS^{-1}) + R_5,
\end{aligned}
$$

where

$$\begin{aligned} R_4 &= R_2/[\rho(X, nS^{-1})D], \\ R_5 &= I\{\rho(X, nS^{-1}) > 0\}\{\frac{R_1}{(1+R_4)\rho(X, nS^{-1})D} - \frac{\alpha R_4}{1+R_4} \\ &\quad - \frac{R_4\pi_0(X, nS^{-1}) \int_{\Omega_\alpha}\int_{S_p} exp(A+B)F \; d\Gamma d\xi}{(1+R_4)\rho(X, nS^{-1}) exp(-M_{X,nS^{-1}})D} \\ &\quad + \frac{\int_{\Omega_\alpha}\int_{S_p} e^{A+B} \sum_{i\le j}[\partial log \rho(X, nS^{-1})/\partial\tau_{ij}](\tau_{ij} - ns^{ij}) \; d\Gamma d\xi}{(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B d\Gamma} \\ &\quad - \frac{(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} R_3 \int_{\Omega_\alpha}\int_{S_p} e^{A+B} F \; d\Gamma d\xi}{\rho(X, nS^{-1}) \int_{S_p} e^B \; d\Gamma[(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B \; d\Gamma + R_3]}\}, \end{aligned}$$

with R_1, R_2, R_3 defined as in Lemmas 2, 3, 4 respectively.

PROOF. It follows from (12), Lemmas 2 and 3 that

$$\begin{aligned} & P_{\pi_*}^{X,S}(\xi \in \Omega_\alpha) \\ =\; & \frac{\rho(X, nS^{-1})N}{\rho(X, nS^{-1})D + R_2} \\ & + \frac{\pi_0(X, nS^{-1})\exp(M_{X,nS^{-1}}) \int_{\Omega_\alpha}\int_{S_p} \exp(A+B)F \; d\Gamma d\xi + R_1}{\rho(X, nS^{-1})D + R_2} \\ =\; & \frac{N}{D(1+R_4)} \\ & + \frac{\pi_0(X, nS^{-1})\exp(M_{X,nS^{-1}}) \int_{\Omega_\alpha}\int_{S_p} \exp(A+B)F \; d\Gamma d\xi + R_1}{\rho(X, nS^{-1})D(1+R_4)} \\ =\; & \alpha + \frac{\pi_0(X, nS^{-1}) \int_{\Omega_\alpha}\int_{S_p} \exp(A+B)F \; d\Gamma d\xi}{\rho(X, nS^{-1})\exp(-M_{X,nS^{-1}})D} \\ & - \frac{R_4\pi_0(X, nS^{-1}) \int_{\Omega_\alpha}\int_{S_p} \exp(A+B)F \; d\Gamma d\xi}{(1+R_4)\rho(X, nS^{-1})\exp(-M_{X,nS^{-1}})D} \\ & + \frac{R_1}{(1+R_4)\rho(X, nS^{-1})D} - \frac{\alpha R_4}{1+R_4}. \end{aligned}$$

The last equality uses (9) and now the result follows easily from Lemma 5. □

For the case where $\rho(X, nS^{-1}) = 0$, we have the following result.

Proposition 2 *If $\pi_*(\xi,\Gamma)$ is twice continuously differentiable, then*

$$P_{\pi_*}(\rho(X,nS^{-1})=0)=O(n^{-3/2}).$$

PROOF. For this and all subsequent proofs, we make the following simplifying observations. X, nS^{-1} are the maximum likelihood estimators for ξ,Γ respectively. From classical large sample theory, the posterior distribution of (ξ,Γ) given (X,nS^{-1}) is asymptotically multivariate normal with error of the order of $n^{-1/2}$. That is to say, for $i,j=1,\dots,p$,

$$\xi_i - x_i = O_p(n^{-1/2}), \qquad \tau_{ij} - ns^{ij} = O_p(n^{-1/2}).$$

However for all the integrals that we shall be dealing with, the contribution arising from those values of $|\ \xi_i - x_i\ |$ or $|\ \tau_{ij} - ns^{ij}\ |$ that exceed $O(n^{-1/2})$ decreases exponentially with n. Hence to the order of accuracy that we are interested in, we shall assume that

$$\xi_i - x_i = O(n^{-1/2}), \qquad \tau_{ij} - ns^{ij} = O(n^{-1/2}), \tag{18}$$

whenever $i,j=1,\dots,p$.

We observe that

$$\begin{aligned}
& P_{\pi_*}(\rho(X,nS^{-1})=0) \\
= & \int_{\rho(X,nS^{-1})=0}\int_{R^p}\int_{S_p} e^{M_{\xi,\Gamma}}\pi_*(\xi,\Gamma)\ d\Gamma d\xi dS dX \\
= & \int_{\rho(X,nS^{-1})=0}\int_{R^p}\int_{S_p} e^{M_{\xi,\Gamma}}[\pi_*(X,nS^{-1}) + \sum_i \frac{\partial\pi_*(X,nS^{-1})}{\partial\xi_i}(\xi_i - x_i) \\
& +\sum_{i\le j}\frac{\partial\pi_*(X,nS^{-1})}{\partial\tau_{ij}}(\tau_{ij}-ns^{ij}) \\
& +\sum_{i,j\le k}(\xi_i - x_i)\frac{\partial^2\pi_*(X,nS^{-1})}{\partial\xi_i\partial\tau_{jk}}(\tau_{jk}-ns^{jk}) \\
& +\sum_{i\le j,k\le l}(\tau_{ij}-ns^{ij})\frac{\partial^2\pi_*(X,nS^{-1})}{\partial\tau_{ij}\partial\tau_{kl}}(\tau_{kl}-ns^{kl}) \\
& +\sum_{i,j}(\xi_i - x_i)\frac{\partial^2\pi_*(X,nS^{-1})}{\partial\xi_i\partial\xi_j}(\xi_j - x_j)]\ d\Gamma d\xi dS dX + O(n^{-3/2}) \\
= & O(n^{-3/2}).
\end{aligned}$$

The second equality follows from (18) and a Taylor expansion about (X, nS^{-1}). The final equality makes use of the observation that since π_* is twice continuously differentiable, on the set $\{X, S : \rho(X, nS^{-1}) = 0\}$, the first and second partial derivatives of π_* vanish except possibly on a subset of probability zero. □

Finally, we shall conclude this section by obtaining an identity which will be needed in the approximations of the next two sections.

Theorem 1 *Suppose $\pi_0(\xi, \Gamma)$ is a nonvanishing continuously differentiable (generalized) prior density for (ξ, Γ) with respect to $d\xi d\Gamma$ such that its posterior distribution exists. Let Ω_α be defined as in (8). Then under any twice continuously differentiable prior density $\pi_*(\xi, \Gamma)$ for (ξ, Γ), the average coverage probability of Ω_α is given by*

$$\begin{aligned} & P_{\pi_*}(\xi \in \Omega_\alpha) \\ = \; & \alpha + (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \int_{R^p} \int_{S_p} \eta'(\Gamma, n)[\nabla \pi_0(\xi, \Gamma)] \pi_0^{-1}(\xi, \Gamma) \\ & \times \pi_*(\xi, \Gamma) \; d\Gamma d\xi + E_{\pi_*} R_5 + \int_{R^p} \int_{S_p} R_6 \pi_*(\xi, \Gamma) \; d\Gamma d\xi + O(n^{-3/2}), \end{aligned} \tag{19}$$

where R_5 is defined as in Proposition 1 and

$$\begin{aligned} R_6 \; = \; & -(2\pi)^{-1/2} E_{\pi_*}^{\xi, \Gamma} \{ e^{-c_\alpha(X,S)^2/2} \eta'(nS^{-1}, n)[\nabla log \; \rho(X, nS^{-1})] \\ & \times I\{\rho(X, nS^{-1}) > 0\}\} + (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \eta'(\Gamma, n) \nabla log \rho(\xi, \Gamma). \end{aligned}$$

PROOF. We observe from Propositions 1 and 2 that

$$\begin{aligned} & P_{\pi_*}(\xi \in \Omega_\alpha) \\ = \; & E_{\pi_*} P_{\pi_*}^{X,S}(\xi \in \Omega_\alpha) I\{\rho(X, nS^{-1}) > 0\} + O(n^{-3/2}) \\ = \; & \alpha - (2\pi)^{-1/2} E_{\pi_*} \{ e^{-c_\alpha(X,S)^2/2} \eta'(nS^{-1}, n)[\nabla \log \rho(X, nS^{-1})] \\ & \times I\{\rho(X, nS^{-1}) > 0\}\} + E_{\pi_*} R_5 + O(n^{-3/2}) \\ = \; & \alpha - (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \int_{R^p} \int_{S_p} \eta'(\Gamma, n) \nabla \log \rho(\xi, \Gamma) \pi_*(\xi, \Gamma) \; d\Gamma d\xi \\ & + E_{\pi_*} R_5 + \int_{R^p} \int_{S_p} R_6 \pi_*(\xi, \Gamma) \; d\Gamma d\xi + O(n^{-3/2}) \\ = \; & \alpha - (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \int_{R^p} \int_{S_p} \eta'(\Gamma, n) (\nabla \pi_* - \frac{\pi_* \nabla \pi_0}{\pi_0}) \; d\Gamma d\xi \\ & + E_{\pi_*} R_5 + \int_{R^p} \int_{S_p} R_6 \pi_*(\xi, \Gamma) \; d\Gamma d\xi + O(n^{-3/2}) \\ = \; & \alpha + (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \int_{R^p} \int_{S_p} \eta'(\Gamma, n) \frac{\pi_* \nabla \pi_0}{\pi_0} \; d\Gamma d\xi \\ & + E_{\pi_*} R_5 + \int_{R^p} \int_{S_p} R_6 \pi_*(\xi, \Gamma) \; d\Gamma d\xi + O(n^{-3/2}). \end{aligned}$$

The last equation follows from the observations that $\eta(\Gamma, n)$ is independent of ξ and that for all Γ,

$$\int_{R^p} \nabla \pi_* \, d\xi = 0,$$

since $\pi_*(\xi, \Gamma)$ is a probability density. This completes the proof. □

4 Second-Order Asymptotics

We shall now obtain a second-order approximation to $P_{\pi_*}(\xi \in \Omega_\alpha)$ defined as in Theorem 1 by imposing a relatively mild boundary condition on π_*. First let us introduce some additional notation. We consider S_p, the set of all $p \times p$ positive definite matrices, as a subset of $R^{p(p+1)/2}$. For each $\epsilon > 0$, we define $\partial S_p(\epsilon)$ as the open subset of S_p that lies within ϵ, in the sense of Euclidean distance, of the boundary of S_p.

Theorem 2 *Suppose $\pi_0(\xi, \Gamma)$ is a nonvanishing twice continuously differentiable (generalized) prior density for (ξ, Γ) satisfying (7). Let Ω_α be defined as in (8) and $\pi_*(\xi, \Gamma)$ be a twice continuously differentiable prior density for (ξ, Γ) satisfying the following boundary condition: There exists an $\epsilon > 0$, such that for all ξ and $\Gamma \in \partial S_p(\epsilon)$, we have $\pi_*(\xi, \Gamma) = 0$. Then the average coverage probability of Ω_α with respect to π_* is given by*

$$\begin{aligned} & P_{\pi_*}(\xi \in \Omega_\alpha) \\ = \; & \alpha + (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \int_{R^p} \int_{S_p} \eta'(\Gamma, n)[\nabla \pi_0(\xi, \Gamma)] \pi_0^{-1}(\xi, \Gamma) \\ & \times \pi_*(\xi, \Gamma) \, d\Gamma d\xi + O(n^{-1}). \end{aligned}$$

PROOF. To prove this theorem, it suffices to show that the third and fourth terms on the right hand side of (19) are of the order of n^{-1}. First we observe from (18) and Lemma 1 that by using appropriate Taylor expansions, we have

$$\pi_0(\xi, \Gamma) - \pi_0(X, nS^{-1}) = O(n^{-1/2}), \tag{20}$$

$$\rho(\xi, \Gamma) - \rho(X, nS^{-1}) - F = O(n^{-1}), \tag{21}$$

$$M_{\xi,\Gamma} - M_{X,nS^{-1}} - A - B = O(n^{-1/2}). \tag{22}$$

Now we consider $E_{\pi_*} R_5$. To do so, we need to take a closer look at R_1 to R_4. We observe from (20), (22) and the definition of R_3 that

$$| S/n^2 |^{-1/2} | (-\frac{\partial^2}{\partial \tau_{ij} \partial \tau_{kl}} M_{X,nS^{-1}}) |^{1/2} R_3 I\{\rho(X, nS^{-1}) > 0\}$$

$$
\begin{aligned}
&= \mid S/n^2 \mid^{-1/2} \mid (-\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}} M_{X,nS^{-1}}) \mid^{1/2} I\{\rho(X,nS^{-1})>0\} \\
&\quad \times \int_{R^p}\int_{S_p} \exp(M_{\xi,\Gamma}-M_{X,nS^{-1}}) \frac{\pi_0(\xi,\Gamma)-\pi_0(X,nS^{-1})}{\pi_0(X,nS^{-1})} \\
&\quad +\exp(M_{\xi,\Gamma}-M_{X,nS^{-1}}) - \exp(A+B)\ d\Gamma d\xi \\
&= \mid S/n^2 \mid^{-1/2} \mid (-\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}} M_{X,nS^{-1}}) \mid^{1/2} I\{\rho(X,nS^{-1})>0\} \\
&\quad \times \int_{R^p}\int_{S_p} e^{A+B}\{\sum_i [\frac{\partial}{\partial\xi_i}\log\pi_0(X,nS^{-1})](\xi_i-x_i) \\
&\quad + \sum_{i\le j} [\frac{\partial}{\partial\tau_{ij}}\log\pi_0(X,nS^{-1})](\tau_{ij}-ns^{ij}) \\
&\quad + \sum_{i,j,k\le l} [\frac{\partial^3}{\partial\xi_i\partial\xi_j\partial\tau_{kl}} M_{X,nS^{-1}}](\xi_i-x_i)(\xi_j-x_j)(\tau_{kl}-ns^{kl}) \\
&\quad + \sum_{i\le j,k\le l,a\le b} [\frac{\partial^3}{\partial\tau_{ab}\partial\tau_{ij}\partial\tau_{kl}} M_{X,nS^{-1}}](\tau_{ab}-ns^{ab})(\tau_{ij}-ns^{ij}) \\
&\quad \times(\tau_{kl}-ns^{kl})\}\ d\Gamma d\xi + O(n^{-1}) \\
&= O(n^{-1}).
\end{aligned}
$$

Here $(-\partial^2 M_{X,nS^{-1}}/\partial\tau_{ij}\partial\tau_{kl})$ denotes the $p(p+1)/2\times p(p+1)/2$ matrix whose ij,klth element is $-\partial^2 M_{X,nS^{-1}}/\partial\tau_{ij}\partial\tau_{kl}$. Also the second last equality uses Lemma 1 and the last equality uses the observation that the boundary condition on π_* and (18) allow us to replace S_p by $R^{p(p+1)/2}$ in the second integral.

Furthermore it follows from (18) that

$$
\mid S/n^2 \mid^{-1/2} \mid (-\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}} M_{X,nS^{-1}}) \mid^{1/2} \int_{\Omega_\alpha}\int_{S_p} e^{A+B} F\ d\Gamma d\xi \ = \ O(n^{-1/2}).
$$

Hence if $\rho(X,nS^{-1})>0$, we have

$$
\begin{aligned}
& \frac{(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} R_3 \int_{\Omega_\alpha}\int_{S_p} e^{A+B} F\ d\Gamma d\xi}{[\rho(X,nS^{-1})\int_{S_p} e^B\ d\Gamma][(2\pi)^{p/2} \mid S/n^2 \mid^{1/2} \int_{S_p} e^B\ d\Gamma + R_3]} \\
(23) \quad &= O(n^{-3/2}).
\end{aligned}
$$

Next it follows from the boundary condition on π_* that with an error which decreases exponentially with n, we can write for $\rho(X, nS^{-1}) > 0$

$$
\begin{aligned}
& \frac{\int_{\Omega_\alpha}\int_{S_p} e^{A+B}\sum_{i\leq j}[\partial \log\rho(X,nS^{-1})/\partial\tau_{ij}](\tau_{ij}-ns^{ij})\ d\Gamma d\xi}{(2\pi)^{p/2}\mid S/n^2\mid^{1/2}\int_{S_p} e^B d\Gamma} \\
\approx\ & \frac{\int_{\Omega_\alpha}\int_{R^{p(p+1)/2}} e^{A+B}\sum_{i\leq j}[\partial \log\rho(X,nS^{-1})/\partial\tau_{ij}](\tau_{ij}-ns^{ij})\ d\Gamma d\xi}{(2\pi)^{p/2}\mid S/n^2\mid^{1/2}\int_{S_p} e^B d\Gamma} \\
=\ & 0.
\end{aligned}
$$

Consequently we conclude that for X, nS^{-1} satisfying $\rho(X, nS^{-1}) > 0$, we have

$$
\frac{\int_{\Omega_\alpha}\int_{S_p} e^{A+B}\sum_{i\leq j}[\partial \log\rho(X,nS^{-1})/\partial\tau_{ij}](\tau_{ij}-ns^{ij})\ d\Gamma d\xi}{(2\pi)^{p/2}\mid S/n^2\mid^{1/2}\int_{S_p} e^B d\Gamma} = O(n^{-3/2}). \tag{24}
$$

We note that this bound is extremely crude. However it suffices for our purposes. Now we look at R_4. We observe that

$$
\begin{aligned}
& \pi_0(X,nS^{-1})e^{M_{X,nS^{-1}}}D^{-1}\int_{R^p}\int_{S_p} e^{A+B}F\ d\Gamma d\xi \\
=\ & \pi_0(X,nS^{-1})e^{M_{X,nS^{-1}}}D^{-1}\int_{R_p}\int_{S_p} e^{A+B} \\
& \times\sum_{i\leq j}[\frac{\partial}{\partial\tau_{ij}}\rho(X,nS^{-1})](\tau_{ij}-ns^{ij})\ d\Gamma d\xi \\
=\ & O(n^{-3/2}). \qquad (25)
\end{aligned}
$$

Again we note that although the bound above is very crude, it is sufficient for our purposes. The final equality follows from an argument similar to that given for (24). With this bound in hand, we have

$$
\begin{aligned}
& R_2 D^{-1} \\
=\ & e^{M_{X,nS^{-1}}}D^{-1}\int_{R^p}\int_{S_p}\{\pi_0(X,nS^{-1})e^{A+B}F \\
& +\pi_0(X,nS^{-1})e^{A+B}[\rho(\xi,\Gamma)-\rho(X,nS^{-1})-F] \\
& +e^{A+B}[\pi_0(\xi,\Gamma)-\pi_0(X,nS^{-1})][\rho(\xi,\Gamma)-\rho(X,nS^{-1})-F] \\
& +e^{A+B}[\pi_0(\xi,\Gamma)-\pi_0(X,nS^{-1})]F \\
& +[e^{M_{\xi,\Gamma}-M_{X,nS^{-1}}}-e^{A+B}][\pi_0(\xi,\Gamma)-\pi_0(X,nS^{-1})]F \\
& +\pi_0(X,nS^{-1})[e^{M_{\xi,\Gamma}-M_{X,nS^{-1}}}-e^{A+B}][\rho(\xi,\Gamma)-\rho(X,nS^{-1})-F]
\end{aligned}
$$

$$
\begin{aligned}
& +[e^{M_{\xi,\Gamma}-M_{X,nS^{-1}}} - e^{A+B}]\pi_0(X, nS^{-1})F \\
& +[e^{M_{\xi,\Gamma}-M_{X,nS^{-1}}} - e^{A+B}][\pi_0(\xi,\Gamma) - \pi_0(X, nS^{-1})] \\
& \times[\rho(\xi,\Gamma) - \rho(X, nS^{-1}) - F]\} \, d\Gamma d\xi \\
= \; & O(n^{-1}).
\end{aligned}
$$

The last equality uses (20), (21), (22) and (25). Now it follows from the definition of R_4 given in Proposition 1 that

(26) $$R_4 I\{\rho(X, nS^{-1}) > 0\} = O(n^{-1}).$$

By an argument similar to that given for R_4, it can easily be shown that

(27) $$R_1 D^{-1} = O(n^{-1}).$$

From (23), (24), (26) and (27), we conlude that $R_5 = O(n^{-1})$. This implies that

(28) $$E_{\pi_*} R_5 = O(n^{-1}).$$

Next from classical large sample theory, we observe that given X, nS^{-1}, the posterior distribution of $\eta'(X, nS^{-1})n^2 S^{-1}(\xi - X)$ is asymptotically standard normal with error of the order of $n^{-1/2}$. Thus it follows from (7) that

$$c_\alpha(X, S) = \Phi^{-1}(\alpha) + O(n^{-1/2}).$$

Also we observe that η_i, $i = 1, \ldots, p$ is of the order of $n^{-1/2}$. Hence it follows from (18) that R_6, defined as in Theorem 1, is of the order of n^{-1}. This implies that

(29) $$\int_{R^p}\int_{S_p} R_6 \pi_*(\xi,\Gamma) \, d\Gamma d\xi = O(n^{-1}).$$

We conclude from (28) and (29) that

$$
\begin{aligned}
& P_{\pi_*}(\xi \in \Omega_\alpha) \\
= \; & \alpha + (2\pi)^{-1/2} e^{-[\Phi^{-1}(\alpha)]^2/2} \int_{R^p}\int_{S_p} \eta'(\Gamma, n)[\nabla \pi_0(\xi,\Gamma)]\pi_0^{-1}(\xi,\Gamma) \\
& \times \pi_*(\xi,\Gamma) \, d\Gamma d\xi + O(n^{-1}).
\end{aligned}
$$

This proves the theorem. □

Corollary 1 *Under the conditions of Theorem 2, we have*

$$P_{\pi_*}(\xi \in \Omega_\alpha) = \alpha + O(n^{-1/2}).$$

PROOF. We observe that the components of η are each of order of $n^{-1/2}$. Hence

$$(2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\int_{R^p}\int_{S_p}\eta'(\Gamma,n)[\nabla\pi_0(\xi,\Gamma)]\pi_0^{-1}(\xi,\Gamma)\pi_*(\xi,\Gamma)\;d\Gamma d\xi$$
$$= O(n^{-1/2}),$$

and the result follows immediately from Theorem 2. □

Corollary 2 *Under the conditions of Theorem 2, suppose that $\pi_0(\xi,\Gamma)$ is independent of ξ. Then*

$$P_{\pi_*}(\xi\in\Omega_\alpha)=\alpha+O(n^{-1}).$$

PROOF. This is immediate from Theorem 2 since $\nabla\pi_0(\xi,\Gamma)=0$. □

Finally Theorem 2 suggests that for most parameter values, especially those that are not too close to the boundary of S_p,

$$\begin{aligned}&P_{\xi,\Gamma}(\xi\in\Omega_\alpha)\\ =\;&\alpha+(2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\eta'(\Gamma,n)[\nabla\pi_0(\xi,\Gamma)]\pi_0^{-1}(\xi,\Gamma)+O(n^{-1}).\end{aligned}$$

5 Third-Order Asymptotics

Here we shall obtain a third-order approximation to $P_{\pi_*}(\xi\in\Omega_\alpha)$ defined as in Theorem 1. First we need two lemmas.

Lemma 6 *Let $c_\alpha(X,S)$ be defined as in (7) and $\epsilon>0$. Then for all X and $nS^{-1}\in S_p\backslash\partial S_p(\epsilon)$, we have*

$$c_\alpha(X,S)=\Phi^{-1}(\alpha)+\eta'(nS^{-1},n)\nabla log\pi_0(X,nS^{-1})+O(n^{-1}).$$

PROOF. We observe that for all X and $nS^{-1}\in S_p\backslash\partial S_p(\epsilon)$,

$$\begin{aligned}&(2\pi)^{-p(p+3)/4}\mid S/n^2\mid^{-1/2}\mid(-\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}}M_{X,nS^{-1}})\mid^{1/2}N\\ =\;&(2\pi)^{-p(p+3)/4}\mid S/n^2\mid^{-1/2}\mid(-\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}}M_{X,nS^{-1}})\mid^{1/2}e^{M_{X,nS^{-1}}}\\ &\times\int_{\Omega_\alpha}\int_{S_p}e^{A+B}\{1+\frac{1}{6}\sum_{i,j,k\le l}[\frac{\partial^3}{\partial\xi_i\partial\xi_j\partial\tau_{kl}}M_{X,nS^{-1}}]\end{aligned}$$

$$
\begin{aligned}
&\times(\xi_i - x_i)(\xi_j - x_j)(\tau_{kl} - ns^{kl}) \\
&+\frac{1}{6}\sum_{i\le j,k\le l,a\le b}[\frac{\partial^3}{\partial\tau_{ij}\partial\tau_{kl}\partial\tau_{ab}}M_{X,nS^{-1}}](\tau_{ij} - ns^{ij})(\tau_{ab} - ns^{ab}) \\
&\times(\tau_{kl} - ns^{kl})\}\{\pi_0(X, nS^{-1}) + \sum_i[\frac{\partial}{\partial\xi_i}\pi_0(X, nS^{-1})](\xi_i - x_i) \\
&+\sum_{i\le j}[\frac{\partial}{\partial\tau_{ij}}\pi_0(X, nS^{-1})](\tau_{ij} - ns^{ij})\}\ d\Gamma d\xi + O(n^{-1}) \\
= \;&(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} e^{M_{X,nS^{-1}}}\pi_0(X, nS^{-1}) \\
&\times\int_{\Omega_\alpha} e^A\{1 + [\nabla\log\pi_0(X, nS^{-1})]'(\xi - X)\}\ d\xi + O(n^{-1}).
\end{aligned}
$$

The second equality uses Lemma 1. Similarly, to the same level of approximation, we have

$$
\begin{aligned}
&(2\pi)^{-p(p+3)/4} \mid S/n^2 \mid^{-1/2} \mid (-\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}}M_{X,nS^{-1}}) \mid^{1/2} D \\
= \;&e^{M_{X,nS^{-1}}}\pi_0(X, nS^{-1}) + O(n^{-1}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\alpha \;&= \;N/D \\
&= \;(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} \\
&\qquad\times\int_{\Omega_\alpha} e^A\{1 + [\nabla\log\pi_0(X, nS^{-1})]'(\xi - X)\}\ d\xi + O(n^{-1}) \\
&= \;(2\pi)^{-p/2} \mid S/n^2 \mid^{-1/2} \int_{\Omega_\alpha} e^{-n^2(\xi-X)'S^{-1}(\xi-X)/2} \\
&\qquad\times\{1 + [\nabla\log\pi_0(X, nS^{-1})]'(\xi - X)\}\ d\xi + O(n^{-1}).
\end{aligned}
\tag{30}
$$

The final equality uses Lemma 1. As in the proof of Lemma 5, we define

$$
\begin{aligned}
Y &= \eta'(nS^{-1}, n)n^2S^{-1}(\xi - X), \\
W &= [\nabla\log\pi_0(X, nS^{-1})]'(\xi - X).
\end{aligned}
$$

Then

$$EY = EW = 0$$

and

$$E\begin{pmatrix} Y \\ W \end{pmatrix}\begin{pmatrix} Y & W \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \eta'\nabla\log\pi_0(X, nS^{-1}) \\ \eta'\nabla\log\pi_0(X, nS^{-1}) & [\nabla\log\pi_0(X, nS^{-1})]'S[\nabla\log\pi_0(X, nS^{-1})]/n^2 \end{pmatrix}.$$

Now it follows from (30) that

$$\begin{aligned} \alpha &= E[WI\{Y \leq c_\alpha(X,S)\} + I\{Y \leq c_\alpha(X,S)\}] + O(n^{-1}) \\ &= -(2\pi)^{-1/2}\eta'(nS^{-1}, n)\nabla\log\pi_0(X, nS^{-1})e^{-c_\alpha(X,S)^2/2} \\ &\quad +\Phi(c_\alpha(X,S)) + O(n^{-1}) \end{aligned}$$

and hence

$$c_\alpha(X,S) = \Phi^{-1}(\alpha) + \eta'(nS^{-1}, n)\nabla\log\pi_0(X, nS^{-1}) + O(n^{-1}).$$

This completes the proof. □

Lemma 7 *Let R_6 be defined as in Theorem 1. Under the conditions of Theorem 2, we have*

$$\begin{aligned} E_{\pi_*}R_6 &= (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\Phi^{-1}(\alpha)\int_{R^p}\int_{S_p}\eta'(\Gamma,n)[\nabla\log\pi_0(\xi,\Gamma)] \\ &\quad \times\eta'(\Gamma,n)[\nabla\pi_*(\xi,\Gamma)] - [\eta'(\Gamma,n)\nabla\log\pi_0(\xi,\Gamma)]^2\pi_*(\xi,\Gamma)\ d\Gamma d\xi \\ &\quad +O(n^{-3/2}). \end{aligned}$$

PROOF. From the definition of R_6 and Lemma 6, we observe that for $\rho(\xi,\Gamma) > 0$

$$\begin{aligned} R_6 &= (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\{\eta'(\Gamma,n)\nabla\log\rho(\xi,\Gamma) \\ &\quad -E_{\pi_*}^{\xi,\Gamma}[1 - \Phi^{-1}(\alpha)\eta'(nS^{-1},n)\nabla\log\pi_0(X,nS^{-1})]\eta'(nS^{-1},n) \\ &\quad \times[\nabla\log\rho(X,nS^{-1})]I\{\rho(X,nS^{-1}) > 0\}\} + O(n^{-3/2}) \\ &= (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\{\Phi^{-1}(\alpha)[\eta'(\Gamma,n)\nabla\log\pi_0(\xi,\Gamma)]\eta'(\Gamma,n) \\ &\quad \times[\nabla\log\rho(\xi,\Gamma)] - E_{\pi_*}^{\xi,\Gamma}[\sum_i\frac{\partial}{\partial\xi_i}(\eta'(\Gamma,n)\nabla\log\rho(\xi,\Gamma))(x_i - \xi_i) \\ &\quad +\sum_{i\leq j}\frac{\partial}{\partial\tau_{ij}}(\eta'(\Gamma,n)\nabla\log\rho(\xi,\Gamma))(ns^{ij} - \tau_{ij})]\} + O(n^{-3/2}) \\ &= (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\Phi^{-1}(\alpha)\eta'(\Gamma,n)[\nabla\log\pi_0(\xi,\Gamma)]\eta'(\Gamma,n) \\ &\quad \times[\nabla\log\rho(\xi,\Gamma)] + O(n^{-3/2}). \end{aligned}$$

Thus

$$\begin{aligned} E_{\pi_*}R_6 &= (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\Phi^{-1}(\alpha)\int_{R^p}\int_{S_p}\eta'(\Gamma,n)[\nabla\log\pi_0(\xi,\Gamma)] \\ &\quad \times\eta'(\Gamma,n)[\nabla\log\rho(\xi,\Gamma)]\pi_*(\xi,\Gamma)\ d\Gamma d\xi + O(n^{-3/2}) \\ &= (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\Phi^{-1}(\alpha)\int_{R^p}\int_{S_p}\eta'(\Gamma,n)[\nabla\log\pi_0(\xi,\Gamma)] \\ &\quad \times\eta'(\Gamma,n)[\nabla\pi_*(\xi,\Gamma)] - [\eta'(\Gamma,n)\nabla\log\pi_0(\xi,\Gamma)]^2\pi_*(\xi,\Gamma)\ d\Gamma d\xi \\ &\quad +O(n^{-3/2}). \end{aligned}$$

This completes the proof. □

For simplicity, we write

$$\begin{aligned} \nu_{ij,kl} &= E(\tau_{ij}-ns^{ij})(\tau_{kl}-ns^{kl}), \\ \nu_{ab,ij,kl,rs} &= E(\tau_{ab}-ns^{ab})(\tau_{ij}-ns^{ij})(\tau_{kl}-ns^{kl})(\tau_{rs}-ns^{rs}), \end{aligned}$$

where Γ is multivariate normal with mean nS^{-1} and precision matrix given by $(-\partial^2 M_{X,nS^{-1}}/\partial\tau_{ij}\partial\tau_{kl})$. Now we are ready to prove the main result of this section.

Theorem 3 *Suppose $\pi_0(\xi,\Gamma)$ is a nonvanishing thrice continuously differentiable (generalized) prior density for (ξ,Γ) satisfying (7). Let Ω_α be defined as in (8) and $\pi_*(\xi,\Gamma)$ be a thrice continuously differentiable prior density for (ξ,Γ) satisfying the following:*

1. *There exists an $\epsilon > 0$, such that for all ξ and $\Gamma \in \partial S_p(\epsilon)$, we have*

$$\pi_*(\xi,\Gamma) = 0.$$

2. *For all ξ, Γ we have for $i,j = 1,\ldots,p$,*

$$\lim_{\xi_j\to-\infty}\pi_*(\xi,\Gamma)\partial log\pi_0(\xi,\Gamma)/\partial\xi_i = \lim_{\xi_j\to+\infty}\pi_*(\xi,\Gamma)\partial log\pi_0(\xi,\Gamma)/\partial\xi_i.$$

Then the average coverage probability of Ω_α with respect to π_* is given by

$$\begin{aligned} &P_{\pi_*}(\xi\in\Omega_\alpha) \\ &= \alpha + (2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\int_{R^p}\int_{S_p}\eta'(\Gamma,n)[\nabla\pi_0(\xi,\Gamma)]\frac{\pi_*(\xi,\Gamma)}{\pi_0(\xi,\Gamma)}\ d\Gamma d\xi \\ &\quad +(2\pi)^{-1/2}e^{-[\Phi^{-1}(\alpha)]^2/2}\Phi^{-1}(\alpha)\int_{R^p}\int_{S_p}\eta'(\Gamma,n)[\nabla\log\pi_0(\xi,\Gamma)] \end{aligned}$$

$$\times\eta'(\Gamma,n)[\nabla\pi_*(\xi,\Gamma)] - [\eta'(\Gamma,n)\nabla\log\pi_0(\xi,\Gamma)]^2\pi_*(\xi,\Gamma)\ d\Gamma d\xi$$
$$+\int_{R^p}\int_{S_p}\sum_{i,j}\{\frac{\partial^2}{\partial\xi_i\partial\xi_j}[\pi_0(\xi,\Gamma)[(\alpha-1)n^{-1}\Gamma^{-1}$$
$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{\pi_*(\xi,\Gamma)}{2\pi_0(\xi,\Gamma)}\ d\Gamma d\xi$$
$$-\int_{R^p}\int_{S_p}\sum_{i,j}\{\frac{\partial}{\partial\xi_j}[(\frac{\partial}{\partial\xi_i}\pi_0(\xi,\Gamma))[(\alpha-1)n^{-1}\Gamma^{-1}$$
$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{\pi_*(\xi,\Gamma)}{\pi_0(\xi,\Gamma)}\ d\Gamma d\xi$$
$$-\int_{R^p}\int_{S_p}\sum_{i\le j,k\le l}\{\frac{\partial}{\partial\tau_{kl}}[\pi_0(\xi,\Gamma)\nu_{ij,kl}(\Gamma)[(\alpha-1)n^{-1}\Gamma^{-1}$$
$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{n\pi_*(\xi,\Gamma)}{3\pi_0(\xi,\Gamma)}\ d\Gamma d\xi$$
$$+O(n^{-3/2}).$$

PROOF. Using Taylor expansions and (18), it can be shown by a straightforward though tedious calculation that for $\rho(X,nS^{-1})>0$,

$$\begin{aligned}
& R_1/[D\rho(X,nS^{-1})] \\
= \ & \frac{\pi_0(X,nS^{-1})}{2\pi_*(X,nS^{-1})}\sum_{i,j}[\frac{\partial^2}{\partial\xi_i\partial\xi_j}\rho(X,nS^{-1})] \\
& \times[\alpha n^{-2}S-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(nS^{-1},n)\eta'(nS^{-1},n)]_{ij} \\
& +\frac{\alpha\pi_0(X,nS^{-1})}{2\pi_*(X,nS^{-1})}\sum_{i\le j,k\le l}[\frac{\partial^2}{\partial\tau_{ij}\partial\tau_{kl}}\rho(X,nS^{-1})]\nu_{ij,kl}(nS^{-1}) \\
& +\pi_*(X,nS^{-1})^{-1}\sum_{i,j}[\frac{\partial}{\partial\xi_i}\pi_0(X,nS^{-1})][\frac{\partial}{\partial\xi_j}\rho(X,nS^{-1})] \\
& \times[\alpha n^{-2}S-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta\eta']_{ij} \\
& +\frac{\alpha}{\pi_*(X,nS^{-1})}\sum_{i\le j,k\le l}[\frac{\partial}{\partial\tau_{ij}}\pi_0(X,nS^{-1})][\frac{\partial}{\partial\tau_{kl}}\rho(X,nS^{-1})]\nu_{ij,kl}(nS^{-1}) \\
& +\frac{\pi_0(X,nS^{-1})}{6\pi_*(X,nS^{-1})}\sum_{i,j,k\le l,a\le b}[\frac{\partial^3}{\partial\xi_i\partial\xi_j\partial\tau_{kl}}M_{X,nS^{-1}}][\frac{\partial}{\partial\tau_{ab}}\rho(X,nS^{-1})] \\
& \times\nu_{kl,ab}(nS^{-1})[\alpha n^{-2}S-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta\eta']_{ij}
\end{aligned}$$

$$+\frac{\alpha\pi_0(X,nS^{-1})}{6\pi_*(X,nS^{-1})}\sum_{i\le j,k\le l,a\le b,r\le s}[\frac{\partial^3}{\partial\tau_{kl}\partial\tau_{ab}\partial\tau_{rs}}M_{X,nS^{-1}}][\frac{\partial}{\partial\tau_{ij}}\rho(X,nS^{-1})]$$
$$\times\nu_{ij,kl,ab,rs}(nS^{-1})+O(n^{-3/2}).$$

Furthermore we observe from (25) that by replacing α by 1 in the above equation, we obtain an approximation for R_4 with the same level of precision. Hence

$$\frac{R_1}{(1+R_4)\rho(X,nS^{-1})D}-\frac{\alpha R_4}{1+R_4}$$
$$= \frac{\pi_0(X,nS^{-1})}{2\pi_*(X,nS^{-1})}\sum_{i,j}[\frac{\partial^2}{\partial\xi_i\partial\xi_j}\rho(X,nS^{-1})][(\alpha-1)n^{-2}S$$
$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(nS^{-1},n)\eta'(nS^{-1},n)]_{ij}$$
$$+\pi_*(X,nS^{-1})^{-1}\sum_{i,j}[\frac{\partial}{\partial\xi_i}\pi_0(X,nS^{-1})][\frac{\partial}{\partial\xi_j}\rho(X,nS^{-1})]$$
$$\times[(\alpha-1)n^{-2}S-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta\eta']_{ij}$$
$$+\frac{\pi_0(X,nS^{-1})}{6\pi_*(X,nS^{-1})}\sum_{i,j,k\le l,a\le b}[\frac{\partial^3}{\partial\xi_i\partial\xi_j\partial\tau_{kl}}M_{X,nS^{-1}}][\frac{\partial}{\partial\tau_{ab}}\rho(X,nS^{-1})]$$
$$\times\nu_{kl,ab}(nS^{-1})[(\alpha-1)n^{-2}S-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta\eta']_{ij}$$
$$+O(n^{-3/2}).$$

Using integration by parts and Lemma 1, we get

$$E_{\pi_*}[\frac{R_1}{(1+R_4)\rho(X,nS^{-1})D}-\frac{\alpha R_4}{1+R_4}]$$
$$= \int_{R^p}\int_{S_p}\sum_{i,j}\{\frac{\partial^2}{\partial\xi_i\partial\xi_j}[\pi_0(\xi,\Gamma)[(\alpha-1)n^{-1}\Gamma^{-1}$$
$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{\pi_*(\xi,\Gamma)}{2\pi_0(\xi,\Gamma)}\,d\Gamma d\xi$$
$$-\int_{R^p}\int_{S_p}\sum_{i,j}\{\frac{\partial}{\partial\xi_j}[(\frac{\partial}{\partial\xi_i}\pi_0(\xi,\Gamma))[(\alpha-1)n^{-1}\Gamma^{-1}$$
$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{\pi_*(\xi,\Gamma)}{\pi_0(\xi,\Gamma)}\,d\Gamma d\xi$$
$$-\int_{R^p}\int_{S_p}\sum_{i\le j,k\le l}\{\frac{\partial}{\partial\tau_{kl}}[\pi_0(\xi,\Gamma)\nu_{ij,kl}(\Gamma)[(\alpha-1)n^{-1}\Gamma^{-1}$$

$$-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{n\pi_*(\xi,\Gamma)}{3\pi_0(\xi,\Gamma)}\,d\Gamma d\xi$$
$$+O(n^{-3/2}).$$

Also we observe from (23), (24) and the definition of R_5 that

$$E_{\pi_*}R_5 = E_{\pi_*}[\frac{R_1}{(1+R_4)\rho(X,nS^{-1})D} - \frac{\alpha R_4}{1+R_4}] + O(n^{-3/2}).$$

This observation together with Lemma 7 give the result. □

Corollary 3 *Under the conditions of Theorem 3, suppose that $\pi_0(\xi,\Gamma)$ is independent of ξ. Then*

$$\begin{aligned} & P_{\pi_*}(\xi\in\Omega_\alpha) \\ = \;& \alpha - \int_{R^p}\int_{S_p}\sum_{i\le j,k\le l}\{\frac{\partial}{\partial\tau_{kl}}[\pi_0(\Gamma)\nu_{ij,kl}(\Gamma)[(\alpha-1)n^{-1}\Gamma^{-1} - (2\pi)^{-1/2} \\ & \times\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\}\frac{n\pi_*(\xi,\Gamma)}{3\pi_0(\Gamma)}\,d\Gamma d\xi + O(n^{-3/2}).\end{aligned}$$

PROOF. This is immediate from Theorem 3. Here we note that condition 2 of Theorem 3 is always satisfied. □

Corollary 4 *Under the conditions of Corollary 3, suppose $\pi_0(\Gamma)$ satisfies*

$$\begin{aligned}&\sum_{i\le j,k\le l}\{\frac{\partial}{\partial\tau_{kl}}[\pi_0(\Gamma)\nu_{ij,kl}(\Gamma)[(\alpha-1)n^{-1}\Gamma^{-1} \\ &-(2\pi)^{-1/2}\Phi^{-1}(\alpha)e^{-[\Phi^{-1}(\alpha)]^2/2}\eta(\Gamma,n)\eta'(\Gamma,n)]_{ij}]\} = 0,\end{aligned}$$

for all Γ. Then

$$P_{\pi_*}(\xi\in\Omega_\alpha) = \alpha + O(n^{-3/2}).$$

PROOF. Immediate from Corollary 3. □

Finally, Corollary 4 suggests that for most parameter values, especially those that are not too close to the boundary of S_p,

$$P_{\xi,\Gamma}(\xi\in\Omega_\alpha) = \alpha + O(n^{-3/2}).$$

6 Acknowledgments

I would like to thank Professor Charles M. Stein for introducing this problem to me and also for the generous advice he offered me. I am also grateful to Professor Michael Woodroofe for alerting me to his work in this area.

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