THE SELF AVOIDING BRANCHING RANDOM WALK

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1. Introduction.

There has been a lot of interest in stochastic growth models in recent years. These models, in discrete or continuous time, generally take the following form. At time 0, a point on a lattice is filled or infected while other lattice points are not. A set of rules then determines how the filled or infected points may fill or infect other points. In the model of Richardson (1973), points become filled at a constant rate once one of the four nearest neighbors is filled. One could also depict bond percolation, as defined in Kesten (1982), in this fashion. At time 0, the origin is filled. At time 1, those nearest neighbors of the origin with open bonds are filled. At the time $n+1$, those points are filled which have open bonds to a point filled at time n. Thus the question of the existence of an infinite path is equivalent to asking when is there positive probability the process will continue indefinitely to grow. Directed percolation is another such example. Some models also have rules for unfilling or uninfecting points. In the model proposed in Williams and Bjerknes (1972), unfilled points are filled at a rate proportional to the number of nearest neighbors which are filled, while filled points become unfilled at a rate proportional to the number of unfilled nearest neighbors. In contact processes, as found in Durrett and Griffeath (1982), points are infected at a rate proportional to the number of infected nearest neighbors, but become uninfected at a constant rate.

All of the above models were motivated by the desire to model some physical phenomena. The self-avoid branching random walk, here-to-fore the SABRW, is also motivated by the desire to model a physical growth phenomena. Consider the growth properties of the branching fungus Aspergillus nidulans. A pellet sprouts hyphae which grow outwards, each tip then branches sprouting new hyphae, which themselves grow and branch again. Assuming that hyphae grow at constant rate, branch at regular intervals, and have a certain distribution of the number of offspring (i.e. new hyphae), one would expect an exponential mass increase over time. However due to crowding and/or substrate limitations, the number of branchings per unit area or volume is limited, so that the exponential phase of growth soon ends and growth becomes linear (for background information see Trinci (1969)).

The preceding life situation is certainly not unique. No population in nature can grow exponentially for very long due to growth limiting factors. To account for this, a branching random walk will be defined, but as opposed to treatments such as Biggins (1978), the process will be restricted to the two-dimensional integer lattice and the paths made self-avoiding. While motivated by a particular life situation, the SABRW is quite general, and one can embed bond percolation and directed percolation in it, though not the other models mentioned.

Section 2 contains definitions and notation for the SABRW. Section 3 contains some basic results. In section 4, the notation of an ergodic random field and its distribution will be defined. This leads to an identity concerning the distribution of an ergodic random field and its "successor". The identity has a non-trivial solution if and only if the probability of non-extinction (i.e. an infinite path) is positive. Section 5 considers the special case of directed percolation and contrasts results here with those of Durrett (1984) where random fields are also used to obtain results, though in a different fashion. In addition it will be pointed out how ergodic random fields may be used to study other processes and obtain similar identities related to non-extinction, and which processes this is likely to work for.

2. Definitions and Notation.

Assume that for each point (x, y) in a Z^2, the two-dimensional integer lattice, there corresponds a random subset $M_{x,y}$ contained in Z^2. Furthermore, if $M_{x,y} - (x, y)$ denotes the translate of the points in $M_{x,y}$ by x and y units respectively (i.e. translate $M_{x,y}$ to the origin), then assume

(2.1) $$\forall\, (x, y) \in Z^2,\ P(M_{x,y} - (x, y) = A) = P(M_{oo} = A).$$

(2.2) For any sequence of points $(x_1, y_1) - -(x_n, y_n)$, and sets $A_1, --, A_n$, the events $\{M_{x_1,y_1} = A_1\}, \ldots, \{M_{x_n,y_n} = A_n\}$ are independent.

Assumptions (2.1) and (2.2) amount to saying that the $M_{x,y}$ suitably centered are independent, identically distributed. Also assume there is a bounded subset A that

(2.3) $$P(M_{oo} \subset A) = 1.$$

Thus all points in $M_{x,y}$ are within a bounded distance of (x, y).

Intuitively, one can describe the development of the process as follows. At time 0, the origin is filled. At time $(n + 1)$ those points first filled at time n fill other points. Filled points remain filled. More formally define

$$Z_o = (o, o)$$

$$Z_1 = (0, 0) U M_{o,o}$$

$$\vdots$$

$$Z_{n+1} = U M_{x,y} U Z_n.$$

$$(x, y) \in \{Z_n - Z_{n-1}\}$$

In order to study the process Z_n, it is useful to consider the duel process $\tilde{Z}_n$ which is defined here as follows. Let

$$\tilde{M}_{x,y} = \{(x', y') | (x, y) \in M_{x',y'}\}$$

$$\tilde{Z}_o = (o, o)$$

$$\tilde{Z}_1 = (o, o) U \tilde{M}_{0,0}$$

$$\vdots$$

$$\tilde{Z}_{n+1} = U \tilde{M}_{x,y} U \tilde{Z}_n.$$

$$(x, y) \in \{\tilde{Z}_n - \tilde{Z}_{n-1}\}$$

The almost sure density of points in Z^2 which satisfy condition " " will mean the following. If A_n denotes the set of lattice points (x,y) where $|x| \leq n$, $|y| \leq n$, and if the sample proportion of such points which satisfy the condition " " converges almost to a constant, then this constant is the almost sure density referred to above.

The notation $(x, y) \to (x', y')$ means that the process begun at (x, y) eventually fills (x', y'), intuitively, a path exists beginning at (x, y) to (x', y'). One writes $(x, y) \leftrightarrow (x', y')$ iff $(x, y) \to (x'y')$ and $(x', y') \to (x, y)$.

Growth is said to be mandatory for the process in the Θ direction if and only if for H, the open half-plane whose edge contains (o, o) and is perpendicular to Θ, we have $P(M_{oo} \cap H \neq \phi) = 1$. That is to say, there will always be growth in the Θ direction.

If growth is not mandatory, it is possible that it will cease or become "extinct". The event "the process becomes extinct" may be written $\bigcup_n \{Z_{n+1} = Z_n\}$.

It is tacitly assumed that the process is "non-arithmetic", that is to say the process Z_n could not fit on a sub lattice. One could insure this by an assumption such as "for all $(x, y) \in Z^2$ there exists an n such that $P((x, y) \in Z_n) > 0$".

A few comments are in order. Z_n may be regarded as the set of points filled in $\leq n$ steps starting from the origin. $\tilde{z}_n$ are the points which would fill (o, o) in $\leq n$ steps had we started the process there. One might just as well choose any point in Z^2 other than the origin to define Z_n and $\tilde{Z}_n$. Clearly one can embed bond percolation in the SABRW by putting the right distribution on the $M_{x,y}$'s. Thus the probability of an infinite path is the probability of non-extinction.

3. Basic Results.

Proposition 3.1 $P(\text{extinction}) = 0$ iff growth is mandatory in some direction.

Proof. One must only show there is positive probability of extinction if growth isn't mandatory in any direction. Since the $M_{x,y}$ are bounded, there exists a constant $r_o > 0$ such that points only within r_o of (x, y) may be filled. Now let r_1 be same number greater than o, and consider the event $E_{r_1} = \{$no lattice point (x, y) such that $r_1 \leq \sqrt{x^2 + y^2} \leq$

$r_1 + r_o$ will fill points in the open half-plane lying in the direction of the nearest point of the circle of radius $r_o + r_1$ about the origin}. $P(E_{r_1}) > 0$ for any $r_1 > 0$. Also it is clear that most points in this annulus, on the event E_{r_1}, have no chance of filling points outside the circle of radius $r_o + r_1$ about the origin. However, lattice points near the edge of this circle might possibly fill points on the outside, even if the edge of the circle is almost "flat", i.e. might approximate the edge of the half-plane. Clearly r_1 may be chosen large enough so that only one or, at most, two points might be filled outside the circle from a bounded number of points within the circle. If only one point is available outside, the half-plane could be slightly turned to include it, and so there is still probability greater than zero that it will not be filled. Now suppose two points outside could be filled. If, conditioned on filling no points in the open half-plane, both points (or one fixed) point were filled with probability one, this would imply growth would be mandatory in some direction. If only one is filled, with probability greater than zero, it would only fill points already filled. Thus a subset of E_{r_1} exists with positive probability where extinction must occur.

In addition to non-arithmetic, i.i.d., and boundedness for $\{M_{x,y}\}$, we heretofore assume symmetry, i.e. realizations of M_{oo} are equiprobable when reflected about one or both axes.

Proposition 3.2. If growth is mandatory, the conditional probability that $(o, o) \rightarrow (x, y)$ given that the dual process begun at (x, y) is never extinct, is one.

Proof. The paths of Z_n and the dual process begun at (x, y) must cross infinitely often so will eventually hit.

This suggests the following results for which we let $\tilde{p} = P(\tilde{Z}_n$ never extinct).

Proposition 3.3. If growth is mandatory, the almost sure density of points is the plane filled eventually by Z_n is $\tilde{p}$.

Proof. In view of the previous proposition it suffices to show that the almost sure proportion of points in the plane whose dual process never dies out is $\tilde{p}$. By the ergodic theorem in two-dimensions, the relative proportion of points whose dual process never dies has an almost sure limit X. In addition, the proportion of points whose dual process lives at least n steps has an almost sure limit equal to $\tilde{p}_n = P(\tilde{Z}_n$ not extinct). Clearly $X \leq \tilde{p}_n$ a.s. for each n and $EX = \tilde{p} = \lim_{n\to\infty} \tilde{p}_n$ so $X = \tilde{p}$ a.s.

From this point on, consider the case where growth is not mandatory for which the additional assumption is made that for any sequence $(x_1, y_1), \ldots, (x_n, y_n)$ and any point (x, y)

$$P\{(x_1, y_1), \ldots, (x_n, y_n)\} M_{x,y}) = \prod_{i=1}^{n} P((x_i, y_i) \in M_{x,y})$$

that is, points are filled independently. In this case $p = P(Z_n$ never extinct$) = \tilde{p}$ since Z_n and $\tilde{Z}_n$ will now have the same distribution. In this case proposition 3.3 becomes (for roughly the same reasons).

Proposition 3.4. Assume $o < p < 1$. Then on the set of non-extinction, the almost sure density of filled points is p.

For the last result of this section, we say a point is self-dual if and only if the SABRW and its dual process are never extinct. Thus a point (x, y) may be self-dual on some realizations and not on others.

Corollary 3.1. The conditional probability that $(x, y) \leftrightarrow (x', y')$ given that (x, y) and (x', y') are self-dual is one.

The analogous result for bond percolation is "there is one infinite path". Self-dual points are also interesting since they can provide a sub-additive sequence to show there is an asymptotic shape. This will be studied in another publication.

4. Ergodic Random Fields

When will the process have positive probability of never dying out? In branching processes a non-trivial fixed point of the generating function for the offspring distribution guarantees this. For the SABRW, a non-trivial fixed point is also a necessary and sufficient condition for positive probability of non-extinction. Here, however, the condition for positive probability of non-extinction. Here, however, the condition is stated in terms of distributional properties of certain random fields, for which some more notation must be introduced.

On the integer lattice, points have three possible states: unfilled (type 0), filled and ready to branch (type 1), filled, but cannot branch (type 2). Thus the set of functions from Z^2 into $\{0, 1, 2\}$ describes all possible configurations. A probability distribution on this set yields a random measure, and let $\mathcal{M}$ denote the set of all such random measures.

As the process develops, the configuration of various types of points (filled or not) about the filled points of type 1 determine the growth behavior. Suppose that the origin is filled (type 1) and consider the finite set of possible configurations of filled and unfilled lattice points in the closed $2n \times 2n$ square centered at the origin. Denote this set Ω_n and its elements ω_n. Now for a particular random field $M \in \mathcal{M}$, it is of interest to know what "proportion" of filled points of type 1 have configuartion ω_n when centered. Define, for $A_m = 2m \times 2m$ closed square

$$\hat{p}_m(\omega_n) = \frac{\#\text{ filled points of type 1 with configuration } \omega_n}{\#\text{ filled points of type 1 if defined,}}$$

$$= 0 \text{ if not.}$$

Now if for all n, and each $\omega_n \in \Omega_n$ the $\lim_{m \to \infty} \hat{p}_m(\omega_n) = P_M(\omega_n)$ almost surely and $P_M(\omega_n)$ is almost surely a constant, then the random field M is <u>ergodic</u>. If the almost sure limit exists but is not a constant, then we say the random field is <u>weakly ergodic</u>. Clearly one way to get weakly ergodic fields is by mixing ergodic fields.

Since the almost sure limit exists for each n, one can use Kolmogorov extension to construct the "centered" or "averaged" random field, denoted $A(M)$, at the origin, with distribution function $P_M(\ )$ yielding probabilities of the individual $\omega_n \in \Omega_n$ for each n. We say that two random fields M and M' are <u>equal in law</u>, denoted $M \stackrel{\ell}{=} M'$, if $A(M)$ and $A(M')$ have the same distribution function.

Given an ergodic random field M, one also can get its successor, i.e. the random field defined by letting the type 1 particles branch. As long as the successor field has type 1 particles, it will also be ergodic (or weakly ergodic). Denote the successor of M by $\phi(M)$. In addition let $X(\omega_n)$ be number of type 1 points in configuration $\omega_n \in \Omega_n$. Let c_M be defined by

$$c_M = \lim_{n \to \infty} \inf \frac{E \int_{\Omega_n} X(\omega_n) dP_M(\omega_n)}{n}. \tag{4.1}$$

It turns out that the question "when does the process have positive probability of non-extinction?" is equivalent to when does there exist a solution to

$$(4.2) \qquad \phi(M) \stackrel{\ell}{=} M \text{ with } c_M > 0?$$

There are certain powerful intuitive reasons for examining (4.2). On the set of non-extinction, growth should occur at the edge, moving outward in some regular fashion as it does in fact in nature. If so, the configurations about type 1 points and their density should stabilize also. It may be, that even if there is probability one of extinction, (4.2) could still be satisfied. However, from the last section one knows that an almost sure fraction of points in the lattice will eventually be filled is equal to the probability of non-extinction. Thus only "sparse" or "trivial" solutions with $c_M = 0$ will exist in this case when extinction is assured. Now in Markov Chain theory, one way to prove a stationary distribution exists is to average the probabilities of being in a certain state over time, take a limit and show that the limit is a solution. Here, we fill the left half-plane of lattice points, take mixtures of the succession random fields, find a limit and show it satisfies equation (4.2).

<u>Theorem 4.1</u>: $p > o$ if and only if a solution of (4.2) exists with $c_M > o$.

<u>Proof</u>: Let $\overline{M}$ denote the random field of filled lattice points in the left half-plane. Let M_o be a realization of $A(\overline{M})$. Successively let M_1 be a realization of $A(\phi(\overline{M}))$, M_2 a realization of $A(\phi(\phi(\overline{M})))$, etc. Clearly $M_o, M_1, \ldots$ are all ergodic. Let A_{ij} $j = 1, 2, \ldots, i$, $i = 1, 2, \ldots$ be an array of events, independent of M_o, $M_1, \ldots$, such that $P(A_{ij}) = 1/i$ and for fixed i, $P(A_{ij} \cap A'_{ij}) = 0$ if $j \neq j'$. Now define a sequence $M'_o, M'_1, \ldots$ of weakly ergodic

random measures by defining their distribution functions

$$P_{M_i'}(\cdot) = \sum_{j=1}^{i} I_{A_{ij}} \cdot P_{M_i}(\cdot).$$

So the new sequence is a mixture of the old.

Clearly for each Ω_n, there is a subsequence for which the vector $P_{M_{i_j}}(\omega_n^1), P_{M_{i_j}}(\omega_n^2)$ $\ldots$, $P_{M_{i_j}}(\omega_n^{m_n})$), where $\omega_n^1, \ldots, \omega_n^{m_n}$ are the m_n elements of Ω_n, converges in law and by diagonalizing this will be so for all Ω_n. So now it remains to show that this limiting distribution, call it $P_M(\cdot)$ of the random field M, is a fixed point and that $c_M > 0$. To do this a metric on the space of distribution functions will be constructed with respect to which $\phi(\ )$ is continuous, where convergence is equivalent to convergence in law, and with respect to which M and $\phi(M)$ can be shown to be arbitrarily close.

Let, for a particular random field M, and vector $\underset{\sim}{x} = (x_1, \ldots, x_{m_n}) F_n(\underset{\sim}{x}) = P(P_M(\omega_n')$ $\leq x_1, \ldots, P_M(\omega_n^{m_n}) \leq x_{m_n})$. Following an idea in Grandell (1977), let, for F_n defined by P_M, F_n' defined by P_M' dist $(F_n, F_n') = \sup_{\underset{\sim}{x}} \inf_t \{t > 0 | F_n(\underset{\sim}{x} - t) - t \leq F_n'(x) \leq F_n(\underset{\sim}{x} + t) + t\}$. Now let the metric $||, ||$ be defined by $||P_M, P_{M'}|| = \sum_{n=1}^{\infty} 2^{-n} \cdot$ dist (F_n, F_n'). It is not difficult to show $P_M \to P_{\phi(M)}$ is a continuous mapping for ergodic random fields, and then extend to weakly ergodic fields. To simplify notation let the sequence of random fields converging in law to M be denoted M_n. Clearly $||M, \Phi(M))|| \leq ||M, M_n|| + ||M_n, M_{n+1}|| +$ $||M_{n+1}, \Phi(M)||$. Each of the three terms on the right hand side can be made arbitrarily small for large n. The first term by convergence in law, the second by the way the M_n were constructed, the third by continuity of $\Phi(\cdot)$.

Now for $c_M > 0$. It is clear from the previous section that in the right half-plane the almost sure proportion of points for which the dual process is never extinct is $p > o$. Since the process grows in at least a linear fashion, the probability (on $\tilde{Z}_n \to \infty$) that a point is filled outside a circle of radius an, in n steps is bounded away from 0, for some constant $a > o$. Thus the E (number of filled points in n steps ini A_n) is at least a positive constant times n. Since M_n is a mixture of past steps, each filled point was counted as open at some time. This bounds the right hand side at (4.3) from below for each n. Thus $c_M > o$.

Now suppose the probability of extinction is 1, and M is a solution of 4.2. In a finite number of steps the expected number of type 1 offspring is less than one. Thus $\phi(\phi(\phi(\ldots\phi(M))\ldots))$ has $c_{\phi(..\phi(M))} < c_M$ but M is a fixed point. Contradiction.

5. Comments

In Durrett (1984), the author considers directed percolation. It is shown that if one fills the points of the horizontal axis to the left of the origin and considers the rightmost filled point about to branch (type 1 in our notation), then the configurations converge in law. Also it was shown that if the probability of extinction is one, the distance to filled points directly the left is arbitrarily large, i.e. infinite in expected value. This corresponds to solutions of the identity with $c_M = 0$. Also, if the probability of an infinite path is greater than zero, one can find a fixed point to the identity by letting all points on the horizontal axis which have a dual process which is never extinct be filled and type 1, all points below the axis with this property be filled and type 2. Denoting this random field M, M is an ergodic solution of the identity with $c_M = P$(infinite path). This is probably true in the more general setting.

Another area where an identity like (4.2) should hold is contact processes. If an infection spreads throughout the plane then the configurations around the averaged infected point should stabilize. In fact, the average infected point may be regarded as the average point about to either infect a neighboring point or become uninfected. So there should be a distribution $P_M(\cdot)$ which is fixed here also.

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