THINNING OF POINTS PROCESSES
– MARTINGALE METHOD*

by

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Abstract

By using martingale method we show that thinning of an arbitrary point process produces independent thinned processes if and only if the original point process is (non-homogeneous) Poisson.

KEY WORDS: point process, Poisson process, dual predictable projection.

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Thinning is a classical problem for point processes. It is well-known that independent homogeneous Poisson processes result from constant Bernoulli thinnings of homogeneous Poisson processes. In fact, the conclusion remains true for non-constant Bernoulli thinnings of nonhomogeneous Poisson processes, though it seems that nowhere one could see its serious proof. It is easy to see that the converse is also true, i.e., if the thinned processes are independent nonhomogeneous Poisson processes, so are the original processes. But if we only suppose the thinned processes are independent, nothing is concerned with their distribution law, the problem whether or not the original processes are (nonhomogeneous) Poisson becomes interesting and challenging. This is just the objective of this paper. It is considerably surprising for us to arrive at the affirmative answer.

So far as to this problem, the most works were done under the renewal assumption. For example, [1] showed that for arbitrary delayed renewal processes the existence of a pair of uncorrelated thinned processes is sufficient to guarantee that the original process is Poisson. It is natural that the mathematical tools to solve the problem in this case be typic ones for renewal theory, such as renewal equations and Laplace–Stieltjes transformations. Obviously, they are not available for non-renewal processes.

We find out that the martingale method is the most efficient one to solve this problem in general case. More precisely, we use mainly the dual predictable projections of point processes. In fact, the distribution law of a point process is determined uniquely by its dual predictable projection (see [4]), and [5] offered us a very useful criterion of independence of jump processes having no common jump time through their dual predictable projections. Based on these results, it is not a distant way to reach at the destination.

1. Preliminaries.

The discussion will be proceeded on a fixed complete probability space $(\Omega, \mathcal{F}, P)$.

Let $(T_n)_{n\geq 1}$ be a sequence of positive random variables (the value $+\infty$ is permitted) such that $(T_0 \equiv 0)$

(1) for each $n \geq 0$, $T_n \leq T_{n+1}$, and $T_n < \infty$ implies $T_n < T_{n+1}$,

(2) $T_n \to \infty$ as $n \to \infty$.

Set

$$X_t = \sum_{n=1}^{\infty} 1_{T_n \leq t}, t \geq 0. \tag{1.1}$$

(Obviously, $X_0 \equiv 0$). Then $X = (X_t)_{t\geq 0}$ or $(T_n)_{n\geq 1}$ is called a point process. Denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the natural filtration of X:

$$\mathcal{F}_t^X = \sigma\{X_s,\ s \leq t\},\ t \geq 0.$$

(We denote by $\sigma\{\cdots\}$ the complete σ-field generated by $\{\cdots\}$.) It is well-known (see [3] or [4]) that each T_n is a stopping time with respect to $(\mathcal{F}_t^X)_{t\geq 0}$, and

$$\mathcal{F}_{T_n}^X = \sigma\{T_1,\cdots,T_n\},\ n\geq 0,$$
$$\mathcal{F}_t^X = \bigcup_{n=0}^{\infty}\mathcal{F}_{T_n}^X\{T_n \leq T < T_{n+1}\},\ t\geq 0.$$

Denote by $H_n(dt)$ the conditional distribution of T_n, given $\mathcal{F}_{T_{n-1}}^X$. Then the dual predictable projection of X with respect to $(\mathcal{F}_t^X)_{t\geq 0}$ is

$$\wedge(dt) = \sum_{n=1}^{\infty} \frac{H_n(dt)}{H_n([t,\infty])} 1_{T_{n-1}<t\leq T_n} \tag{1.2}$$

Let $(\eta_n)_{n\geq 1}$ be a sequence of random variables such that for each $n \geq 1$

$$\eta_n = 0 \text{ on } \{T_n = \infty\}. \tag{1.3}$$

(In fact, it is only needed to suppose η_n is constant on $\{T_n = \infty\}$.) Then $(T_n, \eta_n)_{n\geq 1}$ is called a marked point process. Its natural filtration is defined as follows:

$$\mathcal{F}_t = \sigma\{\sum_{n=1}^{\infty} 1_{T_n\leq s,\ \eta_n\in B}, s\leq t,\ B\in \mathcal{B}\},\ t\geq 0. \tag{1.4}$$

In this case, we have

$$\mathcal{F}_{T_n} = \sigma\{T_1,\cdots,T_n,\ \eta_1,\cdots,\eta_n\},\ n\geq 1,$$
$$\mathcal{F}_{T_n-} = \sigma\{T_1,\cdots,T_n,\ \eta_1,\cdots,\eta_{n-1}\},\ n\geq 1,$$
$$\mathcal{F}_t = \bigcup_{n=0}^{\infty}\mathcal{F}_{T_n}\{T_n\leq t < T_{n+1}\},\ t\geq 0.$$

Obviously, $\mathcal{F}_{T_0} = \mathcal{F}_0 = \sigma\{\phi, \Omega\}$.

An equivalent representation of marked point process $(T_n, \eta_n)_{n\geq 1}$ is the following integer–valued random measure on $(\mathbb{R}_+ \times \mathbb{R},\ \mathcal{B}_+ \times \mathcal{B})$.

$$\mu(dt, dx) = \sum_{n=1}^{\infty} \delta_{(T_n,\eta_n)}(dt, dx) 1_{T_n<\infty} \tag{1.5}$$

where δ_a is the unit measure concentrating on point a. Denote by $\nu(dt, dx)$ the dual predictable projection of $\mu(dt, dx)$ with respect to $(\mathcal{F}_t)_{t\geq 0}$:

$$\nu(dt, dx) = \sum_{n=1}^{\infty} \frac{G_n(dt, dx)}{H_n([t,\infty])} 1_{T_{n-1}<t\leq T_n} \tag{1.6}$$

where $G_n(dt, dx) = P[T_n \in dt,\ \eta_n \in dx | \mathcal{F}_{T_{n-1}}],\ n \geq 1$, is the conditional distribution of (T_n, η_n), given $\mathcal{F}_{T_{n-1}}$.

Set

$$Y_t = \sum_{n=1}^{\infty} \eta_n 1_{T_n \leq t},\ t \geq 0. \tag{1.7}$$

If for each $n \geq 1$

$$\{T_n = \infty\} = \{\eta_n = 0\}, \tag{1.8}$$

then we have

$$\mathcal{F}_t = \sigma\{Y_s, s \leq t\},\ t \geq 0,$$

i.e., $(\mathcal{F}_t)_{t\geq 0}$ is just the natural filtration of $Y = (Y_t)_{t\geq 0}$. At this time, $\mu(dt, dx)$ is the jump measure of Y. Hence, under the assumption (1.8), the jump process $Y = (Y_t)_{t\geq 0}$ is another equivalent representation of marked point process $(T_n, \eta_n)_{n\geq 1}$.

THEOREM 1. *Let $N(x, t, dy)$ be a transition probability kernel from $(\mathbb{R} \times \mathbb{R}_+,\ \mathcal{B} \times \mathcal{B}_+)$ to $(\mathbb{R}, \mathcal{B})$. The following statements are equivalent:*

(1) *For $n \geq 1,\ B_j \in \mathcal{B},\ j = 1, \cdots, n$, on $\{T_n < \infty\}$*

$$P[\eta_j \in B_j,\ j = 1, \cdots, n \,|\, \mathcal{F}_\infty^X] = \int_{\mathbb{R}^n} \prod_{j=1}^{n} 1_{B_j}(y_j) N(y_{j-1} T_j, dy_j) \tag{1.9}$$

(2) (i) *For $n \geq 1,\ A \in \mathcal{B}_+$*

$$P[T_n \in A \,|\, T_1, \cdots, T_{n-1},\ \eta_1, \cdots, \eta_{n-1}] = P[T_n \in A \,|\, T_1, \cdots, T_{n-1}] \tag{1.10}$$

(ii) *For $n \geq 1,\ B \in \mathcal{B}$, on $\{T_n < \infty\}$*

$$P[\eta_n \in B \,|\, T_1, \cdots, T_n,\ \eta_1, \cdots, \eta_{n-1}] = N(\eta_{n-1}, T_n, B) \tag{1.11}$$

(3)

$$\nu(dlt, dx) = N(Y_{t-}, t, dx) \wedge (dt) \tag{1.12}$$

where $\eta_0 = y_0 = 0$.

Proof. By using the standard argument it is not difficult to show (1.9) is equivalent to the following more general statement: for $n \geq 1,\ f_j \in b\mathcal{B}$ (bounded Borel functions), $j = 1, \cdots, n$, on $\{T_n < \infty\}$

$$E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, \mathcal{F}_\infty^X] = \int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(y_j) N(y_{j-1} T_j, dy_j) \tag{1.13}$$

and (1.11) is equivalent to the following one, for $n \geq 1,\ f \in b\mathcal{B}$, on $\{T_n < \infty\}$.

$$E[f(\eta_n) \,|\, T_1, \cdots, T_n,\ \eta_1, \cdots, \eta_{n-1}] = N(\eta_{n-1}, T_n, f) \tag{1.14}$$

where as usual we have

$$N(x,t,f) = \int_{\mathbb{R}} f(y) N(x,t,dy).$$

(1) $\Rightarrow$ (2). Since the right side of (1.13) is only dependent on $\{T_1, \cdots, T_n\}$, conditioning on $\{T_1, \cdots, T_n\}$, from (1.13) we get

$$E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, \mathcal{F}_\infty^X] = E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_n]. \tag{1.15}$$

From (1.15) we also have

$$E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_{n+1}] = E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_n]. \tag{1.16}$$

Notice that (1.16) and (1.10) formulate the same fact: for $n \geq 1$, $\{\eta_1, \cdots, \eta_n\}$ and T_{n+1} are conditionally independent, given $\{T_1, \cdots, T_n\}$. Hence, (1.10) has been shown.

On the other hand, for $n \geq 1,\ f, f_j \in b\mathcal{B},\ j = 1, \cdots, n-1,\ g \in b\mathcal{B}^n$, we have

$$\begin{aligned}
&E\{f(\eta_n) \prod_{j=1}^{n-1} f_j(\eta_j) g(T_1, \cdots, T_n) 1_{T_n<\infty}\} \\
=&E\{g(T_1, \cdots, T_n) \int_{\mathbb{R}^{n-1}} \prod_{j=1}^{n-1} f_j(y_j) N(y_{j-1}, T_j, dy_j) N(y_{n-1}, T_n, f) 1_{T_n<\infty}\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } (1.13)) \\
=&E\{g(T_1, \cdots, T_n) E[\prod_{j=1}^{n-1} f_j(\eta_j) N(\eta_{n-1}, T_n, f) \,|\, T_1, \cdots, T_n] 1_{T_n<\infty}\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } (1.13), (1.15)) \\
=&E\{N(\eta_{n-1}, T_n, f) \prod_{j=1}^{n-1} f_j(\eta_j) g(T_1, \cdots, T_n) 1_{T_n<\infty}\},
\end{aligned}$$

then (1.14) follows.

(2) $\Rightarrow$ (1). By using the conditional independence (1.10) we obtain: for $k > n \geq 1,\ f_j \in b\mathcal{B},\ j = 1, \cdots, n$,

$$\begin{aligned}
E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_k] &= E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_{k-1}] \\
&= \cdots\cdots \\
&= E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_n].
\end{aligned} \tag{1.17}$$

Letting $k \to \infty$ in (1.17) yields (1.15). Now it is sufficient to show: on $\{T_n < \infty\}$

$$E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_n] = \int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(y_j) N(y_{j-1}, T_j, dy_j). \tag{1.18}$$

We'll make it by induction on n. In fact, for $n = 1$, (1.18) is identified with (1.14). Suppose (1.18) holds for $n-1$, then on $\{T_n < \infty\}$

$$\begin{aligned}
&E[\prod_{j=1}^{n} f_j(\eta_j) \,|\, T_1, \cdots, T_n] \\
=&E[\prod_{j=1}^{n-1} f_j(\eta_j) E[f_n(\eta_n) \,|\, T_1, \cdots, T_n, \ \eta_1, \cdots, \eta_{n-1}] \,|\, T_1, \cdots, T_n] \\
=&E[\prod_{j=1}^{n-1} f_j(\eta_j) N(\eta_{n-1}, T_n, f_n) \,|\, T_1, \cdots, T_n] \qquad \text{(by (1.14))} \\
=&E[\prod_{j=1}^{n-1} f_j(\eta_j) N(\eta_{n-1}, t, f_n) \,|\, T_1, \cdots, T_n] \,|_{t=T_n} \\
=&E[\prod_{j=1}^{n-1} f_j(\eta_j) N(\eta_{n-1} t, f_n) \,|\, T_1, \cdots, T_{n-1} \,|_{t=T_n} \\
&\qquad \text{(by conditional independence)} \\
=&\int_{\mathbb{R}^{n-1}} \prod_{j=1}^{n-1} f_j(y_j) N(y_{n-1} t, f_n) \prod_{j=1}^{n-1} N(y_{j-1}, T_j, dy_j) \,|_{t=T_n} \\
&\qquad \text{(by induction)} \\
=&\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(y_j) N(y_{j-1}, T_j, dy_j).
\end{aligned}$$

(2) $\Rightarrow$ (3). From (1.10) and (1.11) we have

$$\begin{aligned}
G_n(dt, dx) &= E[1_{T_n \in dt} E[1_{\eta_n \in dx} \,|\, \mathcal{F}_{T_n -}] \,|\, \mathcal{F}_{T_{n-1}}] \\
&= E[1_{T_n \in dt} N(\eta_{n-1}, T_n, dx) \,|\, \mathcal{F}_{T_n - 1}] \\
&= N(\eta_{n-1}, t, dx) H_n(dt).
\end{aligned}$$

By (1.2) and (1.6) we obtain

$$\begin{aligned}
\nu(dt, dx) &= \sum_{n=1}^{\infty} \frac{N(\eta_{n-1}, t, dx) H_n(dt)}{H_n([t, \infty])} 1_{T_{n-1} < t \le T_n} \\
&= N(Y_{t-}, t, dx) \sum_{n=1}^{\infty} \frac{H_n(dt)}{H_n([t, \infty])} 1_{T_{n-1} < t \le T_n} \\
&= N(Y_{t-}, t, dx) \wedge (dt).
\end{aligned}$$

(3) $\Rightarrow$ (1). No loss generality, we may suppose $(\Omega, \mathcal{F}, P)$ is the standard space of marked point processes, and $\mathcal{F} = \mathcal{F}_\infty$. We construct another probability measure $\overline{P}$ on $\mathcal{F}$ such that:

1) Under P and $\overline{P}$, $(T_n)_{n \ge 1}$ has the same distribution law.
2) (1.9) holds with replacing P by $\overline{P}$.

Then based on the results shown above, we conclude that under $\overline{P}$ the dual predictable projection $\overline{\nu}(dt, dx)$ of $\mu(dt, dx)$ is just $\nu(dt, dx)$:

$$\overline{\nu}(dt, dx) = \nu(dt, dx).$$

Hence, $P = \overline{P}$, because the law of a marked point process is determined uniquely by its dual predictable projection (see [4]). Therefore, (1.9) comes true again. □

COROLLARY. *Suppose (1.8) holds.*

(1) *If $X = (X_t)_{t\geq 0}$ is a process with independent increments and (1.9) holds, then $Y = (Y_t)_{t\geq 0}$ is a Markov process.*

(2) *$Y = (Y_t)_{t\geq 0}$ is a process with independent increments if and only if $X = (X_t)_{t\geq 0}$ is a process with independent increments and for $n \geq 1$, $B_j \in \mathcal{B}$, $j = 1, \cdots, n$, on $\{T_n < \infty\}$*

$$P[\eta_j \in B_j,\ j = 1, \cdots, n \,|\, \mathcal{F}_\infty^X] = \prod_{j=1}^{n} N(T_j, B_j) \tag{1.19}$$

where $N(t, dy)$ is a transition probability kernel from $(\mathbb{R}_+, \mathcal{B}_+)$ to $(\mathbb{R}, \mathcal{B})$.

Proof. A jump process is a process with independent increments if and only if the dual predictable projection of its jump measure is non–random. The conclusions (1) and (2) follow immediately from Theorem 4 in [3] and Theorem 2 in [6] respectively. □

We point out that in general (1.10) cannot be deduced from (1.11). The following is such an example.

Let $(\xi_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, distributed uniformly on $[0, 1)$. Set

$$T_1 = 1, \qquad T_n = n + \xi_{n-1},\ n \geq 2,$$
$$\eta_n = 1_{[\frac{1}{2},1)}(\xi_n), \qquad n \geq 1.$$

Then for $n \geq 1$

$$P[\eta_n = 1 \,|\, T_1, \cdots, T_n,\ \eta_1, \cdots, \eta_{n-1}] = \frac{1}{2},$$

i.e. (1.11) holds with $N(x, t, dy) = \frac{1}{2}\delta_0(dy) + \frac{1}{2}\delta_1(dy)$, and

$$\begin{aligned} &P[T_n \in A \,|\, T_1, \cdots, T_{n-1},\ \eta_1, \cdots, \eta_{n-1}] \\ &= 2m(A \cap [n, n + \frac{1}{2}))1_{\xi_n=0} + 2m(A \cap [n + \frac{1}{2},\ n + 1))1_{\xi_n=1} \end{aligned} \tag{1.20}$$

where m is Lebesgue measure. If (1.10) also holds, then $P[T_n \in A \,|\, T_1, \cdots, T_{n-1},\ \eta_1, \cdots \eta_{n-1}]$ is a function of $\{T_1, \cdots, T_{n-1}\}$, which must be independent of ξ_{n-1}. But this contradicts (1.20). Hence, (1.10) does not hold.

For the sake of simplicity, we consider real mark sequences $(\eta_n)_{n\geq 1}$ in the above. Obviously, more general mark sequences can be considered as in [4].

2. A criterion of independence of jump processes.

Let $X^1 = (X^1_t)_{t\geq 0}$ and $X^2 = (X^2_t)_{t\geq 0}$ be two jump processes, i.e.

$$X^i_t = \sum_{n=1}^{\infty} \eta^i_n 1_{T^i_n \leq t}, t \geq 0,\ i = 1, 2. \tag{2.1}$$

where $(T^i_n)_{n\geq}$ and $(\eta^i_n)_{n\geq 1}$, $i = 1, 2$, satisfy

1) $T^i_n \leq T^i_{n+1}$, and $T^i_n < \infty$ implies $T^i_n < T^i_{n+1}$, $n \geq 0$, $(T^i_0 = 0)$, and $T^i_n \to \infty$ as $n \to \infty$, $i = 1, 2$.
2) $\{T^i_n = \infty\} = \{\eta^i_n \neq 0\}$, $n \geq 1$, $i = 1, 2$.

The jump measure of X^i is

$$\mu_i(dt, dx) = \sum_{n=1}^{\infty} \delta_{(T^i_n, \eta^i_n)}(dt, dx) 1_{T^i_n < \infty}, i = 1, 2. \tag{2.2}$$

Set

$$\mathcal{F}^i_t = \sigma\{X^i_s,\ s \leq t\}, \qquad t \geq 0,\ i = 1, 2, \tag{2.3}$$

$$\mathcal{F}_t = \mathcal{F}^1_t \vee \mathcal{F}^2_t, \qquad t \geq 0. \tag{2.4}$$

Denote by $\nu_i(dt, dx)$ the dual predictable projection of $\mu_i(dt, dx)$ with respect to $(\mathcal{F}^i_t)_{t\geq 0}$.

THEOREM 2. *Suppose $\Delta X^1 \Delta X^2 = 0$. Then X^1 and X^2 are independent if and only if the following conditions are satisfied:*

1) *$\nu_i(dt, dx)$ is the dual predictable projection of $\mu_i(dt, dx)$ with respect to $(\mathcal{F}_t)_{t\geq 0}$, $i = 1, 2$.*
2) *For real $t > 0$, $j, k \geq 1$*

$$P(T^1_j = t) P(T^2_k = t) = 0. \tag{2.5}$$

The proof of "only if" part can be referred to Lemma 4 and 5 in [2]. The proof of "if" part is due to Zhiming Ma [5]. Since [5] is written in Chinese, we sketch the proof as follows.

No loss generality, we may suppose $(\Omega, \mathcal{F}, P)$ is the standard space of two–dimensional marked point processes, i.e.

$$\Omega = \{\omega = (\omega^1_t,\ \omega^2_t)_{t\geq 0}\},$$
$$X^i_t(\omega) = \omega^i_t, \quad t \geq 0,\ i = 1, 2,$$
$$\mathcal{F} = \mathcal{F}^1_\infty \vee \mathcal{F}^2_\infty$$

where ω_t^1 and ω_t^2 are step functions, right continuous and with left hand limits. On the other hand, Ω can be considered as

$$\Omega = \Omega^1 \times \Omega^2, \ \mathcal{F} = \mathcal{F}_\infty^1 \times \mathcal{F}_\infty^2,$$
$$\Omega^i = \{\omega^i = (\omega_t^i)_{t\geq 0}\}, \ i = 1, 2.$$

Denote by P^i the restriction of P on $\mathcal{F}_\infty^i, i = 1, 2$.

Define a probability measure

$$\overline{P} = P^1 \times P^2,$$

on $\mathcal{F} = \mathcal{F}_\infty^1 \times \mathcal{F}_\infty^2$. Because of (2.5), under $\overline{P}$ we also have $\Delta X^1 \Delta X^2 = 0$. Define

$$X_t(\omega) = (X_t^1(\omega), X_t^2(\omega)), \quad t \geq 0.$$

Then under both P and $\overline{P}$ we have

$$\mathcal{F}_t = \sigma\{X_s, s \leq t\} = \mathcal{F}_t^1 \vee \mathcal{F}_t^2, \quad t \geq 0.$$

Set

$$\begin{aligned}
\mu_1(dt, dx_1, dx_2) &= \mu_1(dt, dx_1)\delta_0(dx_2), \\
\mu_2(dt, dx_1, dx_2) &= \mu_2(dt, dx_2)\delta_0(dx_1), \\
\nu_1(dt, dx_1, dx_2) &= \nu_1(dt, dx_1)\delta_0(dx_2), \\
\nu_2(dt, dx_1, dx_2) &= \nu_2(dt, dx_2)\delta_0(dx_1), \\
\mu(dt, dx_1, dx_2) &= \mu_1(dt, dx_1, dx_2) + \mu_2(dt, dx_1, dx_2), \\
\nu(dt, dx_1, dx_2) &= \nu_1(dt, dx_1, dx_2) + \nu_2(dt, dx_1, dx_2).
\end{aligned}$$

Under both P and $\overline{P}$, ν_1 and ν_2 are the dual predictable projections of μ_1 and μ_2 with respect to $(\mathcal{F}_t)_{t\geq 0}$, respectively, and μ is the jump measure of X. Hence, under P and $\overline{P}$ the dual predictable projections of μ with respect to $(\mathcal{F}_t)_{t\geq 0}$ is the same. Therefore, $P = \overline{P}$, i.e. X^1 and X^2 are independent.

3. Thinning of point processes.

Now we turn to the non–constant Bernoulli thinning of point process $X = (X_t)_{t\geq 0}$.

Suppose that $(\eta_n)_{n\geq 1}$ is a sequence of random variables taking values 0 and 1, and for all n

$$\eta_n = 0 \qquad \text{on } \{T_n = \infty\}. \tag{3.1}$$

If for all $k \geq 1, \ 1 \leq n_1 < n_2 < \cdots < n_k$, on $\{T_{n_k} < \infty\}$

$$P[\eta_{n_j} = 1, \ j = 1, \cdots, k \,|\, \mathcal{F}_\infty^X] = \prod_{j=1}^{k} p(T_{n_j}) \tag{3.2}$$

where $p(t),\ t > 0$, is a Borel function:

$$0 \le p(t) \le 1 \tag{3.3}$$

then the following two point processes:

$$X_t^1 = \sum_{n=1}^{\infty} \eta_n 1_{T_n \le t}, \qquad t \ge 0, \tag{3.4}$$

$$X_t^2 = \sum_{n=1}^{\infty} (1-\eta_n) 1_{T_n \le t}, \qquad t \ge 0, \tag{3.5}$$

are called thinned processes obtained by non–constant Bernoulli thinning of X. Obviously, we always have

$$\Delta X^1 \Delta X^2 = 0.$$

It is not difficult to verify that (3.2) is equivalent to (1.9) with the transition kernel

$$N(x,t,dy) = p(t)\delta_1(dy) + (1-p(t))\delta_0(dy). \tag{3.6}$$

In this cae, we still denote by $(\mathcal{F}_t)_{t\ge0}$ the natural filtration of marked point process $(T_n,\eta_n)_{n\ge1}$, and by $(\mathcal{F}_t^i)_{t\ge0},\ i=1,2$:

$$\mathcal{F}_t^i = \sigma\{X_s^i,\ s \le t\},\ t \ge 0,\ i = 1,2.$$

LEMMA 1. *For all* $t \ge 0$

$$\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2. \tag{3.7}$$

Proof. Because for all $n \ge 1$, T_n is a stopping time with respect to $(\mathcal{F}_t)_{t\ge0}$, and $\eta_n \in \mathcal{F}_{T_n}$, X^1 and X^2 are adapted to $(\mathcal{F}_t)_{t\ge0}$. Hence

$$\mathcal{F}_t^1 \vee \mathcal{F}_t^2 \subset \mathcal{F}_t, \qquad t \ge 0.$$

On the other hand, $X = X^1 + X^2$ is obviously adapted to $(\mathcal{F}_t^1 \vee \mathcal{F}_t^2)_{t\ge0}$. Therefore, each $T_n,\ n \ge 1$, is a stopping time with respect to $(\mathcal{F}_t^1 \vee \mathcal{F}_t^2)_{t\ge0}$, either. Because of (3.1) we have

$$\eta_n = \Delta X_{T_n}^1 1_{T_n < \infty}.$$

Hence,

$$\mathcal{F}_t \subset \mathcal{F}_t^1 \vee \mathcal{F}_t^2,\ t \ge 0.$$

□

LEMMA 2. *The dual predictable projections of the thinned processes X^1 and X^2 with respect to $(\mathcal{F}_t)_{t\geq 0}$ are*

$$\Lambda_t^1 = \int_0^t p(s)\wedge(ds),\ \Lambda_t^2 = \int_0^t (1-p(s))\wedge(ds),\ t\geq 0, \tag{3.8}$$

respectively.

Proof. Immediately from (1.12) and (3.6). □

THEOREM 3. *If X is a process with independent increments (with respect to $(\mathcal{F}_t^X)_{t\geq 0}$), then the thinned processes $(X_t^1,\ \mathcal{F}_t)_{t\geq 0}$ and $(X_t^2,\ \mathcal{F}_t)_{t\geq 0}$ are processes with independent increments.*

Proof. Since Λ is non–random, Λ^1 and Λ^2 are non–random immediately from (3.8). □

THEOREM 4. *If X is a (nonhomogeneous) Poisson process, then the thinned processes X^1 and X^2 are independent (nonhomogeneous) Poisson processes.*

Proof. In this time, Λ is non–random and continuous, so are Λ^1 and Λ^2. Hence, X^1 and X^2 are Poisson processes. Obviously, Λ^i is the dual predictable projection of X^i with respect to both $(\mathcal{F}_t^i)_{t\geq 0}$ and $(\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2)_{t\geq 0}$, $i = 1,2$. On the other hand, all jump times of X^i are totally inaccessible, $i = 1,2$, and their distributions are continuous. Therefore, two conditions in Theorem 2 are satisfied. And independence between X^1 and X^2 follows from Theorem 2. □

Note that under the assumption of Theorem 3 we know only Λ is non–random, but Λ may be discontinuous, and the independence between X^1 and X^2 cannot be guaranteed. Here is a simple example:

Suppose that $T_n \equiv n$, $n \geq 1$, and $p(t) \equiv p$, $0 < p < 1$. Then $\Lambda(dt) = \sum_{n=1}^{\infty} \delta_n(dt)$ is non–random but discontinuous. We have

$$\begin{aligned} &P(X_1^1 = 1) = P(X_1^2 = 0) = P(X_1^1 = 1,\ X_1^2 = 0) \\ =&P(\eta_1 = 1) = p. \end{aligned}$$

If X^1 and X^2 are independent, then

$$P(X_1^1 = 1,\ X_1^2 = 0) = P(X_1^1 = 1)P(X_1^2 = 0),$$

i.e., $p^2 = p$. This is impossible.

THEOREM 5. *Suppose that*

$$0 < p(t) < 1, \qquad t > 0 \tag{3.9}$$

and the thinned processes X^1 and X^2 are independent. Then the original point process X is Poisson.

Proof. Again using Theorem 2, we know that Λ^i is the dual predictable projection of X^i with respect to $(\mathcal{F}_t^i)_{t\geq 0}$, $i=1,2$. Hence, $(\Lambda_t^1)_{t\geq 0}$ and $(\Lambda_t^2)_{t\geq 0}$ are independent. Because of (3.8) anad (3.9), Λ_t is independent of itself. Therefore, each Λ_t, $t\geq 0$, is constant, and Λ is non–random.

Now we show Λ is continuous. For each $t>0$, if $r_t=\Delta\Lambda_t=E\Delta X_t=P(\Delta X_t=1)>0$, then

$$\begin{aligned} P(\Delta X_t^1=1) &= P(\Delta X_t^1=1,\ \Delta X_t^2=0) \\ &= P(\Delta X_t^1=1)P(\Delta X_t^2=0) \\ &= [P(\Delta X_t^1=1)]^2 \end{aligned} \tag{3.10}$$

On the other hand, by (3.9) we have

$$P(\Delta X_t^1=1)=P(\Delta X_t^1=\Delta X_t=1)=p(t)r_t<1.$$

From (3.9) and (3.10) it must be

$$\begin{aligned} P(\Delta X_t^1=1)=p(t)r_t=0, \\ r_t=0. \end{aligned}$$

Therefore, X is Poisson. □

It is understandable that for the validity of Theorem 5, the condition (3.9) is indispensable.

Theorem 4 and 5 can be extended to multinomial thinnings of point processes. Suppose that $\eta_n^1,\cdots,\eta_n^m$, $n\geq 1$ are random variables taking values 0 and 1, and satisfy

$$\eta_n^1+\cdots+\eta_n^m=1,\ n\geq 1$$

and

$$\eta_n^1=\cdots=\eta_n^{m-1}=0, \quad \text{on } \{T_n=\infty\},\ n\geq 1.$$

Put

$$X_t^k=\sum_{n=1}^{\infty}\eta_n^k 1_{T_n\leq t}, t\geq 0, k=1,\cdots,m.$$

Then $X^1,\cdots,X^m$ are the thinned processes obtained through multinomial thinning of point process X, if $(\eta_n^1,\cdots,n_n^m)_{n\geq 1}$ are conditionally independent, given $\mathcal{F}_\infty^X$, and on $\{T_n<\infty\}$

$$P[\eta_n^k=1\,|\,\mathcal{F}_\infty^X]=p^k(T_n),\ n\geq 1,\ k=1,\cdots,m$$

where $p^k(t)$, $t>0$, $k=1,\cdots,m$, are positive Borel functions satisfying

$$p^1(t)+\cdots+p^m(t)=1,\ t>0.$$

THEOREM 6. *If X is (nonhomogeneous) Poisson, then the thinned processes $X^1, \cdots, X^m$ are mutually independent (nonhomogeneous) Poisson processes. Conversely, if $0 < p^k(t) < 1$, $t > 0$, $k = 1, \cdots, m$, and $X^1, \cdots, X^m$ are mutually independent, then X is (nonhomogeneous) Poisson.*

The proof of Theorem 6, including the generalization of Theorem 2 to the case of several jump processes, is completely similar. We omit it.

If in Theorem 6, instead of the mutual independence of $X^1, \cdots, X^m$, we only assume that there exists a pair of index (k, j), $1 \leq k < j \leq m$, such that X^k and X^j are independent, is the original process X still a Poisson process? This is an interesting open problem.

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